Vertex-Disjoint Copies of $K_1 + (K_1 \cup K_2)$ in Graphs

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Abstract

Let S denote the graph obtained from K_4 by removing two edges which have an endvertex in common. Let k be an integer with $k \geq 2$. Let G be a graph with $|V(G)| \geq 4k$ and $\sigma_2(G) \geq |V(G)|/2 + 2k - 1$, and suppose that G contains k vertex-disjoint triangles. In the case where $|V(G)| = 4k + 2$, suppose further that $G \ncong K_{4t+3} \cup K_{4k-4t-1}$ for any t with $0 \leq t \leq k - 1$. Under these assumptions, we show that G contains k vertex-disjoint copies of S.

1 Introduction

In this paper, we consider only finite, simple, undirected graphs with no loops and no multiple edges. For a graph G, we denote by $V(G)$, $E(G)$ and $\delta(G)$ the vertex set, the edge set and the minimum degree of G, respectively. For a vertex x of a graph G, the neighborhood of x in G is denoted by $N_G(x)$, and we let $d_G(x) := |N_G(x)|$. For a noncomplete graph G, let $\sigma_2(G)$:=min$\{d_G(x) + d_G(y) |\ xy \notin E(G)\}$; if G is a complete graph, let $\sigma_2(G) := \infty$. For a subset L of $V(G)$, the subgraph induced by L is denoted by $\langle L \rangle$. For a subset M of $V(G)$, we let $G - M = \langle V(G) - M \rangle$ and, for a vertex x of G, we let $G - x = \langle V(G) - \{x\} \rangle$. For subsets L and M of $V(G)$, we let $E(L, M)$ denote the set of edges of G joining a vertex in L and a vertex in M. When L or M consists of a single vertex, say $L = \{x\}$ or $M = \{y\}$, we write $E(x, M)$ or $E(L, y)$ for $E(L, M)$.

Let K_n denote the complete graph of order n, and let K_n^- be the graph obtained from K_n by removing one edge. Also let S be the graph obtained from K_4 by removing two edges which have an endvertex in common; thus $S = K_1 + (K_1 \cup K_2)$.

In [1], Kawarabayashi proved the following theorem.

Theorem 1 *Let k be an integer with $k \geq 2$, and let G be a graph with $|V(G)| \geq 4k$ and $\sigma_2(G) \geq |V(G)| + k$. Then G contains k vertex-disjoint copies of S.*

In [1], it is also shown that the bound on $\sigma_2(G)$ in Theorem 1 is sharp. But this is simply because there exists a graph G with $|V(G)| \geq 4k$ and $\sigma_2(G) = |V(G)| + k - 1$ such that G does not even contain k vertex-disjoint triangles. Based on this observation, Kawarabayashi and Ota[2] suggested the possibility of lowering the bound on $\sigma_2(G)$ by adding the assumption that G contains k vertex-disjoint triangles. Along this line, we prove the following theorem.

Theorem 2 *Let k be an integer with $k \geq 2$. Let G be a graph with $|V(G)| \geq 4k$ and $\sigma_2(G) \geq |V(G)|/2 + 2k - 1$, and suppose that G contains k vertex-disjoint triangles. In the case where $|V(G)| = 4k + 2$, suppose further that $G \not\cong K_{4t+3} \cup K_{4k-4t-1}$ for any t with $0 \leq t \leq k - 1$. Then G contains k vertex-disjoint copies of S.*

We show that in Theorem 2, the bound on $\sigma_2(G)$ is sharp. For reference in the discussion of the sharpness of Theorem 3, which we state toward the end of this section, we construct three families of examples.

Example 1. Let k, n be integers with $k \geq 2$ and $n \geq 4k$, and let s be an integer with $0 \leq s \leq k - 1$. We construct a graph $F(n, k, s)$ of order n as follows. Let A, B, C, D be vertex-disjoint graphs with $|V(A)| = \lceil (n+1)/2 \rceil - 2k$, $|B| = \lfloor (n+1)/2 \rfloor - 2k + s$, $|V(C)| = s$ and $|V(D)| = 4k - 2s - 1$ such that A, B and C have no edge and D is a complete graph. Join A completely to B, i.e., join each vertex of A to all vertices of B. Further join B completely to C, and C completely to D. Let $F(n, k, s)$ denote the resulting graph. Then $F(n, k, s)$ satisfies $\sigma_2(F(n, k, s)) = \lfloor (n - 1)/2 \rfloor + 2k - 1 (= \lceil n/2 \rceil + 2k - 2)$ and contains k vertex-disjoint triangles, but does not contain k vertex-disjoint copies of S.

Example 2. Let k, n be integers with $k \geq 2$ and $n \geq 4k$, and let r be an integer with $0 \leq r \leq k - 1$. We construct a graph $G(n, k, r)$ of order n as follows. Let A, B, C, D, E be vertex-disjoint graphs with $|V(A)| = \lceil (n - 2r - 3)/2 \rceil - (2k - r - 2)$, $|B| = \lfloor (n - 2r - 3)/2 \rfloor$, $|V(C)| = 2(k - 1 - r)$, $|V(D)| = r$, $|V(E)| = 2r + 3$ such that A and B have no edge and C, D and E are complete graphs. Join A completely to B, B completely to $C \cup D$, and $C \cup D$ completely to E. Let $G(n, k, r)$ denote the resulting graph. Then

$G(n,k,r)$ satisfies $\sigma_2(G(n,k,r)) = \lceil n/2 \rceil + 2k - 2$ and contains k vertex-disjoint triangles, but does not contain k vertex-disjoint copies of S.

Example 3. Let k, n be integers with $k \geq 2$ and $n \geq 4k$ such that n is even, and let q be an integer with $0 \leq q \leq k-2$. We construct a graph $H(n,k,q)$ of order n as follows. Let A, B, C, D, E, F be vertex-disjoint graphs with $|V(A)| = n/2 - 2k + 2$, $|V(B)| = n/2 - q - 2$, $|V(C)| = 2k - 2q - 4$, $|V(D)| = q$, $|V(E)| = 2q + 3$ and $|V(F)| = 1$ such that A, B and D have no edge and C and E are complete graphs. Join A completely to B, B completely to $C \cup D$, $C \cup D$ completely to E, and $A \cup B \cup C \cup D \cup E$ completely to F. Let $H(n,k,q)$ denote the resulting graph. Then $H(n,k,q)$ satisfies $\sigma_2(H(n,k,r)) = n/2 + 2k - 2$ and contains k vertex-disjoint triangles, but does not contain k vertex-disjoint copies of S.

It is easy to verify that if a graph G with $|V(G)| \geq 4k$ and $\delta(G) \geq 4k - 1$ contains k vertex-disjoint triangles, then it contains k vertex-disjoint copies of S. Thus as an immediate corollary of Theorem 2, we obtain the following theorem.

Theorem 3 *Let k be an integer with $k \geq 2$. Let G be a graph with $|V(G)| \geq 4k$ and $\delta(G) \geq min\{|V(G)|/4 + k - 1/2, 4k - 1\}$, and suppose that G contains k vertex-disjoint triangles. In the case where $|V(G)| = 4k + 2$ and k is odd, suppose further that $G \not\cong K_{2k+1} \cup K_{2k+1}$. Then G contains k vertex-disjoint copies of S.*

We conclude this section by showing that in Theorem 3, the bound on $\delta(G)$ is sharp. First we consider the case where $n \geq 8k - 1$. In this case, let $s = 0$ or $3k - \lfloor (n+1)/4 \rfloor - 1$ in Example 1, according as $n \geq 12k - 5$ or $8k - 1 \leq n \leq 12k - 6$. Then $F(n,k,s)$ has minimum degree $4k - 2$ or $\lfloor (n+1)/4 \rfloor + k - 1 (= \lceil (n-2)/4 \rceil + k - 1)$ according as $n \geq 12k - 5$ or $8k - 1 \leq n \leq 12k - 6$, which means that the bound on $\delta(G)$ in Theorem 3 is sharp. Next we consider the case where $4k + 1 \leq n \leq 8k - 2$. In this case, let $r = \lfloor (n-1)/4 \rfloor - k$ in Example 2. Then $G(n,k,r)$ has minimum degree $\lfloor (n-3)/2 \rfloor - \lfloor (n-1)/4 \rfloor + k (= \lceil (n-2)/4 \rceil + k - 1)$. Finally we consider the case where $4k \leq n \leq 8k - 6$ and n is even (this includes the case where $n = 4k$, which is excluded from the preceding case). In this case, let $q = \lfloor n/4 \rfloor - k$ in Example 3. Then $H(n,k,q)$ has minimum degree $\lfloor n/4 \rfloor + k - 1$.

2 Preparation for the proof of Theorem 2

Let k, G be as in Theorem 2. Write $|V(G)| = 4k + l$. By assumption, G has k vertex-disjoint triangles. Let $S_1, \dots, S_k$ be k vertex-disjoint induced subgraphs of G such that for each i, either $|V(S_i)| = 4$ and S_i contains S as a spanning subgraph, or $S_i \cong K_3$. We may assume that there exists k' such that $S_i \supset S$ and $|V(S_i)| = 4$ for each i with $1 \le i \le k'$ and $S_i \cong K_3$ for each i with $k' + 1 \le i \le k$. We choose $S_1, \dots, S_k$ so that k' is maximum and, subject to the condition that k' is maximum, $\sum_{i=1}^{k} |E(S_i)|$ is maximum. If $k' = k$, then the desired conclusion holds. Hence we may assume that $k' \le k-1$. Let $L := \cup_{i=1}^{k'} V(S_i)$ and $M := \cup_{i=k'+1}^{k-1} V(S_i)$. Let v be a vertex in $G - L - M - V(S_k)$. For a subgraph N of G, let $d_N = 3|E(v, V(N))| + \sum_{x \in V(S_k)} |E(x, V(N))|$. Note that $d_G = \sum_{x \in V(S_k)} (|E(v, V(G))| + |E(x, V(G))|) \ge 3\sigma_2(G)$ because $E(v, V(S_k)) = \emptyset$. Let $Z := G - L - M - V(S_k) - v$. For each i with $1 \le i \le k'$, write $V(S_i) = \{a_i, b_i, c_i, d_i\}$ so that $d_{S_i}(b_i) \ge d_{S_i}(c_i) \ge d_{S_i}(d_i) \ge d_{S_i}(a_i)$; thus $d_{S_i}(a_i) = 1, d_{S_i}(b_i) = 3, d_{S_i}(c_i) = d_{S_i}(d_i) = 2$ if $S_i \cong S$, and $d_{S_i}(b_i) = d_{S_i}(c_i) = 3$ and $d_{S_i}(a_i) = d_{S_i}(d_i) = 2$ if $S_i \cong K_4^-$.

The main aim of this section is to prove that $d_{S_i} \le 13$ for each $1 \le i \le k'$ (see Lemma 2.4). We start with easy lemmas.

Lemma 2.1. *Let i be an integer with $1 \le i \le k'$. Then the following statements hold:*

(i) *Suppose that there exists a subgraph X of S_i such that $X \cong K_3$ and $N_G(v) \supset V(X)$. Then $S_i \cong K_4$.*

(ii) *Suppose that there exists a subgraph X of S_i such that $X \cong K_3$ and $|N_G(v) \cap V(X)| \ge 2$. Then $S_i \not\cong S$.*

(iii) *If $S_i \cong K_4^-$, then $|E(v, V(S_i))| \le 3$.*

(iv) *If $S_i \cong K_4^-$ and $|E(v, V(S_i))| = 3$, then $|N_G(v) \cap \{b_i, c_i\}| = 1$.*

(v) *If $S_i \cong S$, then $|E(v, V(S_i))| \le 2$.*

(vi) *If $S_i \cong S$ and $|E(v, V(S_i))| = 2$, then $a_i v \in E(G)$.*

Proof. If $S_i \not\cong K_4$, and there exists a subgraph X of S_i such that $X \cong K_3$ and $N_G(v) \supset V(X)$, then by replacing S_i by $\langle V(X) \cup \{v\} \rangle$, we get a contradiction to the maximality of $\sum_{i=1}^{k} |E(S_i)|$ because $\langle V(X) \cup \{v\} \rangle \cong K_4$. Thus (i) holds, and we can similarly prove (ii). Now (iii) and (iv) immediately follow from (i), and (v) and (vi) follow from (ii). □

Lemma 2.2. *Let $x \in V(S_k)$, and let i be an integer with $1 \leq i \leq k'$. Then the following statements hold:*

(i) *If $S_i \cong K_4$, then there exist no independent edges $xy, vz \in E(G)$ with $y, z \in V(S_i)$; in particular, $|E(\{x, v\}, V(S_i))| \leq 4$.*

(ii) *If $S_i \cong K_4^-$, then $|E(\{x, v\}, V(S_i))| \leq 4$.*

Proof. Suppose that $S_i \cong K_4$ and there exist two independent edges $xy, vz \in E(G)$ with $y, z \in V(S_i)$. Then each of $\langle \{y\} \cup V(S_k) \rangle$ and $\langle \{v\} \cup V(S_i - y) \rangle$ contains a copy of S, and these two copies of S are vertex-disjoint, which contradicts the maximality of k'. Thus (i) follows. Next suppose that $S_i \cong K_4^-$ and $|E(\{v, x\}, V(S_i))| \geq 5$. Then there exist independent edges $xy, vz \in E(G)$ with $y, z \in V(S_i)$. If $y \in \{a_i, d_i\}$, then $\langle \{v\} \cup V(S_i - y) \rangle \supset S$ and $\langle \{y\} \cup V(S_k) \rangle \supset S$, which contradicts the maximality of k'. Thus there are no independent edges xy, vz with $y, z \in V(S_i)$ such that $y \in \{a_i, d_i\}$. Since $|E(\{x, v\}, V(S_i))| \geq 5$, this implies $N_G(x) \cap V(S_i) \subseteq \{b_i, c_i\}$ and $|N_G(v) \cap V(S_i)| \geq 3$. In view of Lemma 2.1(iii), this forces $N_G(x) \cap V(S_i) = \{b_i, c_i\}$ and $|N_G(v) \cap V(S_i)| = 3$. By Lemma 2.1(iv), we may assume $N_G(v) \cap V(S_i) = \{a_i, b_i, d_i\}$. But then each of $\langle \{c_i\} \cup V(S_k) \rangle$ and $\langle \{v\} \cup V(S_i - c_i) \rangle$ contains S, which contradicts the maximality of k'. □

Lemma 2.3. *Let i be an integer with $1 \leq i \leq k'$. If $S_i \cong S$, then $|E(a_i, V(S_k)$ $)| \leq 1$, and equality holds only if $E(v, V(S_i - a_i)) = \emptyset$.*

Proof. Otherwise, we can easily get a contradiction to the maximality of k' or $\sum_{i=1}^{k} |E(S_i)|$. □

Lemma 2.4. *Let $1 \leq i \leq k'$. Then $d_{S_i} \leq 13$, and equality holds only if $S_i \cong S$, c_i or d_i, say, c_i, is adjacent to v, $E(v, V(S_i)) = \{a_i v, c_i v\}$, $N_G(a_i) \cap V(S_k) = \emptyset$, $N_G(d_i) \supset V(S_k)$, $N_G(b_i) \cap V(S_k) = N_G(c_i) \cap V(S_k)$ and $|N_G(b_i) \cap V(S_k)| = 2$.*

Proof. If $S_i \cong K_4$ or K_4^-, then by Lemma 2.2, $|E(\{v, x\}, V(S_i))| \leq 4$ for any $x \in V(S_k)$, which implies $d_{S_i} \leq 12$. Thus we may assume $S_i \cong S$. If $E(v, V(S_i)) = \emptyset$, then $d_{S_i} = \sum_{x \in V(S_k)} |E(x, V(S_i))| \leq 12$. Hence by Lemma 2.1(v), we may assume $1 \leq |E(v, V(S_i))| \leq 2$. Suppose that

$|E(v,V(S_i))|=1$. If $a_iv \notin E(G)$, then by Lemma 2.3, $E(a_i,V(S_k))=\emptyset$, and hence $d_{S_i}=3|E(v,V(S_i))|+\sum_{x\in V(S_k)}|E(x,V(S_i))|\le 3+9=12$. Thus we may assume $a_iv\in E(G)$. If $|E(V(S_i),V(S_k))|\ge 10$, then it follows from Lemma 2.3 that there exists $x\in V(S_k)$ such that $N_G(x)\supset V(S_i)$, and we have $N_G(y)\supset V(S_i-a_i)$ for each $y\in V(S_k-x)$, and hence $\langle\{x,v,a_i,b_i\}\rangle\supset S$ and $\langle V(S_k-x)\cup\{c_i,d_i\}\rangle\supset S$, a contradiction. Thus $|E(V(S_i),V(S_k))|\le 9$, and hence $d_{S_i}\le 12$. Consequently we may assume $|E(v,V(S_i))|=2$. If $|E(V(S_i),V(S_k))|\le 6$, then $d_{S_i}\le 12$. Thus we may assume $|E(V(S_i),V(S_k))|\ge 7$. Note that by Lemma 2.1(vi) and Lemma 2.3, $a_iv\in E(G)$ and $E(a_i,V(S_k))=\emptyset$. Hence $|E(y,V(S_i))|\le 3$ for each $y\in V(S_k)$, and there exists $x\in V(S_k)$ such that $|E(x,V(S_i))|=3$ and $N_G(x)\cap V(S_i)=\{b_i,c_i,d_i\}$. If $vb_i\in E(G)$, then $\langle\{v,a_i,b_i,c_i\}\rangle\supset S$ and $\langle\{d_i\}\cup V(S_k)\rangle\supset S$, a contradiction. Thus we may assume $N_G(v)\cap V(S_i)=\{a_i,c_i\}$. If $|E(b_i,V(S_k))|=3$, then $\langle\{a_i,b_i\}\cup V(S_k-x)\rangle\supset S$, $\langle V(S_i-\{a_i,b_i\})\cup\{x,v\}\rangle\supset S$, a contradiction; similarly, if $|E(c_i,V(S_k))|=3$, then $\langle\{v,c_i\}\cup V(S_k-x)\rangle\supset S$ and $\langle V(S_i-c_i)\cup\{x\}\rangle\supseteq S$, a contradiction. Thus $|E(b_i,V(S_k))|\le 2$ and $|E(c_i,V(S_k))|\le 2$. Since $|E(V(S_i),V(S_k))|\ge 7$, this forces $|E(b_i,V(S_k))|=2$, $|E(c_i,V(S_k))|=2$ and $|E(d_i,V(S_k))|=3$, and hence $d_{S_i}=13$. Now if $(N_G(b_i)\cap V(S_k))\neq(N_G(c_i)\cap V(S_k))$, say, $N_G(b_i)\cap V(S_k)=\{x,y\}$ amd $N_G(c_i)\cap V(S_k)=\{x,z\}$, then $\langle\{a_i,b_i,x,y\}\rangle\supset S$ and $\langle\{v,z,c_i,d_i\}\rangle\supset S$, a contradiction. Thus the lemma follows. □

Lemma 2.5. $G-L-M-V(S_k)\not\supseteq K_3$.

Proof. We see from the maximality of k' that in $G-L-M-V(S_k)$, there is no subgraph isomorphic to S. Thus it suffices to show that there is no triangle component in $G-L-M-V(S_k)$. By way of contradiction, let S_{k+1} be a triangle component in $G-L-M-V(S_k)$, and take $y\in V(S_{k+1})$ and $x\in V(S_k)$. Note that by the maximality of k', $E(V(S_i),V(G-L-V(S_i)))=\emptyset$ for each i with $k'+1\le i\le k+1$. We separate the following point of the proof, and present it as a subclaim.

Subclaim. *Let $1\le i\le k'$. Then there exist no independent edges $xu,yw\in E(G)$ such that $u,w\in V(S_i)$.*

Proof. If there exist independent edges $xu,yw\in E(G)$ such that $u,w\in V(S_i)$, then by replacing S_i by $\langle\{u\}\cup V(S_k)\rangle$ and $\langle\{w\}\cup V(S_{k+1})\rangle$, we get a contradiction to the maximality of k'. □

Now by the subclaim, $|E(\{x,y\},V(S_i))|\le 4$ for each i with $1\le i\le k'$. Consequently $d_G(x)+d_G(y)\le 4k'+2+2\le 4(k-1)+4=4k$. On the other hand, since $xy\notin E(G)$ by the maximality of k', it follows from the assumption of

Theorem 2 that $d_G(x)+d_G(y) \geq \sigma_2(G) \geq 4k+\frac{l}{2}-1$. Hence $k'=k-1, l=2$ and, for each i with $1 \leq i \leq k-1$, $|E(\{x,y\},V(S_i))|=4$. By the subclaim, this implies that for each i with $1 \leq i \leq k-1$, either $|E(x,V(S_i))|=4$ and $E(y,V(S_i))=\emptyset$ or $|E(y,V(S_i))|=4$ and $E(x,V(S_i))|=\emptyset$. We may assume there exists t such that $E(x,V(S_i))=\emptyset$ for each $1 \leq i \leq t$ and $|E(x,V(S_i))|=4$ for each $t+1 \leq i \leq k-1$. Since $y \in V(S_{k+1})$ is arbitrary, for each $z \in V(S_{k+1})$, we have $|E(\{x,z\},V(S_i))|=4$ for each $1 \leq i \leq k-1$, and hence $|E(z,V(S_i))|=4$ for each $1 \leq i \leq t$ and $E(z,V(S_i))=\emptyset$ for each $t+1 \leq i \leq k-1$. Thus $N_G(z)=V(S_{k+1}-\{z\}) \cup (\cup_{i=1}^{t}V(S_i))$ for each $z \in V(S_{k+1})$. Now let $1 \leq i \leq t$. Applying Lemma 2.1(i) to y, we see that $S_i \cong K_4$. Take $u \in V(S_i)$. Then arguing as above with S_i and S_{k+1} replaced by $\langle V(S_i-u) \cup \{y\}\rangle$ and $\langle V(S_{k+1}-\{y\}) \cup \{u\}\rangle$, we obtain $N_G(u)=((\cup_{i=1}^{t}V(S_i))-\{u\}) \cup V(S_{k+1})$. Since i $(1 \leq i \leq t)$ and $u \in V(S_i)$ are arbitrary, this means that $\langle(\cup_{i=1}^{t}V(S_i)) \cup V(S_{k+1})\rangle$ is a component of G, and is isomorphic to K_{4t+3}. Arguing similarly with the roles of S_k and S_{k+1} replaced by each other, we also see that $\langle \cup_{i=t+1}^{k}V(S_i)\rangle$ is a component of G and isomorphic to $K_{4k-4t-1}$. Therefore, $G \cong K_{4t+3} \cup K_{4k-4t-1}$, which contradicts the assumption of Theorem 2. □

3 Proof of Theorem 2

We continue with the notation of the preceding section. Note that Lemmas 2.1 through 2.4 hold for any choice of $v \in V(G-L-M-V(S_k))$. In this section, we assume that we have chosen v so that $|E(v,V(Z))|$ is minimum.

Lemma 3.1. $|E(v,V(Z))| \leq \frac{|V(Z)|+1}{2}$.

Proof. If $N_{G-L-M-V(S_k)}(v)=\emptyset$, then the assertion of the lemma obviously holds. Hence we may assume there exists an edge $vw \in E(G-L-M-V(S_k))$. By Lemma 2.5, it follows that $N_{G-L-M-V(S_k)}(v) \cap N_{G-L-M-V(S_k)}(w)=\emptyset$. Consequently $|E(v,V(Z))|+|E(w,V(Z))| \leq |V(Z)|+1$. Hence by the choice of v, the assertion holds. □

Lemma 3.2. *The following statements hold:*

(i) *For each i with $k'+1 \leq i \leq k-1$, $d_{S_i} \leq 9$.*

(ii) $d_Z \leq 3|E(v, V(Z))|$.

Proof. It follows from the maximality of k' that $E(v, V(S_i)) = \emptyset$ for i with $k'+1 \leq i \leq k-1$ and $E(V(Z), V(S_k)) = \emptyset$. Hence the desired results obviously hold. □

Lemma 3.3. *There exists i with $1 \leq i \leq k'$ such that $d_{S_i} = 13$.*

Proof. Suppose that $d_{S_i} \leq 12$ for all i with $1 \leq i \leq k'$. Then by Lemma 3.1 and Lemma 3.2, $d_G = \sum_{i=1}^{k-1} d_{S_i} + d_Z + 2 + 2 + 2 \leq 12k' + 9(k-1-k') + \frac{3(|V(Z)|+1)}{2} + 6 = 3k' + 9k - 3 + \frac{3}{2}\{4k + l - 4k' - 3(k-1-k') - 4 + 1\} = 9k + \frac{3}{2}(k+k') + \frac{3}{2}l - 3 \leq 9k + \frac{3}{2}(k+k-1) + \frac{3}{2}l - 3 = 12k + \frac{3}{2}l - \frac{9}{2}$. On the other hand, by assumption, $d_G \geq 3\sigma_2(G) \geq 12k + \frac{3}{2}l - 3$. This is a contradiction. □

By Lemma 2.4 and Lemma 3.3, we may assume that $d_{S_1} = 13, S_1 \cong S$, and $N_G(v) \cap V(S_1) = \{a_1, c_1\}$. Write $V(S_k) = \{a, b, c\}$. By Lemma 2.4, we may assume $N_G(a) \cap V(S_1) = \{d_1\}$, and $N_G(b) \cap V(S_1) = N_G(c) \cap V(S_1) = \{b_1, c_1, d_1\}$. For a subgraph N of G, let $d'_N = 2|E(v, V(N))| + |E(a_1, V(N))| + \sum_{x \in V(S_k)} |E(x, V(N))|$. Since $E(\{a_1, v\}, V(S_k)) = \emptyset$, it follows from the assumption of Theorem 2 that

$$d'_G \geq 3\sigma_2(G) \geq 12k + \tfrac{3}{2}l - 3. \tag{A}$$

Also, note that by the symmetry of the roles of v and a_1 in $\langle V(S_1) \cup V(S_k) \cup \{v\}\rangle$, we can apply Lemmas 2.1 through 2.4 to a_1 as well; i.e., we can apply those lemmas with S_1 and v replaced by $\langle\{v, b_1, c_1, d_1\}\rangle$ and a_1.

Lemma 3.4. *For each i with $2 \leq i \leq k'$, $d'_{S_i} \leq 12$.*

Proof. Suppose that $d'_{S_i} \geq 13$. Let $p = 3|E(a_1, V(S_i))| + |E(V(S_k), V(S_i))|$. Applying Lemma 2.4 to v and a_1, we get $d_{S_i} \leq 13$ and $p \leq 13$. Since $d'_{S_i} = \frac{2}{3}d_{S_i} + \frac{1}{3}p$, this implies $d_{S_i} = 13$ and $p = 13$. Hence, again applying Lemma 2.4 to v or a_1, we see that $S_i \cong S$ and $a_i v, a_i a_1 \in E(G)$. Consequently, by replacing S_1, S_i, S_k by $\langle\{v, a_1, a_i, b_1\}\rangle$, $\langle\{d_1, a, b, c\}\rangle$, $\langle\{b_i, c_i, d_i\}\rangle$, respectively, we get a contradiction to the maximality of $\sum_{i=1}^{k} |E(S_i)|$ because $\langle\{d_1, a, b, c\}\rangle \cong K_4$. □

Lemma 3.5. *The following statements hold:*

(i) *For each i with $k'+1 \leq i \leq k-1$, $d'_{S_i} \leq 9$.*

(ii) *For each* $z \in V(Z)$, $|E(\{a_1, v\}, z)| \leq 1$.

(iii) $d'_Z \leq \frac{3|V(Z)|+1}{2}$.

Proof. It follows from the maximality of k' that $E(v, V(S_i)) = \emptyset$ for each i with $k'+1 \leq i \leq k-1$. Also, by symmetry, we have $E(a_1, V(S_i)) = \emptyset$ for each i with $k'+1 \leq i \leq k-1$. Hence (i) obviously holds. To show (ii), suppose that $|E(\{a_1, v\}, z)| \geq 2$. Then $\langle\{a_1, b_1, v, z\}\rangle \supset S$ and $\langle\{d_1, a, b, c\}\rangle \supset K_4$, which contradicts the maximality of k'. Thus (ii) holds. Now by (ii), $|E(a, V(Z))| \leq |V(Z)| - |E(v, V(Z))|$. Since $E(V(S_k), V(Z)) = \emptyset$, this together with Lemma 3.1 implies $d'_Z = 2|E(v, V(Z))| + |E(a, V(Z))| \leq |E(v, V(Z))| + |V(Z)| \leq \frac{|V(Z)|+1}{2} + |V(Z)|$. This proves (iii). $\square$

By Lemma 3.4 and (i) and (iii) of Lemma 3.5, we now obtain

$d'_G \leq 12(k'-1) + 9(k-1-k') + \frac{3}{2}\{4k + l - 4k' - 3(k-k') - 1\} + \frac{1}{2} + 4 + 2 + 5 + 5 + 3 = 9k + \frac{3}{2}(k+k') + \frac{3}{2}l - 3 \leq 9k + \frac{3}{2}(k+k-1) + \frac{3}{2}l - 3 = 12k + \frac{3}{2}l - \frac{9}{2}$, which contradicts (A). This completes the proof of Theorem 2. $\square$

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