Heavy cycles in hamiltonian weighted graphs

Jun Fujisawa[1] Shinya Fujita[2] Tomoki Yamashita[3]

Department of Mathematics

Keio University

Yokohama, 223-8522 Japan

Abstract

Let G be a 2-connected weighted graph such that the minimum weighted degree is at least d. In [1], Bondy and Fan proved that either G contains a cycle of weight at least $2d$ or every heaviest cycle in G is a hamiltonian cycle. If G is not hamiltonian, this theorem implies the existence of a cycle of weight at least $2d$, but in case of G is hamiltonian we cannot decide whether G has a heavy cycle or not. In this paper, we prove that if G is triangle-free, then G has a cycle of weight at least $2d$ even in case of G is hamiltonian.

Keywords. weighted graph, heavy cycle, triangle-free.

AMS classification. 05C38, 05C45

1 Introduction

We only consider undirected graphs which have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of a graph G, respectively. A weighted graph is one in which every edge e is assigned a nonnegative real number $w(e)$, called the *weight* of e. For a subgraph H of G, the *weight* of H is defined by

$$w(H) = \sum_{e \in E(H)} w(e).$$

For each vertex $v \in V(G)$, $N_G(v)$ is the set, and $d_G(v)$ the number, of neighbors of v in G. We define the *weighted degree* of v in G by

$$d^w_G(v) = \sum_{u \in N_G(v)} w(uv).$$

[1]e-mail: fujisawa@math.keio.ac.jp

[2]e-mail: shinyaa@comb.math.keio.ac.jp

[3]e-mail: tomoki@comb.math.keio.ac.jp

Let x and y be distinct vertices in G. We define

$$\varepsilon_G(xy) = \begin{cases} 0 & \text{if } xy \notin E(G) \\ 1 & \text{if } xy \in E(G). \end{cases}$$

When no confusion occurs, we denote $N_G(v)$, $d_G(v)$, $d^w_G(v)$ and $\varepsilon_G(xy)$ by $N(v)$, $d(v)$, $d^w(v)$ and $\varepsilon(xy)$, respectively.

A cycle which contains all the vertices of a graph is called a *hamiltonian cycle*. If a graph G has a hamiltonian cycle, then we call G *hamiltonian*. A *triangle-free* graph is one which contains no cycle of length 3. Moreover, we call a path P a *longest heaviest path* of G if

(i) $w(P)$ is maximum, and

(ii) P is a longest path of G subject to (i).

In [1], Bondy and Fan proved the following theorem.

Theorem 1 (Bondy and Fan [1]). *Let G be a 2-connected weighted graph and let d be a nonnegative real number. If $d^w(v) \geq d$ for every vertex v in G, then*

(a) G has a cycle of weight at least $2d$, or

(b) every heaviest cycle in G is a hamiltonian cycle.

If we consider the weighted complete graph in which every edge has weight 1, we know that conclusion (b) of Theorem 1 cannot be dropped. However, there are a lot of graphs in which both (a) and (b) of Theorem 1 hold and we cannot obtain the existence of a heavy cycle by this theorem. In this paper, we prove that if G is a triangle-free weighted graph, we can find a heavy cycle even if G is hamiltonian. Our main result is the following.

Theorem 2. *Let G be a 2-connected triangle-free weighted graph and let d be a nonnegative real number. If $d^w(v) \geq d$ for every vertex v in G, then G has a cycle of weight at least $2d$.*

2 Proof of Theorem 2

In our proof of Theorem 2, the following lemma is essential.

Lemma 1. *Let G be a weighted graph and let P be a longest heaviest path of G with endvertices x and y. Assume that*

$$d(x) + d(y) - \varepsilon(xy) \leq |E(P)|.$$

Then

- *if $xy \notin E(G)$, then P has weight at least $d^w_G(x) + d^w_G(y)$, and*

- *if $xy \in E(G)$, then the cycle $xPyx$ has weight at least $d^w_G(x)+d^w_G(y)$.*

Proof. Let $P = a_1a_2\dots a_p$ be a longest heaviest path of G where $a_1 = x$ and $a_p = y$. Then we have $N(a_1) \subseteq V(P)$ and $N(a_p) \subseteq V(P)$. Let

- $N_1 = \{a_i \mid a_i \in N_G(a_1),\ a_{i-1} \notin N_{G-a_1}(a_p)\}$,
- $N_2 = \{a_i \mid a_i \in N_G(a_1),\ a_{i-1} \in N_{G-a_1}(a_p)\}$,
- $N_3 = \{a_i \mid a_i \in N_{G-a_1}(a_p),\ a_{i+1} \notin N_G(a_1)\}$ and
- $N_4 = \{a_i \mid a_i \in N_{G-a_1}(a_p),\ a_{i+1} \in N_G(a_1)\}$.

Moreover, let

$$E_1 = \{a_1v \mid v \in N_1\},\ E_2 = \{a_1v \mid v \in N_2\},\ E_3 = \{va_p \mid v \in N_3\} \text{ and } E_4 = \{va_p \mid v \in N_4\}.$$

Now we define a mapping φ_1 of $\bigcup_{i=1}^3 E_i$ to $E(P)$ such that

- for $e = a_1a_i \in E_1 \cup E_2$, $\varphi_1(e) = a_{i-1}a_i$ and
- for $e = a_ia_p \in E_3$, $\varphi_1(e) = a_ia_{i+1}$,

and let $F_i = \{\varphi_1(e) \mid e \in E_i\}$ for $i = 1,2,3$. Now it is easy to see that $F_1 \cap F_2 = \emptyset$. And by the definition of E_3, $a_{i+1} \notin N(a_1)$ if $a_ia_p \in E_3$, hence F_1, F_2 and F_3 are disjoint.

It follows from the fact $d(x) + d(y) - \varepsilon(xy) \le |E(P)|$ that

$$\begin{aligned}
\sum_{i=1}^{3}|F_i| &= |E_1| + |E_2| + |E_3| \\
&= |N_1| + |N_2| + |N_3| \\
&= |N(a_1)| + |N(a_p) \setminus \{a_1\}| - |N_4| \\
&\le |E(P)| - |N_4| \\
&= |E(P)| - |E_4|.
\end{aligned}$$

Thus $|E(P)\setminus\bigcup_{i=1}^3 F_i| \ge |E_4|$. Let φ_2 be an injection of E_4 to $E(P)\setminus\bigcup_{i=1}^3 F_i$ and let $F_4 = \{\varphi_2(e) \mid e \in E_4\}$. Note that F_1, F_2, F_3 and F_4 are disjoint.

Assume that $a_1a_i \in E_1$ and $Q_1 = a_{i-1}a_{i-2}\dots a_1a_ia_{i+1}\dots a_p$. Then, since $w(Q_1) \le w(P)$, $w(a_1a_i) \le w(\varphi_1(a_1a_i))$. By the similar argument as above, we have $w(e) \le w(\varphi_1(e))$ for all $e \in E_1 \cup E_3$. Suppose $a_j \in N_2$. Then we have $a_{j-1}a_p \in E_4$. Let C be a cycle $a_ja_1a_2\dots a_{j-1}a_pa_{p-1}\dots a_j$ and $e = \varphi_2(a_{j-1}a_p)$. Since $e \in E(C)$, $Q_2 = C - e$ is a path in G. Then it follows from the fact $w(Q_2) \le w(P)$ that $w(a_1a_j) + w(a_{j-1}a_p) \le w(\varphi_1(a_1a_j)) + w(\varphi_2(a_{j-1}a_p))$ for all $a_j \in N_2$. Therefore, if $a_1a_p \notin E(G)$,

$$d^w(a_1) + d^w(a_p) = \sum_{v\in N_1} w(a_1v) + \sum_{v\in N_2} w(a_1v) + \sum_{v\in N_3} w(va_p) + \sum_{v\in N_4} w(va_p)$$

$$
\begin{aligned}
&= \sum_{e\in E_1} w(e) + \sum_{e\in E_3} w(e) + \sum_{a_j\in N_2} (w(a_1a_j) + w(a_{j-1}a_p)) \\
&\le \sum_{e\in F_1} w(e) + \sum_{e\in F_3} w(e) + \sum_{e\in F_2} w(e) + \sum_{e\in F_4} w(e) \\
&\le w(P),
\end{aligned}
$$

which implies the assertion. And in case of $a_1a_p \in E(G)$,

$$
\begin{aligned}
d^w(a_1) + d^w(a_p) &= \sum_{v\in N_1} w(a_1v) + \sum_{v\in N_2} w(a_1v) + \sum_{v\in N_3} w(va_p) + \sum_{v\in N_4} w(va_p) + w(a_1a_p) \\
&= \sum_{e\in E_1} w(e) + \sum_{e\in E_3} w(e) + \sum_{a_j\in N_2} (w(a_1a_j) + w(a_{j-1}a_p)) + w(a_1a_p) \\
&\le \sum_{e\in F_1} w(e) + \sum_{e\in F_3} w(e) + \sum_{e\in F_2} w(e) + \sum_{e\in F_4} w(e) + w(a_1a_p) \\
&\le w(P) + w(a_1a_p).
\end{aligned}
$$

Hence the cycle $a_1Pa_pa_1$ has weight at least $d^w(a_1) + d^w(a_p)$, which implies the assertion. □

Now we prove Theorem 2 by using Lemma 1 and the following lemma.

Lemma 2 (Bondy and Fan [2]). *Let G be a 2-connected weighted graph and let P be a heaviest path in G with endvertices x and y. Then there exists a cycle C in G such that $w(C) > w(P)$ or $w(C) \ge d^w(x) + d^w(y)$.*

Proof of Theorem 2. Let P be a longest heaviest path in G, and let x, y be endvertices of P. Since G is triangle-free and $N(x), N(y) \subseteq V(P)$, $|N(x)| \le |V(P)|/2$ and $|N(y)| \le |V(P)|/2$. Moreover, if $xy \notin E(G)$, $|N(x)| \le (|V(P)| - 1)/2$ and $|N(y)| \le (|V(P)| - 1)/2$. Hence, whether x and y are adjacent or not, we have $d(x) + d(y) - \varepsilon(xy) \le |E(P)|$. In case of $xy \in E(G)$, Lemma 1 implies the existence of a cycle of weight at least $d^w(x) + d^w(y) \ge 2d$, which is a required cycle. Thus we may assume $xy \notin E(G)$, then Lemma 1 implies that $w(P) \ge d^w(x) + d^w(y) \ge 2d$. Now it follows from Lemma 2 that there exists a cycle C in G such that $w(C) > w(P) \ge 2d$ or $w(C) \ge d^w(x)+d^w(y) \ge 2d$, which is a required cycle. □

Acknowledgments

The authors would like to thank Professor Katsuhiro Ota for his valuable suggestions.

References

[1] J.A. Bondy and G. Fan, Optimal paths and cycles in weighted graphs, *Annals of Discrete Mathematics* **41** (1989), 53–69.

[2] J.A. Bondy and G. Fan, Cycles in weighted graphs, *Combinatorica* **11** (1991), no. 3, 191–205.