$K_{1,3}$-factors in graphs

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Abstract

Let k be a positive integer. It is shown that if G is a graph of order $4k$ with minimum degree at least $2k$, then G contains k vertex-disjoint copies of $K_{1,3}$, unless G is isomorphic to $K_{2k,2k}$ with k being odd.

1 Introduction

We consider only finite undirected graphs without loops or multiple edges. Let G be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set, respectively. For $x \in V(G)$, the neighborhood of x in G is denoted by $N_G(x)$, and the degree x in G is denoted by $\deg_G(x)$; thus $\deg_G(x) = |N_G(x)|$. We let $\delta(G)$ denote the minimum degree of G, i.e., the minimum of $\deg_G(x)$ as x ranges over $V(G)$. Let H be a fixed connected graph, and assume that $|V(G)|$ is a multiple of $|V(H)|$. In this situation, we say that G has an H-factor if G contains k pairwise vertex-disjoint copies of H.

Many studies have been made concerning minimum degree conditions for a graph to have an H-factor. In this paragraph, it is assumed that $|V(G)|$ is a multiple of $|V(H)|$. Three general results are known. For $H = K_t (t \geq 2)$, Hajnal and Szemerédi [7] proved that if $\delta(G) \geq \frac{t-1}{t}|V(G)|$, then G has a K_t-factor (for the case where $t = 3$, see also Corrádi and Hajnal [2]). For $H = C_t (t \geq 4)$, a special case of a conjecture of El-Zahar

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[4] asserts that the condition $\delta(G) \geq \frac{\lceil t/2 \rceil}{t}|V(G)|$ will suffices for the existence of a C_t-factor, and it is proved in Abassi [1] that the conjecture of El-Zahar is true if $|V(G)|$ is sufficiently large (for the case where $t = 4$, see Erdős and Faudree [6] and Komlós, Sárkőzy and Szemerédi [9]). For $H = P_t (t \geq 3)$, it is proved in Johansson [10] that the condition $\delta(G) \geq \frac{\lceil t/2 \rceil}{t}|V(G)|$ suffices (for the case where $t = 3$, see Enomoto, Kaneko and Tuza [5]). We now consider the case where $|V(F)| \leq 4$.

Let K_4^- and K_4^{--} denote the graphs obtained from K_4 by removing one edge and two adjacent edges, respectively (thus $K_4^- = K_2 + 2K_1$ and $K_4^{--} = (K_2 \cup K_1) + K_1$). There are three cases with $|V(H)| \leq 4$ which are not covered by the results mentioned so far: the case where $H = K_4^-, K_4^{--}$ and $K_{1,3}$. For the case where $H = K_4^-$ and K_4^{--}, Kawarabayashi proved that the condition $\delta(G) \geq \frac{5}{8}|V(G)|$ suffices. In this paper, we consider the case where $H = K_{1,3}$, and prove the following theorem, which was announced without proof in [3].

Main Theorem *Let k be a positive integer. Let G be a graph of order $4k$ and suppose that $\delta(G) \geq 2k$. Then G contains k pairwise vertex-disjoint copies of $K_{1,3}$, unless G is isomorphic to $K_{2k,2k}$ with k being odd.*

Our notation is standard except possibly for the following. Let G be a graph. For a subset L of $V(G)$, we let $\langle L \rangle$ denote the subgraph induced by L and let $e(L) = |E(\langle L \rangle)|$, and define $G - L$ by $G - L = \langle V(G) - V(L) \rangle$. For subsets L and M of $V(G)$ with $L \cap M = \emptyset$, we let $e(L, M)$ denote the number of edges of G joining a vertex in L and a vertex in M. A vertex x is often identified with the set $\{x\}$. Thus if $x \in V(G)$, then $G - x$ means $G - \{x\}$, and $e(x, M)$ means $e(\{x\}, M)$ for $M \subseteq V(G - x)$. Also, for $x \in V(G)$, we often write $N(x)$ for $N_G(x)$, and let $N(x, L) = N(x) \cap L$ for $L \subseteq V(G - x)$.

2 Lemmas on the Graphs of Order Eight

In this section, we present several lemmas which we use in the proof of the main theorem. These lemmas all concern graphs of order eight, in which the vertex set is partitioned into two parts evenly. One of the parts contains $K_{1,3}$ as a subgraph, and the other does not contain $K_{1,3}$. But these lemmas assure that if there are many edges between these parts, then we can find another partition of the vertex set which is 'preferable' to the original one.

We define a partial order $\prec$ on the family of graphs of order four by the lexicographic order of the nonincreasing degree sequences of graphs. (In fact, this order $\prec$ is a total order by chance.) Thus in particular, we have

$$\cdots \prec P_3 \cup K_1 \prec P_4 \prec K_3 \cup K_1 \prec C_4 \prec K_{1,3} \prec \cdots.$$

Lemma 2.1 *Let F be a graph on eight vertices. Let $V(F) = C \cup R$ be a partition such that $\langle C \rangle \supseteq K_{1,3}$ and $\langle R \rangle \simeq P_3 \cup K_1$, and suppose that $e(C, R) \geq 9$. Then there exists another partition $V(F) = C' \cup R'$ such that $\langle C' \rangle \supseteq K_{1,3}$ and $\langle R' \rangle \supsetneq \langle R \rangle$, or $\langle C' \rangle \supseteq K_{1,3}$, $e(C') > e(C)$ and $\langle R' \rangle \simeq \langle R \rangle$,*

Lemma 2.2 *Let F be a graph on eight vertices. Let $V(F) = C \cup R$ be a partition such that $\langle C \rangle \supseteq K_{1,3}$ and $\langle R \rangle \simeq P_4$, and suppose that $e(C,R) \geq 9$. Then there exists another partition $V(F) = C' \cup R'$ such that $\langle C' \rangle \supseteq K_{1,3}$ and $\langle R' \rangle \supsetneqq \langle R \rangle$.*

Lemma 2.3 *Let F be a graph on eight vertices, and let $V(F) = C \cup R$ be a partition such that $\langle C \rangle \supseteq K_{1,3}$ and $\langle R \rangle \simeq K_3 \cup K_1$. Let x be the isolated vertex in $\langle R \rangle$, and suppose that*

$$e(C, R - \{x\}) + 3\, e(C, x) \geq 13.$$

Then there exists another partition $V(F) = C' \cup R'$ such that $\langle C' \rangle \supseteq K_{1,3}$ and $\langle R' \rangle \supsetneqq \langle R \rangle$.

Lemma 2.4 *Let F be a graph on eight vertices, and let $V(F) = C \cup R$ be a partition such that $\langle C \rangle \supseteq K_{1,3}$ and $\langle R \rangle \simeq C_4$. Write $C = \{a, b_1, b_2, b_3\}$ with $ab_1, ab_2, ab_3 \in E(F)$, and $R = \{v_1, v_2, v_3, v_4\}$ with $v_1v_2, v_2v_3, v_3v_4, v_4v_1 \in E(F)$. Suppose that $e(C,R) \geq 8$. Then there exists another partition $V(F) = C' \cup R'$ such that $\langle C' \rangle \supseteq K_{1,3}$ and $\langle R' \rangle \supseteq K_{1,3}$, unless $e(C,R) = 8$ and one of the following holds:*

(I) *for some $i \in \{1,2\}$, $N(v_i, C) = N(v_{i+2}, C) = \{a\}$ and $N(v_{i+1}, C) = N(v_{i+3}, C) = \{b_1, b_2, b_3\}$;*

(II) *for some $j \in \{1,2,3\}$, $N(v_i, C) = \{b_1, b_2, b_3\} - \{b_j\}$ for every $i \in \{1,2,3,4\}$;*

(III) *for some labeling $\{1,2,3\} = \{j_1, j_2, j_3\}$, $N(v_1, C) = N(v_3, C) = \{b_{j_1}, b_{j_2}\}$, and $N(v_2, C) = N(v_4, C) = \{b_{j_2}, b_{j_3}\}$; or*

(IV) *for some $i \in \{1,2,3,4\}$ and some $j \in \{1,2,3\}$, $N(v_i, C) = \{b_j\}$, $N(v_{i+2}, C) = \{b_1, b_2, b_3\}$, and $N(v_{i+1}, C) = N(v_{i+3}, C) = \{b_1, b_2, b_3\} - \{b_j\}$.*

(The indices for v are taken modulo 4.)

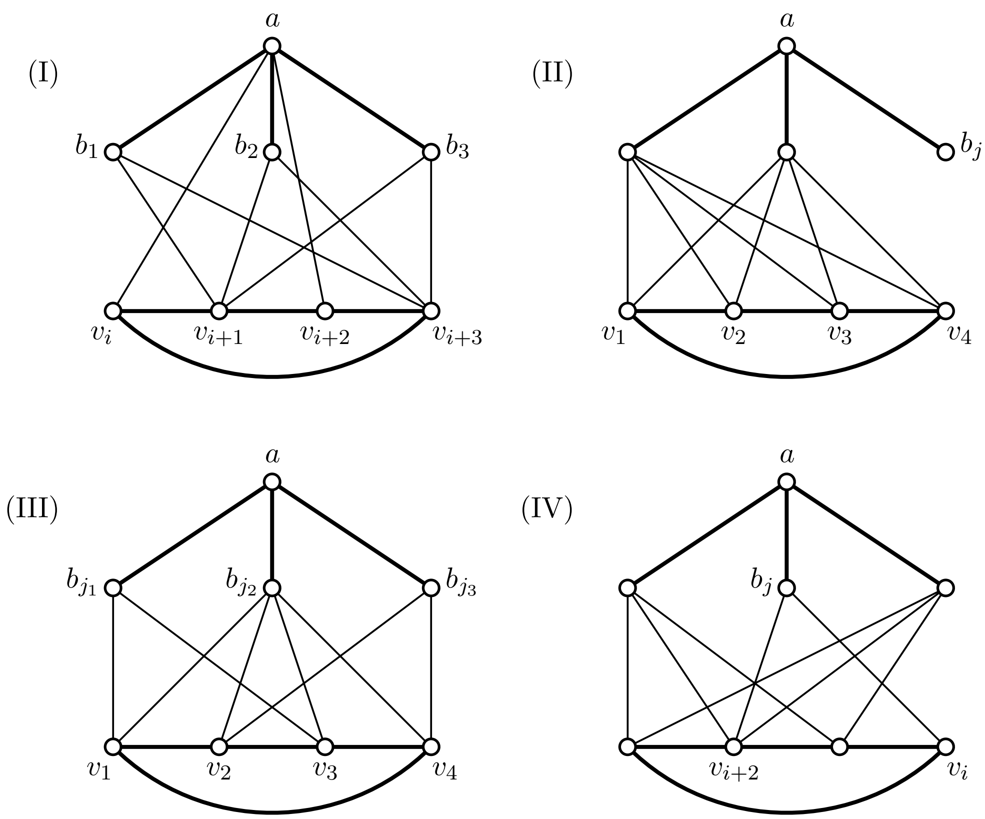
The proofs of the lemmas are not difficult but somewhat tedious. So we shall give the proofs in Section 4.

3 Proof of the Main Theorem

In this section, we prove our main theorem, using the lemmas stated in the preceding section.

Let G be a graph of order $4k$ with minimum degree at least $2k$. Suppose that G does not contain k vertex-disjoint $K_{1,3}$'s. Our aim is to show that k is odd and $G \simeq K_{2k,2k}$. Note that in $K_{2k,2k}$, each vertex has degree precisely $2k$. Thus in showing that k is odd and $G \cong K_{2k,2k}$, we may assume that G is edge-maximal subject to the condition that G does not contain k vertex-disjoint $K_{1,3}$'s.

Claim 1 *$V(G)$ can be partitioned into $S_1, \ldots, S_{k-1}$ and R so that $\langle S_i \rangle \supseteq K_{1,3}$ for every $1 \leq i \leq k-1$ and $\langle R \rangle \supseteq C_4$. Moreover, for any such a partition, $e(S_i, R) = 8$ for every i with $1 \leq i \leq k-1$, and $\deg_G(v) = 2k$ for each $v \in R$.*



Proof. Note that G is not complete. Thus by joining any nonadjacent pair of vertices, we obtain k vertex-disjoint $K_{1,3}$'s. This implies that $V(G)$ can be partitioned into S_1, …, S_{k-1} and R so that $\langle S_i \rangle \supseteq K_{1,3}$ for every $1 \le i \le k-1$ and $\langle R \rangle \supseteq P_3 \cup K_1$. We choose such a partition $V(G) = S_1 \cup \cdots \cup S_{k-1} \cup R$ so that $\langle R \rangle$ is maximal with respect to the order $\prec$, which is defined in Section 2 and, subject to this condition, $\sum_{i=1}^{k-1} e(S_i)$ is as large as possible.

Since $\langle R \rangle \not\supseteq K_{1,3}$, we have $\langle R \rangle \simeq P_3 \cup K_1$, P_4, $K_3 \cup K_1$, or C_4. If $\langle R \rangle \simeq K_3 \cup K_1$, then let x be the isolated vertex in $\langle R \rangle$. Then by Lemma 2.3 and our choice of R, we have $e(S_i, R - \{x\}) + 3e(S_i, x) \le 12$ for every i $(1 \le i \le k-1)$. This implies that

$$\begin{aligned} 12k &\le \sum_{z \in R - \{x\}} \deg_G(z) + 3\deg_G(x) \\ &= \sum_{i=1}^{k-1} \Big(e(S_i, R - \{x\}) + 3e(S_i, x) \Big) + 2e(R) \\ &\le 12(k-1) + 6 \ < \ 12k, \end{aligned}$$

a contradiction. Thus $\langle R \rangle \simeq P_3 \cup K_1$, P_4, or C_4. Then, by Lemmas 2.1, 2.2 and 2.4, and our choice of R, we have $e(S_i, R) \le 8$ for every i with $1 \le i \le k-1$. This implies that

$$8k \le \sum_{z\in R} \deg_G(z) = \sum_{i=1}^{k-1} e(S_i, R) + 2e(R) \le 8(k-1) + 2\cdot 4 = 8k.$$

Hence equality holds throughout. Namely, $e(S_i, R) = 8$ for every i, $\deg_G(z) = 2k$ for every $z \in R$, and $e(R) = 4$, i.e., $\langle R\rangle \simeq C_4$. □

Take an arbitrary partition $V(G) = S_1 \cup \cdots \cup S_{k-1} \cup R$ such that $\langle S_i\rangle \supseteq K_{1,3}$ for $1 \le i \le k-1$ and $\langle R\rangle \supseteq C_4$. Let $S_i = \{a_i, b_{i1}, b_{i2}, b_{i3}\}$ with $a_ib_{i1}, a_ib_{i2}, a_ib_{i3} \in E(G)$, and let $R = \{v_1, v_2, v_3, v_4\}$ with $v_1v_2, v_2v_3, v_3v_4, v_4v_1 \in E(G)$.

Then by Claim 1, $e(S_i, R) = 8$ for each i with $1 \le i \le k-1$. Hence by Lemma 2.4, for each i, $C = S_i$ satisfies one of the conditions (I), (II), (III) and (IV) of Lemma 2.4. We say that S_i is Type (I) ((II), (III) or (IV), resp.) with respect to R, if $C = S_i$ and $R = \{v_1, v_2, v_3, v_4\}$ satisfy (I) ((II), (III) or (IV), resp.) of Lemma 2.4.

We define

$$\begin{aligned} I_1 &= \{i | 1 \le i \le k-1,\ S_i \text{ is Type (I)}\},\\ I_2 &= \{i | 1 \le i \le k-1,\ S_i \text{ is Type (II)}\},\\ I_3 &= \{i | 1 \le i \le k-1,\ S_i \text{ is Type (III)}\},\\ I_4 &= \{i | 1 \le i \le k-1,\ S_i \text{ is Type (IV)}\}; \end{aligned}$$

thus $I_1 \cup I_2 \cup I_3 \cup I_4 = \{1, 2, \ldots, k-1\}$. Also we define

$$\begin{aligned} I_{11} &= \{i \in I_1 | e(v_1, S_i) = e(v_3, S_i) = 3,\ e(v_2, S_i) = e(v_4, S_i) = 1\},\\ I_{12} &= \{i \in I_1 | e(v_1, S_i) = e(v_3, S_i) = 1,\ e(v_2, S_i) = e(v_4, S_i) = 3\},\\ I_{41} &= \{i \in I_4 | e(v_1, S_i) = 3\},\\ I_{42} &= \{i \in I_4 | e(v_2, S_i) = 3\},\\ I_{43} &= \{i \in I_4 | e(v_3, S_i) = 3\},\\ I_{44} &= \{i \in I_4 | e(v_4, S_i) = 3\}; \end{aligned}$$

thus $I_1 = I_{11} \cup I_{12}$ and $I_4 = I_{41} \cup I_{42} \cup I_{43} \cup I_{44}$.

Claim 2 $|I_{11}| = |I_{12}|$, $|I_{41}| = |I_{43}|$, *and* $|I_{42}| = |I_{44}|$.

Proof. Since $\deg_G(v_1) = 2k + |I_{11}| - |I_{12}| + |I_{41}| - |I_{43}| = 2k$ and $\deg_G(v_3) = 2k + |I_{11}| - |I_{12}| + |I_{43}| - |I_{41}| = 2k$ by Claim 1, we have $|I_{11}| = |I_{12}|$ and $|I_{41}| = |I_{43}|$. Similarly, considering the degrees of v_2 and v_4, we obtain $|I_{42}| = |I_{44}|$. □

Claim 3 $I_4 = \emptyset$.

Proof. Suppose that $I_4 \ne \emptyset$. We may assume $I_{41} \ne \emptyset$. Then by Claim 2, $I_{43} \ne \emptyset$. We may also assume that $1 \in I_{41}$ and $2 \in I_{43}$, and $N(v_3, S_1) = \{b_{13}\}$ and $N(v_1, S_2) = \{b_{23}\}$. Let $S_1' = \{b_{11}, a_1, v_2, v_4\}$, $S_2' = \{v_1, b_{12}, b_{13}, b_{23}\}$ and $R' = \{a_2, b_{21}, v_3, b_{22}\}$. Then $\langle S_1'\rangle, \langle S_2'\rangle \supseteq K_{1,3}$ and $\langle R'\rangle \supseteq C_4$. Thus, by considering the partition $V(G) = S_1' \cup S_2' \cup S_3 \cup \cdots \cup S_{k-1} \cup R'$, S_1' satisfies one of the conditions (I), (II), (III) or (IV) of Lemma 2.4. Since $e(v_2, R') = e(v_4, R') = 3$, S_1' must be Type (IV) with respect to R', which forces a_1 to be adjacent to v_3, contrary to the assumption that S_1 is Type (IV) with respect to R. □

Claim 4 $I_3 = \emptyset$.

Proof. Suppose that $1 \in I_3$. We may assume that $N(v_1, S_1) = N(v_3, S_1) = \{b_{11}, b_{12}\}$ and $N(v_2, S_1) = N(v_4, S_1) = \{b_{11}, b_{13}\}$. Let $S_1' = \{b_{12}, a_1, v_1, v_3\}$ and $R' = \{b_{11}, v_2, b_{13}, v_4\}$. Then $\langle S_1' \rangle \supseteq K_{1,3}$ and $\langle R' \rangle \supseteq C_4$. It is easy to see that S_1' is Type (IV) with respect to R'. But since Claim 3 holds for the partition $V(G) = S_1' \cup S_2 \cup \cdots \cup S_{k-1} \cup R'$ as well, this is a contradiction. □

By Claims 3 and 4, $\{1, 2, \ldots, k-1\}$ is the disjoint union $I_1 \cup I_2$. For each $i \in I_2$, we may assume that $e(b_{i1}, R) = e(b_{i2}, R) = 4$ and $e(b_{i3}, R) = 0$.

Claim 5 *If $i \in I_2$ and $j \in I_1$, then $e(\{a_i, b_{i3}\}, S_j) = 0$.*

Proof. Without loss of generality, we let $i = 1$ and $j = 2$, and assume that $N(v_1, S_2) = N(v_3, S_2) = \{b_{21}, b_{22}, b_{23}\}$ and $N(v_2, S_2) = N(v_4, S_2) = \{a_2\}$. Let $S_2' = \{v_3, v_4, b_{22}, b_{23}\}$ and $R' = \{a_2, b_{21}, v_1, v_2\}$. Then $\langle S_2' \rangle \supseteq K_{1,3}$ and $\langle R' \rangle \supseteq C_4$. Applying Claims 3 and 4 to the new partition, we see that S_1 is either Type (I) or Type (II) with respect to R'. Since b_{11} and b_{12} are adjacent to a consecutive pair of vertices in $\langle R' \rangle$, it follows that S_1 is Type (II). Thus $e(\{a_1, b_{13}\}, \{a_2, b_{21}\}) = 0$. By the symmetry of the roles of b_{21}, b_{22} and b_{23}, we also obtain $e(\{a_1, b_{13}\}, \{b_{22}, b_{23}\}) = 0$. □

Claim 6 *If $i, j \in I_2$ $(i \neq j)$, then $e(\{a_i, b_{i3}\}, \{a_j, b_{j3}\}) \leq 1$.*

Proof. Suppose that $1, 2 \in I_2$ and $e(\{a_1, b_{13}\}, \{a_2, b_{23}\}) \geq 2$. It is easy to see that $\langle R \cup \{b_{11}, b_{12}, b_{21}, b_{22}\} \rangle \supseteq K_{4,4} \supseteq 2K_{1,3}$. Since G does not contain a $K_{1,3}$-factor, this implies that $R' = \{a_1, b_{13}, a_2, b_{23}\}$ induces C_4. Let $S_1' = \{b_{11}, v_1, v_2, v_3\}$ and $S_2' = \{v_4, b_{12}, b_{21}, b_{22}\}$. Then $\langle S_1' \rangle, \langle S_2' \rangle \supseteq K_{1,3}$. By Claims 3 and 4, S_1' is either Type (I) or Type (II) with respect to R'. Since the center b_{11} of a $K_{1,3} \subseteq \langle S_1' \rangle$ is adjacent to a vertex $a_1 \in R'$, S_1' is Type (I). But then, b_{13} must be adjacent to v_1, v_2 and v_3, which is a contradiction. This shows that $e(\{a_1, b_{13}\}, \{a_2, b_{23}\}) \leq 1$. □

Claim 7 $I_2 = \emptyset$.

Proof. Suppose that $I_2 \neq \emptyset$. For each $i \in I_2$, we set $X_i = \{b_{i1}, b_{i2}\}$ and $Y_i = \{a_i, b_{i3}\}$, and define $X = \bigcup_{i \in I_2} X_i$ and $Y = \bigcup_{i \in I_2} Y_i$. By Claims 5 and 6, for each $i \in I_2$,

$$\begin{aligned} 4k &\leq \deg_G(a_i) + \deg_G(b_{i3}) \\ &= e(a_i, X) + e(b_{i3}, X) + e(a_i, Y) + e(b_{i3}, Y) \\ &\leq e(a_i, X) + e(b_{i3}, X) + (|I_2| + 1). \end{aligned}$$

Since $|I_2| \leq k - 1$, we have $e(a_i, X) + e(b_{i3}, X) \geq 3k$. This implies that for each $i \in I_2$, there exists an index $j = \tau(i) \in I_2$ such that $X_j \subseteq N(a_i) \cap N(b_{i3})$.

Since I_2 is finite, there exists a sequence of distinct indices $i_1, i_2, \ldots, i_\ell \in I_2$ $(\ell \geq 1)$ such that $i_{j+1} = \tau(i_j)$ for $1 \leq j \leq \ell$ (subscripts are taken modulo ℓ). We may assume that $i_j = j$ for $1 \leq j \leq \ell$. Now we define $S_j' = Y_j \cup X_{j+1}$ for $1 \leq j \leq \ell - 1$, $R' = \{b_{11}, a_\ell, v_1, v_2\}$ and $R'' = \{b_{12}, b_{\ell 3}, v_3, v_4\}$. Then each of $S_1', \ldots, S_{\ell-1}', R'$ and R'' induces a subgraph containing $K_{1,3}$. Hence G contains $K_{1,3}$-factor, a contradiction. □

Thus we have proved that for any partition $V(G) = S_1 \cup \cdots \cup S_{k-1} \cup R$ with $\langle S_i \rangle \supseteq K_{1,3}$ for $1 \le i \le k-1$ and $\langle R \rangle \supseteq C_4$, each S_i is Type (I) with respect to R.

Claim 8 *If $i, j \in I_{11}$ (or $i, j \in I_{12}$, by symmetry) with $i \neq j$, then $a_i a_j \notin E(G)$ and $e(\{b_{i1}, b_{i2}, b_{i3}\}, \{b_{j1}, b_{j2}, b_{j3}\}) = 0$.*

Proof. Suppose that $i, j \in I_{11}$. If $a_i a_j \in E(G)$, then consider the partition $\{a_i, a_j, v_2, v_4\} \cup \{v_1, b_{i1}, b_{i2}, b_{i3}\} \cup \{v_3, b_{j1}, b_{j2}, b_{j3}\}$. If, for example, $b_{i1} b_{j1} \in E(G)$, then consider the partition $\{b_{i1}, a_i, b_{j1}, v_1\} \cup \{v_3, v_4, b_{i2}, b_{i3}\} \cup \{a_j, v_2, b_{j2}, b_{j3}\}$. In either case, the partition shows that $\langle S_i \cup S_j \cup R \rangle$ contains $3K_{1,3}$, a contradiction. □

Claim 9 *If $i \in I_{11}$ and $j \in I_{12}$, then $e(a_i, \{b_{j1}, b_{j2}, b_{j3}\}) = e(a_j, \{b_{i1}, b_{i2}, b_{i3}\}) = 0$.*

Proof. Suppose not. We may assume that $a_i b_{j1} \in E(G)$. Then, considering the partition $\{a_i, b_{j1}, v_2, v_4\} \cup \{a_j, b_{j2}, b_{j3}, v_3\} \cup \{v_1, b_{i1}, b_{i2}, b_{i3}\}$, we find $3K_{1,3}$ in $\langle S_i \cup S_j \cup R \rangle$, a contradiction. □

Recall that we have shown that $|I_{11}| = |I_{12}|$ and $|I_1| = |I_{11}| + |I_{12}| = k-1$. Thus k must be odd. We note that for each i, $i \in I_1$ implies that $\langle S_i \cup R \rangle \supseteq K_{3,5}$. It is easy to see that $K_{3,5}$ with an extra edge contains $2K_{1,3}$. Hence we have $\langle S_i \cup R \rangle \simeq K_{3,5}$. Consequently it follows from Claims 8 and 9 that G is a bipartite graph with partite sets $\bigcup_{i \in I_{11}} \{a_i\} \cup \bigcup_{i \in I_{12}} \{b_{i1}, b_{i2}, b_{i3}\} \cup \{v_1, v_3\}$ and $\bigcup_{i \in I_{12}} \{a_i\} \cup \bigcup_{i \in I_{11}} \{b_{i1}, b_{i2}, b_{i3}\} \cup \{v_2, v_4\}$. Therefore, the minimum degree condition implies that G is isomorphic to $K_{2k,2k}$. This completes the proof of the main theorem. □

4 Proof of the Lemmas

In this section, we prove the four lemmas presented in Section 2. We make use of the following two lemmas.

Lemma 4.1 *Let B be a graph with vertex set $\{a, b_1, b_2, u_1, u_2, u_3\}$, and suppose that*

$$e(\{a, b_1, b_2\}, \{u_1, u_2, u_3\}) \ge 7.$$

Then there exist $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$ such that $a u_j \in E(B)$ and $b_i u_l \in E(B)$ for each $l \in \{1, 2, 3\} - \{j\}$.

Proof. Suppose not. Then for each $i \in \{1, 2\}$ and each $j \in \{1, 2, 3\}$, we have

$$e(a, u_j) + e(b_i, \{u_1, u_2, u_3\} - \{u_j\}) \le 2.$$

By summing up the inequalities over all possibilities for i and j, we obtain

$$\begin{aligned} 12 \;&\ge\; \sum_{i=1}^{2} \sum_{j=1}^{3} \Big(e(a, u_j) + e(b_i, \{u_1, u_2, u_3\} - \{u_j\}) \Big) \\ &=\; 2\, e(\{a, b_1, b_2\}, \{u_1, u_2, u_3\}) \;\ge\; 2 \cdot 7 \;=\; 14, \end{aligned}$$

a contradiction. □

Lemma 4.2 *Let F be a graph with the vertex set $\{b_1, b_2, b_3, u_1, u_2, u_3\}$, and suppose that*

$$e(\{b_1, b_2, b_3\}, \{u_1, u_2, u_3\}) \geq 7.$$

Then there exist $i, j \in \{1, 2, 3\}$ with $i \neq j$ such that $b_1u_i, b_2u_j, b_3u_j \in E(F)$.

Proof. Suppose not. Then we have

$$\begin{aligned} &e(b_1, u_1) + e(\{b_2, b_3\}, u_2) \leq 2, \\ &e(b_1, u_2) + e(\{b_2, b_3\}, u_3) \leq 2, \text{ and} \\ &e(b_1, u_3) + e(\{b_2, b_3\}, u_1) \leq 2. \end{aligned}$$

These three inequalities imply $e(\{b_1, b_2, b_3\}, \{u_1, u_2, u_3\}) \leq 6$, a contradiction. □

Proof of Lemma 2.1. Let $C = \{a, b_1, b_2, b_3\}$ with $ab_1, ab_2, ab_3 \in E(F)$, and let $R = \{u, v, w, x\}$ with $uv, vw \in E(F)$. Set $B = \{b_1, b_2, b_3\}$. Suppose that F has no desired partition.

Since $e(C, R) \geq 9$, we have $e(B, \{u, v, w\}) > 0$. If $ax \in E(F)$, then taking $b_i \in B$ such that $e(b_i, \{u, v, w\}) \neq \emptyset$, we get $\langle \{a, x\} \cup (B - \{b_i\}) \rangle \supseteq K_{1,3}$ and $\langle b_i, u, v, w \rangle \supsetneqq P_3 \cup K_1$, a contradiction. Thus $ax \notin E(F)$.

Suppose that $au \in E(F)$. Then for each $i \in \{1, 2, 3\}$, $e(b_i, \{v, w, x\}) \leq 1$, for otherwise $\langle b_i, v, w, x \rangle \supsetneqq P_3 \cup K_1$ while $\langle \{a, u\} \cup (B - \{b_i\}) \rangle \supseteq K_{1,3}$. Thus in particular, $e(b_i, R) \leq 2$ for $i = 1, 2, 3$. By the condition $e(C, R) \geq 9$, the equalities $e(b_i, R) = 2$ for $i = 1, 2, 3$ and $e(a, R) = 3$ must hold. This implies that $av, aw \in E(F)$ and $ub_1, ub_2, ub_3 \in E(F)$. Hence $\langle u, b_1, b_2, b_3 \rangle \supseteq K_{1,3}$ and $\langle a, v, w, x \rangle \supsetneqq P_3 \cup K_1$, a contradiction. Thus, $au \notin E(F)$. By symmetry, we also obtain $aw \notin E(F)$. Consequently $e(a, R) \leq 1$, and hence $e(B, R) \geq 8$.

We here prove the following claim.

Claim 1 Let $i \in \{1, 2, 3\}$, and suppose that $e(b_i, R) \geq 3$. Take $Z \subseteq N(b_i, R)$ with $|Z| = 3$, and write $R - Z = \{y\}$. Then $e(B - \{b_i\}, y) = 0$.

To prove the claim, suppose that $e(B - \{b_i\}, y) > 0$. Then $\langle \{b_i\} \cup Z \rangle \supseteq K_{1,3}$ and $\langle \{a, y\} \cup (B - \{b_i\}) \rangle \supsetneqq P_3 \cup K_1$, a contradiction. Thus the claim is proved.

Now if $e(b_i, R) = 4$ for some i, then it follows from Claim 1 that $e(B - \{b_i\}, R) = 0$, and hence $e(B, R) = 4$, a contradiction. Consequently $e(b_i, R) \leq 3$ for each i. Thus we may assume $e(b_1, R) = e(b_2, R) = 3$ and $e(b_3, R) \geq 2$. Then $N(b_1, R) = N(b_2, R) \supseteq N(b_3, R)$ by Claim 1.

We show that $v \in N(b_1, R)$. Suppose that $v \notin N(b_1, R)$. Then $N(b_1, R) = \{u, w, x\}$. By the symmetry of u and w, we may assume $u \in N(b_3, R)$. But then $\langle b_1, a, w, x \rangle \supseteq K_{1,3}$ and $\langle u, b_2, b_3, v \rangle \supseteq K_{1,3}$, a contradiction. Thus $v \in N(b_1, R)$. Also we may assume $u \in N(b_1, R)$. If $x \in N(b_1, R)$, then $\langle v, b_2, u, w \rangle \supseteq K_{1,3}$ and $\langle a, b_1, b_3, x \rangle \supsetneqq P_3 \cup K_1$, a contradiction. Thus $x \notin N(b_1, R)$, and hence $N(b_1, R) = \{u, v, w\}$. Therefore $\langle b_1, u, v, w \rangle \supseteq K_{1,3}, e(\{b_1, u, v, w\}) \geq 5$ and $\langle a, b_2, b_3, x \rangle \succ P_3 \cup K_1$. This implies that $\langle a, b_2, b_3, x \rangle \cong P_3 \cup K_1$, i.e., $b_2b_3 \notin E(F)$, and that $e(C) \geq e(\{b_1, u, v, w\}) \geq 5$. Now since we similarly obtain $b_1b_3 \notin E(F)$ by the symmetry of b_1 and b_2, we see that $e(B) \leq 1$. On the other hand, from $e(C) \geq 5$, it follows that $e(B) \geq 2$, a contradiction. $\Box$

Proof of Lemma 2.2. Let $C = \{a, b_1, b_2, b_3\}$ with $ab_1, ab_2, ab_3 \in E(F)$, and let $R = \{u, v, w, x\}$ with $uv, vw, wx \in E(F)$. Set $B = \{b_1, b_2, b_3\}$. Suppose that F does not have a desired partition. If $B \subseteq N(x)$ and $av \in E(F)$, then $\langle \{x\} \cup B \rangle \supseteq K_{1,3}$ and $\langle (R - \{x\}) \cup \{a\} \rangle \supseteq K_{1,3}$, a contradiction. Thus

$$e(B, x) + e(a, v) \leq 3,$$

and we similarly obtain

$$e(B, u) + e(a, w) \leq 3.$$

We first claim that $au \notin E(F)$. Suppose to the contrary that $au \in E(F)$. If $b_iw \in E(F)$ or $e(b_i, \{v, x\}) \geq 2$ for some i, then $\langle \{a, u\} \cup (B - \{b_i\}) \rangle \supseteq K_{1,3}$ and $\langle b_i, v, w, x \rangle \supsetneqq P_4$, a contradiction. Thus for each $i \in \{1, 2, 3\}$, $b_iw \notin E(F)$ and $e(b_i, \{v, x\}) \leq 1$. Now if we also have $ax \in E(F)$, then we similarly get $b_iv \notin E(F)$ for each $i \in \{1, 2, 3\}$, and hence $e(C, R) = 2 + e(a, \{v, w\}) + e(B, \{u, x\}) = 2 + (e(a, v) + e(B, x)) + (e(a, w) + e(B, u)) \leq 8$, a contradiction. Thus we have $ax \notin E(F)$. Since $e(B, \{v, w, x\}) = \sum_{i=1}^{3} e(b_i, \{v, x\}) \leq 3$, it follows that

$$\begin{aligned} 9 \ \leq \ e(C, R) \ &= \ e(a, \{u, v, w\}) + e(B, \{v, w, x\}) + e(B, u) \\ &= \ e(a, \{u, v\}) + e(B, \{v, w, x\}) + e(a, w) + e(B, u) \\ &\leq \ 2 + 3 + 3 \ = \ 8. \end{aligned}$$

This is a contradiction. Thus we have proved $au \notin E(F)$. By symmetry, we also obtain $ax \notin E(F)$.

We now prove two claims.

Claim 1 *Suppose that $e(B, u) \geq 2$, and take $A \subseteq N(u, B)$ with $|A| = 2$. Write $B - A = \{b_i\}$. Then $b_iw \notin E(F)$.*

To prove the claim, suppose that $b_iw \in E(F)$. Then $\langle w, b_i, v, x \rangle \supseteq K_{1,3}$ and $\langle \{a, u\} \cup A \rangle \supseteq C_4$, a contradiction. Thus the claim is proved.

Claim 2 $e(B, \{u, w\}) \leq 4$. *Moreover, if equality holds, then (a) $e(B, u) = 1$ and $e(B, w) = 3$, or (b) $N(u, B) = N(w, B)$.*

To prove the claim, suppose that $e(B, \{u, w\}) \geq 4$. If $e(B, u) \leq 1$, (a) clearly holds. Thus we may assume $e(B, u) \geq 2$. If $e(B, u) = 3$, then $e(B, w) = 0$ by Claim 1, and hence $e(B, \{u, w\}) = 3$, a contradiction. Thus $e(B, u) = e(B, w) = 2$, and it follows from Claim 1 that (b) holds.

Now since $e(C, R) \geq 9$, without loss of generality, we may assume that $e(C, \{u, w\}) \geq 5$. Since $au \notin E(F)$, it follows from Claim 2 that

$$5 \ \leq \ e(C, \{u, w\}) \ = \ e(a, w) + e(B, \{u, w\}) \ \leq \ 1 + 4 \ = \ 5,$$

and hence $aw \in E(G)$ and $e(B, \{u, w\}) = 4$. Therefore, (a) or (b) of Claim 2 holds.

Case 1. $e(B,u)=1$ and $e(B,w)=3$.

We may assume that $N(u,B)=\{b_1\}$. If $av\in E(F)$, then $\langle w,x,b_2,b_3\rangle\supseteq K_{1,3}$ and $\langle a,b_1,u,v\rangle\supseteq C_4$. Hence $av\notin E(G)$. Consequently

$$9\ \le\ e(C,R)\ =\ e(C,\{u,w\})+e(C,\{v,x\})\ =\ 5+e(B,\{v,x\}),$$

and hence $e(B,\{v,x\})\ge 4$. Applying Claim 2 to $\{v,x\}$, we see that this in particular implies that $e(B,v)\ge 2$. Let B' be a subset of $N(v,B)$ with $|B'|=2$. Then $\langle\{u,v\}\cup B'\rangle\supseteq K_{1,3}$ and $\langle\{w,x\}\cup(B-B')\rangle\supseteq K_{1,3}$, a contradiction.

Case 2. $e(B,u)=e(B,w)=2$ and $N(u,B)=N(w,B)$.

We may assume $N(u,B)=N(w,B)=\{b_1,b_2\}$. If $xb_3\in E(F)$, then $\langle u,v,b_1,b_2\rangle\supseteq K_{1,3}$ and $\langle a,w,x,b_3\rangle\supseteq C_4$. Hence $xb_3\notin E(F)$. If $vb_1\in E(F)$, then $\langle a,b_2,b_3,w\rangle\supseteq K_{1,3}$ and $\langle b_1,u,v,x\rangle\supseteq K_3\cup K_1$. Hence $vb_1\notin E(F)$ and, by symmetry, we also obtain $vb_2\notin E(F)$. Therefore

$$\begin{aligned}9\ \le\ e(C,R)\ &=\ e(C,\{u,w\})+e(C,\{v,x\})\\ &=\ 5+e(\{a,b_3\},v)+e(\{b_1,b_2\},x).\end{aligned}$$

This implies that $va,vb_3,xb_1,xb_2\in E(F)$. But then $\langle u,v,w,b_3\rangle\supseteq K_{1,3}$ and $\langle a,b_1,b_2,x\rangle\supseteq C_4$, a contradiction.

This completes the proof of Lemma 2.2. □

Proof of Lemma 2.3. Let $C=\{a,b_1,b_2,b_3\}$ with $ab_1,ab_2,ab_3\in E(F)$. Set $B=\{b_1,b_2,b_3\}$. By assumption,

$$e(C,R-\{x\})+3\,e(C,x)\ \ge\ 13. \qquad (*)$$

Suppose that F has no desired partition. We show that $xa\notin E(F)$. Suppose that $xa\in E(F)$. Then it is easy to see that $e(B,R-\{x\})=0$, for otherwise F contains two vertex-disjoint $K_{1,3}$'s. Hence $e(C,R-\{x\})=e(a,R-\{x\})\le 3$. This together with $(*)$ implies that $e(C,x)=4$. Again by $(*)$, we have $e(a,R-\{x\})>0$. But then $C'=B\cup\{x\}$ and $R'=(R-\{x\})\cup\{a\}$ satisfy the conclusion.

Thus $ax\notin E(F)$, and hence $e(C,x)=e(B,x)\le 3$. As in the proof of Lemma 2.2, the following claim holds.

Claim 1 Suppose that $e(B,x)\ge 2$, and take $A\subseteq N(x,B)$ with $|A|=2$. Write $B-A=\{b_i\}$. Then $e(b_i,R-\{x\})=0$.

Now if $e(C,x)=e(B,x)=3$, then it follows from Claim 1 that $e(B,R-\{x\})=0$, and hence $e(C,R-\{x\})+3e(C,x)\le 3+9$, which contradicts $(*)$. Thus $e(C,x)\le 2$.

Assume first that $e(C,x)=2$. We may assume that $b_1x,b_2x\in E(F)$. Then we have $e(C,R-\{x\})\ge 7$ by $(*)$, while $e(b_3,R-\{x\})=0$ by Claim 1. Hence by Lemma 4.1, there exists $v\in R-\{x\}$ and $b_i\in\{b_1,b_2\}$ such that $av\in E(F)$ and $e(b_i,R-\{x,v\})=2$. But then $\langle(R-\{v\})\cup\{b_i\}\rangle\supseteq K_{1,3}$ and $\langle\{a,v\}\cup(B-\{b_i\})\rangle\supseteq K_{1,3}$.

Thus we may assume that $e(C, x) \leq 1$. Then by $(*)$, $e(C, x) = 1$, say $xb_1 \in E(F)$. By $(*)$, we also have $e(C, R - \{x\}) \geq 10$, and hence $e(\{b_1, b_2, b_3\}, R - \{x\}) \geq 7$. Then by Lemma 4.2, there exist distinct vertices u and v in $R - \{x\}$ such that $ub_1, vb_2, vb_3 \in E(F)$. Then $\langle a, b_1, u, x\rangle \supseteq K_{1,3}$ and $\langle (R - \{x, u\}) \cup \{b_2, b_3\}\rangle \supseteq K_{1,3}$, a contradiction. □

Proof of Lemma 2.4. Suppose that F does not contain two vertex-disjoint $K_{1,3}$'s. Set $B = \{b_1, b_2, b_3\}$.

Case 1. $e(a, R) > 0$.

We first prove two claims.

Claim 1 *Suppose that $v_i a \in E(F)$. Then,*

(i) $e(B, v_{i+2}) = 0$,

(ii) $e(C, \{v_i, v_{i+2}\}) \leq 4$, *and*

(iii) *if* $e(C, \{v_i, v_{i+2}\}) \geq 3$, *then* $e(C, v_{i+2}) = 0$.

Statement (i) is trivial, for otherwise we can find two vertex-disjoint $K_{1,3}$'s. If $e(C, v_{i+2}) = 0$, then both (ii) and (iii) clearly hold. Thus suppose that $e(C, v_{i+2}) > 0$. Then by (i), $v_{i+2}a \in E(F)$, which in turn implies that $e(B, v_i) = 0$ by (i). Consequently $e(C, \{v_i, v_{i+2}\}) \leq 2$. This implies (ii) and (iii).

Claim 2 *No two vertices u and w in R have the following property;*

$$e(C, u) \geq 2,\ e(C, w) \geq 2, \quad \textit{and} \quad C \subseteq N(u) \cup N(w). \qquad (*)$$

Suppose u and w in R satisfy $(*)$. Then we can divide C into $B_1 \subseteq N(u)$ and $B_2 \subseteq N(w)$ such that $|B_1| = |B_2| = 2$. Also, since $\langle R\rangle \simeq C_4$, we can choose $u' \in N(u) \cap R - \{u\}$ and $w' \in N(w) \cap R - \{u, u'\}$. Then both $\langle B_1 \cup \{u, u'\}\rangle$ and $\langle B_2 \cup \{w, w'\}\rangle$ contain $K_{1,3}$.

Now without loss of generality, we may assume that $v_1 a \in E(F)$. Assume first that $e(C, \{v_2, v_4\}) \geq 5$. Then by Claim 1(ii), neither v_2 nor v_4 can be adjacent to a. Thus we may assume that $e(C, v_2) = e(B, v_2) = 3$. Then $N(v_1) \cup N(v_2) \supseteq C$ and hence $e(C, v_1) = 1$ by Claim 2. On the other hand, since $e(C, R) \geq 8$ and $e(C, \{v_2, v_4\}) = e(B, \{v_2, v_4\}) \leq 6$, we have $e(C, \{v_1, v_3\}) \geq 2$. Consequently $e(C, v_3) \geq 1$. In view of Claim 1(i), this forces $e(C, v_3) = e(a, v_3) = 1$, which in turn implies that $e(B, \{v_2, v_4\}) = 6$. Hence F satisfies (I). Assume now that $e(C, \{v_2, v_4\}) \leq 4$. Then $e(C, \{v_1, v_3\}) \geq 4$. By Claim 1(ii), this implies $e(C, \{v_1, v_3\}) = 4$. Hence by Claim 1(iii), $e(C, v_3) = 0$ and v_1 is adjacent to all vertices in C. Therefore by Claim 2, we have $e(C, v_2) \leq 1$ and $e(C, v_4) \leq 1$. Consequently $e(C, R) = e(C, \{v_1, v_3\}) + e(C, v_2) + e(C, v_4) \leq 6$, a contradiction.

Case 2. $e(a, R) = 0$.

Claim 3 *If $b_j v_i, b_j v_{i+1} \in E(F)$, then $e(v_{i+2}, B - \{b_j\}) \leq 1$ and $e(v_{i+3}, B - \{b_j\}) \leq 1$.*

If not, then both $\langle a, b_j, v_i, v_{i+1}\rangle$ and $\langle\{v_{i+2}, v_{i+3}\} \cup (B - \{b_j\})\rangle$ contain $K_{1,3}$.

Now since $e(C, R) = e(B, R) \geq 8$, we may assume that $e(b_1, R) \geq 3$, say $b_1v_1, b_1v_2, b_1v_3 \in E(F)$. Then by Claim 3, for each $i \in \{1, 3, 4\}$, $e(v_i, \{b_2, b_3\}) \leq 1$.

Assume first that b_1v_4 is also an edge. Then we also have $e(v_2, \{b_2, b_3\}) \leq 1$, and hence

$$8 \leq e(C, R) = e(b_1, R) + e(\{b_2, b_3\}, R) \leq 4 + 4 = 8.$$

Thus equality must hold. This implies that every vertex v_i is adjacent to exactly one of b_2 and b_3. If b_2 or b_3, say, b_2, is adjacent to a consecutive pair of vertices in R, then by Claim 3, the rest of the vertices in R cannot be adjacent to b_3 because they are already adjacent to b_1, which implies that F satisfies (II). Otherwise, b_2 is adjacent to a nonadjacent pair of vertices in R, and b_3 is adjacent to the rest, and hence F satisfies (III).

Finally assume that $b_1v_4 \notin E(F)$. Then

$$8 \leq e(C, R) = e(b_1, R) + e(\{b_2, b_3\}, \{v_1, v_3, v_4\}) + e(\{b_2, b_3\}, v_2) \leq 3 + 3 + 2 = 8.$$

Thus equality holds. In particular, v_2 is adjacent to both b_2 and b_3. Since $e(\{b_2, b_3\}, v_3) = 1$, we may assume that $v_3b_3 \in E(F)$. Then since $e(\{b_1, b_2\}, v_2) = 2$, it follows from Claim 3 that v_4 cannot be adjacent to b_3. Since $e(\{b_2, b_3\}, v_4) = 1$, this implies $b_2v_4 \in E(G)$. By Claim 3, we also have $e(\{b_1, b_2\}, v_1) \leq 1$. Since $e(\{b_2, b_3\}, v_1) = 1$ and $b_1v_1 \in E(G)$, this implies $b_3v_1 \in E(G)$. Consequently F is of type (IV). □

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