Degree conditions for the partition of a graph into cycles, edges and isolated vertices

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Abstract

Let k, n be integers with $2 \leq k \leq n$, and let G be a graph of order n. We prove that if $\max\{d_G(x), d_G(y)\} \geq (n-k+1)/2$ for any $x, y \in V(G)$ with $x \neq y$ and $xy \notin E(G)$, then G can be partitioned into k subgraphs $H_1, \ldots, H_k$ such that H_i is a cycle or K_1 or K_2 for each $1 \leq i \leq k$ unless $k = 2$ and $G = C_5$ or $k = 3$ and $G = K_1 \cup C_5$.

Keywords: vertex-disjoint cycles, degree conditions, partition of a graph.

1 Introduction

In this paper, we consider simple finite undirected graphs with no loops and no multiple edges. For a vertex x of a graph G, the neighborhood of x in G is denoted by $N_G(x)$, and $d_G(x) = |N_G(x)|$ is the degree of x in G. With a slight abuse of notation, for a subgraph H of G and a vertex $x \in V(G) - V(H)$,

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we also denote $N_H(x) = N_G(x) \cap V(H)$ and $d_H(x) = |N_H(x)|$. For a subset S of $V(G)$, the subgraph induced by S is denoted by $\langle S \rangle$, and for a subset S of $V(G)$, $G - S = \langle V(G) - S \rangle$. For a graph G, $\delta(G)$ is the minimum degree of G, and $\sigma_2(G) = \min\{d_G(x) + d_G(y) | x, y \in V(G), x \neq y, xy \notin E(G)\}$ is the minimum degree sum of nonadjacent vertices. (When G is a complete graph, we define $\sigma_2(G) = \infty$.) For subsets L and M of $V(G)$ with $L \cap M = \emptyset$, we let $E(L, M)$ denote the set of edges of G joining a vertex in L and a vertex in M. A vertex x is often identified with the set $\{x\}$. Thus if $x \in V(G)$, then $G - x$ means $G - \{x\}$, and $E(x, M)$ means $E(\{x\}, M)$ for $M \subset V(G - x)$. In this paper, "disjoint" means "vertex-disjoint," since we only deal with partitions of the vertex set. For a cycle $C = x_1 x_2 \dots x_{|V(C)|} x_1$ and for a vertex $x = x_i \in V(C)$, we define $x^{+j} = x_{i+j}$ and $x^{-j} = x_{i-j}$ (indices are to be read modulo $|V(C)|$). Also, we let $x^+ = x^{+1}, x^- = x^{-1}$.

In this paper, we are concerned with degree conditions for the partition of a graph G into k disjoint subgraphs $H_1, \dots, H_k$ (i.e., $V(G) = V(H_1) \cup \dots V(H_k)$). In [3], Enomoto and Li proved the following theorem:

Theorem 1.1 *Let G be a graph of order n, and let k be an integer with $1 \leq k \leq n$. If $\sigma_2(G) \geq n - k + 1$, then G can be partitioned into k disjoint subgraphs $H_1, \dots, H_k$ such that H_i is a cycle or K_1 or K_2 for each i with $1 \leq i \leq k$ unless $G = C_5$ and $k = 2$.*

In this paper, we improve the degree condition "$\sigma_2(G) \geq n - k + 1$" in the avobe theorem except for the case where $k = 1$ and the case where $k = 3$ and $G = K_1 \cup C_5$. Our result is the following:

Theorem 1.2 *Let G be a graph of order n, and let k be an integer with $2 \leq k \leq n$. If $\max\{d_G(x), d_G(y)\} \geq (n - k + 1)/2$ for any $x, y \in V(G)$ with $x \neq y$ and $xy \notin E(G)$, then one of the following holds:*

(i) *G can be partitioned into k disjoint subgraphs $H_1, \dots, H_k$ such that H_i is a cycle or K_1 or K_2 for each i with $1 \leq i \leq k$.*

(ii) *$G = C_5$ and $k = 2$.*

(iii) *$G = K_1 \cup C_5$ and $k = 3$.*

There are several known results concerning this kind of partition problems under degree sum conditions. Here, we list some related results.

Theorem 1.3 (Brandt et al. [1]) *Let k, n be positive integers. If G is a graph of order $n \geq 4k$ with $\sigma_2(G) \geq n$, then G can be partitioned into k disjoint cycles.*

Theorem 1.4 (Kawarabayashi [7]) *Let k, n be positive integers with $k \geq$*

$2, n \geq 4k$. *If G is a graph of order n with $\sigma_2(G) \geq n-1$, then G can be partitioned into k disjoint subgraphs $H_1, \ldots, H_k$ such that H_i is a cycle for each $1 \leq i \leq k-1$ and H_k is a cycle or K_1.*

Theorem 1.5 (Fujita [5]) *Let k, r, n be positive integers with $2 \leq r \leq k-2, n \geq 7k$. If G is a graph of order n with $\sigma_2(G) \geq n-r$, then G can be partitioned into k disjoint subgraphs $H_1, \ldots, H_k$ such that H_i is a cycle or K_1 for each $1 \leq i \leq r$ and H_j is a cycle for each $r+1 \leq j \leq k$.*

Theorem 1.6 (Hu & Li [6]) *Let k, n be positive integers with $n \geq 10k+3$. If G is a graph of order n with $\sigma_2(G) \geq n-k+1$, then G can be partitioned into k disjoint subgraphs $H_1, \ldots, H_k$ such that H_i is a cycle or K_1 for each $1 \leq i \leq k$.*

Combining Theorems 1.3 - 1.6, we obtain the following corollary:

Corollary 1.7 *Let k, r, n be positive integers with $0 \leq r \leq k, n \geq 10k+3$. If G is a graph of order n with $\sigma_2(G) \geq n-r$, then G can be partitioned into k disjoint subgraphs $H_1, \ldots, H_k$ such that H_i is a cycle or K_1 for each $1 \leq i \leq r$ and H_j is a cycle for each $r+1 \leq j \leq k$.*

On the other hand, this thesis is deeply linked with the existence of a spanning tree with bounded degrees in connected graphs. In [2], Broersma and Tuinstra obtained the following result:

Theorem 1.8 ([2]) *Let k be an integer with $k \geq 2$ and let G be a connected graph of order $n \geq 3$. If $\sigma_2(G) \geq n-k+1$, then G has a spanning tree with at most k end vertices.*

Let k, n, G be as in Theorem 1.8. Then we see from Theorem 1.1 that G can be partitioned into k disjoint subgraphs $H_1, \ldots, H_k$ such that H_i is a cycle or K_1 or K_2. Contract each $H_i (1 \leq i \leq k)$ into a vertex and consider the spanning tree on k vertices in the resulting graph. We see from the structure that G has a spanning tree with at most k end vertices. Thus we see that Theorem 1.1 implies Theorem 1.8.

Likewise, as an immediate corollary of Theorem 1.2, we obtain the following:

Corollary 1.9 *Let G be a connected graph of order n, and let k be an integer with $2 \leq k \leq n$. If $\max\{d_G(x), d_G(y)\} \geq (n-k+1)/2$ for any $x, y \in V(G)$ with $x \neq y$ and $xy \notin E(G)$, then G has a spanning tree with at most k end vertices.*

We conclude this section by listing known results which we use in the proof of Theorem 1.2.

Theorem 1.10 (Fan [4]) *Let $n \geq 3$ be an integer. Let G be a 2-connected graph of order n, and suppose that $\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$ for any $x, y \in V(G)$ such that x and y are at distance 2 apart. Then G has a hamiltonian cycle.*

Theorem 1.11 (Ore [8]) *Let $n \geq 3$ be an integer, and let G be a graph of order n. If $\sigma_2(G) \geq n$, then G has a hamiltonian cycle.*

2 Preparation for the proof of Theorem 1.2

We prove the theorem by induction on n. First we prove the following lemma.

Lemma 2.1 *If G has $n-k$ independent edges, then there is a desired partition.*

Proof. If G has $n-k$ independent edges, then these edges and the remaining $2k-n$ isolated vertices in G forms a desired partition. □

It is easy to check that the conclusion holds for $n \leq 5$. Hence we may assume that $n \geq 6$ and the conclusion holds for any graph of order less than n. Also, it is easy to check that the conclusion holds if $n \leq k+1$. Suppose that $n = k+2$. Since now we have $\max\{d_G(x), d_G(y)\} \geq 3/2$ for any $x, y \in V(G)$ with $x \neq y$ and $xy \notin E(G)$, there is a path $P = uvw$ of length 2 in G. If $uw \in E(G)$, then $\langle V(P) \rangle$ and the remaining isolated vertices in $G - P$ forms a desired partition. So, we may assume that $uw \notin E(G)$. Then, since $\max\{d_G(u), d_G(w)\} \geq 3/2$, there is a vertex $v' \in V(G-P)$ such that $E(v', \{u, w\}) \neq \emptyset$. By symmetry, we may assume that $v'w \in E(G)$. Then $\langle \{u, v\} \rangle$, $\langle \{v', w\} \rangle$ and the remaining isolated vertices in $G-P$ forms a desired partition. Hence, in the following argument, we may assume that $n \geq k+3$.

Lemma 2.2 *There exists a cycle of order at most $n-k+1$ in G.*

Proof. Since $\max\{d_G(x), d_G(y)\} \geq (n-k+1)/2 \geq 2$ for any $x, y \in V(G)$ with $x \neq y$ and $xy \notin E(G)$, it is easy to check that there is a cycle in G. Let C be a shortest cycle in G. Then C has no chord. By contradiction, we may assume that $|V(C)| \geq n-k+2 \geq 5$. Assume for a while that $n = k+3$. By Lemma 2.1, G does not contain three independent edges. This implies that $|V(C)| = 5$. Since $n \geq 6$, take a vertex $w \in V(G-C)$. Again by Lemma 2.1, $G-C$ has no edge and $E(w, V(C)) = \emptyset$. This forces $k = 3$ and $G = K_1 \cup C_5$. Thus we may assume that $n \geq k+4$. Since now we have $(n-k+1)/2 > 2$, there are $|V(C)|-2$ vertices $v_1, v_2, \ldots, v_{|V(C)|-2} \in V(C)$ such that $d_G(v_i) \geq 3$

for each i with $1 \le i \le |V(C)|-2$. Since C is a shortest cycle in G, note that $N_G(v_i) \cap N_G(v_j) \cap V(G-C) = \emptyset$ for any i, j with $1 \le i < j \le |V(C)|-2$. Hence there are $|V(C)|-2$ independent edges in $E(\{v_1, v_2, \ldots, v_{|V(C)|-2}\}, V(G-C))$. Then by Lemma 2.1, G has a desired partition because $|V(C)| - 2 \ge n - k$.□

Put $L = \{x \in V(G) \mid d_G(x) < (n-k+1)/2\}$. Note that for any $x, y \in L$, $xy \in E(G)$. Take a cycle $C = c_1 c_2 \ldots c_p c_1$ so that

(a) $p \le n - k + 1$, and

(b) subject to the condition (a), p is maximum, and

(c) subject to the condition (b), $|V(C) \cap L|$ is maximum.

Let $R = G - C$, and let $r = |V(R)|$. Then $r = n - p \ge k - 1$. If $r = k - 1$, then C and the remaining isolated vertices forms a desired partition. Thus we may assume that $r \ge k$. Hence $p = n - r < n - k + 1$. Now we have the following lemma.

Lemma 2.3 *Let $x \in V(R)$. Then the following two statements hold:*

(i) *x has no consecutive neighbors in C (i.e., $|E(x, V(C))| \le p/2$).*

(ii) *If there are two distinct vertices $c_i, c_j \in V(C)$ with $c_j \cap \{c_i^+, c_i^-\} = \emptyset$ such that $xc_i, xc_j \in E(G)$, then $c_i^+ c_j^+ \notin E(G)$.*

Proof. Otherwise, there exists a cycle C' in $\langle V(C) \cup \{x\} \rangle$ such that $|V(C')| = p + 1 \le n - k + 1$, which contradicts the maximality of p. □

Assume for a while that $r = k$. If there is an edge $e = xy$ in R, then $C, \langle \{x, y\} \rangle$ and the remaining isolated vertices forms a desired partition. Thus we may assume that R has no edge. Since $r = k \ge 2$, this implies $R - L \ne \emptyset$. Thus there exists a vertex $x \in V(R)$ such that $|E(x, V(C))| \ge (n-k+1)/2 > p/2$. This contradicts Lemma 2.3(i).

Thus we may assume that $r \ge k + 1$. Then $p \le n - k - 1$.

3 Proof of Theorem 1.2

We devide the proof into two cases:

Case 1: For any $x, y \in V(R)$ with $x \ne y$ and $xy \notin E(R)$,
$$\max\{d_R(x), d_R(y)\} \ge (r - (k-1) + 1)/2.$$

Suppose that $k \ge 3$. Then by the induction hypothesis, one of the followings holds:

(I) R can be partitioned into $k - 1$ subgraphs H_i such that H_i is a cycle or

K_1 or K_2 for each i with $1 \le i \le k-1$;

(II) $k-1=2$ and $R=C_5$ or $k-1=3$ and $R=K_1 \cup C_5$.

If (I) holds, then $\{C, H_1, \dots, H_{k-1}\}$ forms a desired partition of G. Thus we may assume that (II) holds. Note that now we have $p = n-k-2$. Since R contains C_5, we see from the maximality of p that $p \ge 5$. Let F_1 be a component which is isomorphic to C_5 in R (i.e., $R = F_1$ when $k=3$). Clearly F_1 has a vertex u such that $u \notin L$ because F_1 is not complete. Now we claim that there exist two vertices $c_i, c_i^{+2} \in N_G(u) \cap V(C)$. If $5 \le p \le 7$, then by Lemma 2.3(i), it is easy to check that the above claim holds. Hence we may assume that $p \ge 8$. Since $n-k = p+2 \ge 10$, it follows that $d_G(u) \ge (n-k+1)/2 > (p+2)/3 + 2 \ge \lceil p/3 \rceil + 2$. This together with Lemma 2.3(i) implies that there exist two vertices $c_i, c_i^{+2} \in N_G(u) \cap V(C)$, as claimed. (Note that if $|E(u, \{c_i^-, c_i, c_i^+\})| \le 1$ for each $1 \le i \le p$, then $|E(u, V(C))| \le \lceil p/3 \rceil$.) Hence we may assume that $c_1, c_3 \in N_G(u) \cap V(C)$. In view of Lemma 2.3(i)(ii), we see that $c_2c_4 \notin E(G)$. This implies $\{c_2, c_4\} - L \ne \emptyset$. Suppose that $k=4$, and let $V(R-F_1) = \{w\}$. By Lemma 2.3(i), $d_G(w) \le p/2 < (n-k+1)/2$. Thus $w \in L$ holds. If $c_2 \in L$, then combining three independent edges in $\langle \{c_2\} \cup V(R-u) \rangle$ and a hamiltonian cycle in $\langle \{u\} \cup V(C - c_2) \rangle$, we have a desired partition. Hence it concludes $c_2 \notin L$ when $k=4$. Let x be a vertex such that $x \in \{c_2, c_4\} - L$ and $x = c_2$ when $k=4$. It is easy to check that $E(x, V(F_1)) = \emptyset$ since otherwise we can find a cycle C' in G such that $p < |V(C')| \le n-k+1$, a contradiction.

Let $K = \{c_i^+ \mid uc_i \in E(G), 1 \le i \le p\}$. By Lemma 2.3(ii), it follows that $N_G(x) \cap K = \emptyset$. Consequently, we have

$n-k+1 \le d_G(u) + d_G(x) \le 2 + |N_G(u) \cap V(C)| + |N_G(x) \cap V(C)| \le 2 + |N_G(x) \cap V(C)| + |K| \le 2 + p = 2 + n - k - 2$, a contradiction.

Thus we may assume that $k=2$. Note that for any $x, y \in V(R)$ with $x \ne y$ and $xy \notin E(R)$,

$$\max\{d_R(x), d_R(y)\} \ge r/2. \tag{1}$$

If R is 2-connected, then by Theorem 1.10, R has a hamiltonian cycle, which means that $\{R, C\}$ forms a desired partition. Thus we may assume that R is not 2-connected. We further devide the proof into two cases:

Subcase 1.1: R is disconnected.

In this case, we see from (1) that R consists of two components F_1, F_2 with $|V(F_1)| \geq r/2+1 > r/2-1 \geq |V(F_2)|$. We see from Lemma 2.3 that for each vertex $v \in V(F_2)$, $d_G(v) \leq r/2-2+(n-r)/2 < (n-1)/2$. This concludes $V(F_2) \subset L$ and hence F_2 is a complete graph. Since $V(F_2) \subset L$, it follows that $V(F_1) \cap L = \emptyset$.

Take $u \in V(F_1)$ and fix it.

Claim 1 *There exist positive integers i, j with $i \neq j$ such that $c_j \cap \{c_i^+, c_i^-\} = \emptyset$ and $N_G(u) \cap V(C) \supset \{c_i, c_j\}$. Furthermore, either $E(c_i^+, V(F_2)) = \emptyset$ or $E(c_j^+, V(F_2)) = \emptyset$ holds.*

Proof. Suppose that $|E(u, V(C))| \leq 1$. Since $d_G(u) \geq (n-1)/2$, it follows that $|V(F_1)| \geq (n-1)/2$. Note that for every vertex x in F_1, $d_{F_1}(x) \geq |V(F_1)|/2$. Hence, by Theorem 1.11, F_1 has a hamiltonian cycle. Note that $p \leq n-3$. Hence, by the maximality of p, we have $p \geq |V(F_1)| \geq (n-1)/2$. This forces $p = (n-1)/2, |V(F_1)| = (n-1)/2$ and $|V(F_2)| = 1$. Let $C' = u_1 u_2 \dots u_{(n-1)/2} u_1$ be a hamiltonian cycle in F_1, and let $V(F_2) = \{w\}$. Since $d_G(u) \geq (n-1)/2$ and $|V(F_1)| = (n-1)/2$, we may assume that $u = u_1$ and $u_1 c_2 \in E(G)$. Since $w \in L$, we see from the maximality of p or $|V(C) \cap L|$ that either $c_1 w \notin E(G)$ or $c_3 w \notin E(G)$ holds. By symmetry, we may assume that $c_1 w \notin E(G)$. Then we have $d_G(c_1) \geq (n-1)/2$ because $c_1 \notin L$. Since $p = (n-1)/2$, this forces $E(c_1, V(F_1)) \neq \emptyset$. Let u_i be a vertex in F_1 such that $c_1 u_i \in E(G)$. Then there exists a cycle C' in $G - F_2$ such that $p < |V(C')| \leq n-1$. This contradicts the maximality of p. Thus $|E(u, V(C))| \geq 2$ holds. By Lemma 2.3(i), we may assume that there exist c_i, c_j with $c_j \cap \{c_i^+, c_i^-\} = \emptyset$ such that $N_G(u) \cap V(C) \supset \{c_i, c_j\}$. Then clearly, either $E(c_i^+, V(F_2)) = \emptyset$ or $E(c_j^+, V(F_2)) = \emptyset$ holds since otherwise we can easily find a cycle C' in $\langle V(C) \cup V(F_2) \rangle$ such that $p < |V(C')| \leq n-1$. Thus the claim holds. □

By Claim 1, we may assume that $uc_1 \in E(G)$ and $E(c_2, V(F_2)) = \emptyset$. Note that $c_2 \notin L$. Let $I = \{c_i^+ \mid uc_i \in E(G), 1 \leq i \leq p\}$.

Claim 2 *The following two statements hold:*

(i) *$I \cap N_G(c_2) = \emptyset$ (i.e., $|I| + |N_G(c_2) \cap V(C)| \leq p$).*

(ii) *$N_G(c_2) \cap V(F_1) = \emptyset$.*

Proof. (i) holds from Lemma 2.3(ii). If there exists a vertex x in $N_G(c_2) \cap V(F_1)$, then by replacing C by a cycle C' contained in $\langle V(F_1) \cup V(C) \rangle$ such that $|V(C')| > p$, we get a contradiction to the maximality of p because $|V(C')| \leq n-1$. Thus (ii) holds. □

Now we estimate $|N_G(u)|+|N_G(c_2)|$. Since u has no consecutive neighbors in C by Lemma 2.3(i), note that $uc_2 \notin E(G)$ and $(N_G(u) \cap V(C)) \cap I = \emptyset$. Since $\{u, c_2\} \cap L = \emptyset$, we have $|N_G(u)|+|N_G(c_2)| \geq n-1$. On the other hand, by Claim 2(i)(ii),

$|N_G(u)|+|N_G(c_2)| = |N_G(u) \cap V(F_1)|+|N_G(u) \cap V(C)|+|N_G(c_2) \cap V(C)|$
$\leq |V(F_1)|-1+|I|+|N_G(c_2) \cap V(C)| \leq |V(F_1)|-1+p \leq n-2.$
This is a contradiction. This completes the proof of Subcase 1.1. □

Subcase 1.2: R is connected and R has a cut vertex v.

In this case, we see from (1) that $R-v$ consists of two components F_1, F_2 with $|V(F_1)| \geq r/2 > r/2-1 \geq |V(F_2)|$. Note that for each vertex $x \in V(F_2)$, $d_G(x) \leq r/2-2+(n-r)/2 < (n-1)/2$. This concludes $V(F_2) \subset L$ and hence F_2 is a complete graph. Since $V(F_2) \subset L$, it follows that $V(F_1) \cap L = \emptyset$.

Claim 3 *There exists a vertex $u \in V(F_1)$ such that $E(u, V(C)) \neq \emptyset$. Moreover, there exists a vertex $c_i \in V(C)$ with $uc_i \in E(G)$ such that either $E(c_i^+, V(F_2) \cup \{v\}) = \emptyset$ or $E(c_i^-, V(F_2) \cup \{v\}) = \emptyset$ holds.*

Proof. Suppose that every vertex in F_1 is not adjacent to C. Take $x \in V(F_1)$. Since $x \notin L$, it follows that $(n-1)/2 \leq d_G(x) = d_R(x) \leq |V(F_1)-\{x\}|+1$. Hence $|V(F_1)| \geq (n-1)/2$. Since $|V(F_1)| \leq n-p-|\{v\} \cup V(F_2)| \leq n-5$ and for every vertex $z \in V(F_1)$, $d_{F_1}(z) \geq (n-1)/2-1 > (n-5)/2$, it follows from Theorem 1.11 that F_1 has a hamiltonian cycle C' in F_1. However, since $p \leq n-|V(F_1)|-2 \leq n-(n-1)/2-2 < |V(F_1)|$, replacing C by C', we get a contradiction to the maximality of p. Thus there exists a vertex $u \in V(F_1)$ such that $E(u, V(C)) \neq \emptyset$. Let c_i be a vertex in C such that $c_iu \in E(G)$. Suppose that $E(c_i^-, V(F_2)) \neq \emptyset$ and $E(c_i^+, V(F_2)) \neq \emptyset$. Since F_2 is a complete graph, we see from the maximality of p that $N_G(c_i^-) \cap V(F_2) = N_G(c_i^+) \cap V(F_2)$ and $|V(F_2)| = 1$. In view of Lemma 2.3(i), it follows that $E(c_i, V(F_2)) = \emptyset$ and hence $c_i \notin L$. Then by replacing C by a hamiltonian cycle in $\langle V(F_2) \cup V(C-c_i)\rangle$, we get a contradiction to the maximality of $|V(C) \cap L|$. Thus $E(c_i^-, V(F_2)) = \emptyset$ or $E(c_i^+, V(F_2)) = \emptyset$ holds. Hence, to show the second assertion, we may assume that $c_i^-v \in E(G)$ or $c_i^+v \in E(G)$ holds. Then $\langle V(C) \cup V(F_1)\rangle$ contains a cycle C' such that $p < |V(C')| \leq n-1$. This contradicts the maximality of p, and hence the second assertion holds. □

By Claim 3, we may assume that there exists $u \in V(F_1)$ such that $uc_1 \in E(G)$ and $E(c_2, V(F_2) \cup \{v\}) = \emptyset$. Note that $c_2 \notin L$ and $uc_2 \notin E(G)$. Let $J = \{c_j^+ \mid c_ju \in E(G), 1 \leq i \leq p\}$. Arguing similarly as in the proof of Claim 2, we see that $N_G(c_2) \cap J = \emptyset$ (i.e., $|J|+|N_G(c_2) \cap V(C)| \leq p$) and

$N_G(c_2) \cap V(F_1) = \emptyset$. Then we have $n-1 \le d_G(u)+d_G(c_2) \le |N_G(c_2) \cap V(C)| + |N_G(u) \cap V(C)| + |N_G(u) \cap V(R)| \le |N_G(c_2) \cap V(C)| + |J| + |V(R-F_2)| - 1 \le p+r-2 = n-2$. This is a contradiction. This completes the proof of Subcase 1.2. □

Case 2: There exist $x, y \in V(R)$ with $x \neq y$ and $xy \notin E(R)$ such that

$$\max\{d_R(x), d_R(y)\} < (r-(k-1)+1)/2.$$

We may assume that $d_G(x) \ge d_G(y)$. Since $\max\{d_G(x), d_G(y)\} \ge (n-k+1)/2$, it follows that $d_C(x) \ge (n-k+1)/2 - (r-k+1)/2 = (n-r)/2 = p/2$. Hence by Lemma 2.3(i), we have $d_C(x) = p/2$. Note that p is even and hence $p \ge 4$. Let $H = \{w \in V(R) \mid N_G(w) \cap V(C) = \{c_1, c_3, c_5, \ldots, c_{p-1}\}\}$, and let $R' = \langle V(R) - V(H) \rangle$ and $r' = |R'|$. We may assume that $x \in H$ (so, note that $H \neq \emptyset$).

Claim 4 *Let $z \in V(R')$. If $E(z, H \cup \{c_2, c_4, \ldots, c_p\}) \neq \emptyset$, then $|N_G(z) \cap (H \cup V(C))| = 1$.*

Proof. Otherwise, we can easily find a cycle C' in $\langle \{z\} \cup H \cup V(C) \rangle$ such that $p < |V(C')| \le n-k+1$, which contradicts the maximality of p. □

Claim 5 *$H \cup \{c_2, c_4, \ldots, c_p\}$ is independent (i.e., $|\{c_2, c_4, \ldots, c_p\} \cap L| \le 1$).*

Proof. Otherwise, we can easily find a cycle C' in $\langle H \cup V(C) \rangle$ such that $p < |V(C')| \le n-k+1$, which contradicts the maximality of p. □

Claim 6 *$H \cap L = \emptyset$.*

Proof. Suppose that there exists a vertex $x \in H \cap L$. Then we see from Claim 5 that $c_2 \notin L$. Then by replacing C by a cycle $C' = c_1 x c_3 \ldots c_p c_1$, we get a contradiction to the maximality of $|V(C) \cap L|$. □

Claim 7 *For each $z \in H$, $E(z, R') \neq \emptyset$.*

Proof. Let $z \in H$. If $E(z, R') = \emptyset$, then by Claims 5 and 6, we have $(n-k+1)/2 \le d_G(z) = p/2$. Since now we have assumed $p \le n-k-1$, this is a contradiction. □

Claim 8 *For each $z \in \{c_2, c_4, \ldots, c_p\} - L$, $E(z, R') \neq \emptyset$.*

Proof. Let $z \in \{c_2, c_4, \ldots, c_p\} - L$. If $E(z, R') = \emptyset$, then by Claim 5, we have $(n-k+1)/2 \le d_G(z) = p/2$. Since now we have assumed $p \le n-k-1$, this is a contradiction. □

Claim 9 $|H| \leq k-1$

Proof. By contradiction, suppose that $|H| \geq k \geq 2$. Take $x, y \in H$. By Claims 4 and 5, it follows that

$$n-k+1 \leq d_G(x)+d_G(y) \leq p+d_R(x)+d_R(y) \leq p+r' = n-|H|.$$

Hence we have $|H| \leq k-1$. This is a contradiction. □

By Claim 9, note that $r' = n-p-|H| \geq k+1-|H| \geq 2$. Suppose that there exists a pair of non-adjacent vertices $u, v \in R'$ such that $\max\{d_{R'}(u), d_{R'}(v)\} \leq \{r'-(k-1-|H|)\}/2 = (n-p-k+1)/2$. Then it follows that

$$\max\{d_{C\cup H}(u), d_{C\cup H}(v)\} \geq \{(n-k+1)-(n-p-k+1)\}/2 = p/2.$$

We may assume that $d_{C\cup H}(v) \geq p/2 (\geq 2)$. Then by Claim 4, we have $N_G(v) \cap V(C) = \{c_1, c_3, c_5, \ldots, c_{p-1}\}$. However, this forces $v \in H$. Since $v \in R'$, this is a contradiction. Thus we have

$$\max\{d_{R'}(u), d_{R'}(v)\} \geq (r'-(k-1-|H|)+1)/2 \quad \text{for every pair } u, v \in R' \text{ with } uv \notin E(G). \tag{2}$$

Claim 10 $|H| \leq k-3$.

Proof. Suppose that $|H| \geq k-2$. By Claim 9, $|H| = k-2$ or $k-1$. By Claim 5, we may assume that $c_2 \notin L$. Take $x \in H$. Then by Claims 4 and 5, it follows that $n-k+1 \leq d_G(x)+d_G(c_2) \leq p+d_{R'}(x)+d_{R'}(c_2) \leq p+r' = n-|H| \leq n-k+2$. This implies that $d_{R'}(x)+d_{R'}(c_2) = r'-1$ or r'. Hence in view of Claims 4, 5, 7 and 8, we see that $|H| \leq 2$ and $p \leq 6$. Note that if $|H| = 2$ or $p = 6$ holds, then $d_{R'}(x)+d_{R'}(c_2) = r'-1$. Moreover, note that by Claims 7 and 8, $|H|+p \leq 7$ holds because now we have $d_{R'}(x)+d_{R'}(c_2) = r'-1$ or r'. Suppose that there exists $y \in (H-x) \cup (\{c_4, \ldots, c_p\}) - L$. Note that by Claims 7 and 8, $E(y, V(R')) \neq \emptyset$. This forces $d_{R'}(x)+d_{R'}(c_2) = r'-1$. Then this together with Claim 4 implies that $(p+k-2+r'-k+1)/2 \leq (n-k+1)/2 \leq d_G(y) \leq p/2+1$, and hence $r' \leq 3$. Then by Claims 7 and 8, we have $r' = 3$. In view of (2), we see that $R' \cong K_3$. Since $\langle \{c_1, c_2, x\} \cup V(R')\rangle$ contains a cycle of length 5, this together with $|H|+p \leq 7$ implies that $|H| = 1$ and $y \notin H$. Thus we have $p = 6$. We may assume that $y = c_4$. Then $\langle V(G) - \{c_5, c_6\}\rangle$ contains a cycle of length 8. This contradicts the maximality of p. Hence in view of Claims 4 - 8, we may assume that $|H| = 1$, $p = 4$ and $c_4 \in L$. Let z_1, z_2 be distinct vertices such that $xz_1, c_2z_2 \in E(G)$. If $z_1z_2 \in E(G)$, then by replacing C by a cycle in $\langle \{x, z_1, z_2, c_1, c_2\}\rangle$, we get a contradiction to the maximality of p. Since $p = 4$, we see from the maximality of p that there is no cycle of length 6. Hence it follows that $N_G(z_1) \cap N_G(z_2) \cap V(R') = \emptyset$. This implies that $\min\{d_{R'}(z_1), d_{R'}(z_2)\} \leq (r'-2)/2$. Since $c_4 \in L$, note that by Claim 4,

$\{z_1, z_2\} \cap L = \emptyset$. Take $z \in \{z_1, z_2\}$ such that $d_{R'}(z) \le (r'-2)/2$. Consequently, we have $(4+1+r'-k+1)/2 = (n-k+1)/2 \le d_G(z) \le 1+(r'-2)/2$, and hence we have $k \ge 6$. On the other hand, since $|H| = 1$ and $k-2 \le |H| \le k-1$ hold, we have $k \le 3$. This is a contradiction.

□

In view of (2) and Claim 10, we see from the induction hypothesis that one of the followings holds:

(I) R' can be partitioned into $k-1-|H|(\ge 2)$ subgraphs H_i such that H_i is a cycle or K_1 or K_2 for each i with $1 \le i \le k-1-|H|$;

(II) $k-1-|H| = 3$ and $R' = K_1 \cup C_5$ or $k-1-|H| = 2$ and $R' = C_5$.

If (I) holds, then $\{C, H_1, H_2, \ldots, H_{k-1-|H|}, \bigcup_{v \in H}\{v\}\}$ is a desired partition of G. Thus we may assume that (II) holds.

Let $C' = v_1v_2v_3v_4v_5v_1$ be a cycle of length 5 in R'. If $E(V(H), V(C')) \ne \emptyset$, say, $v_1y \in E(G)$ where $y \in H$, then

$$\{C, \langle\{v_1, y\}\rangle, \langle\{v_2, v_3\}\rangle, \langle\{v_4, v_5\}\rangle, \textstyle\bigcup_{z \in V(R-C'-y)}\{z\}\}$$

forms a desired partition of G. Hence by Claim 7, we see that $|H| = 1$, $k = 5$ and $E(H, V(R'-C')) \ne \emptyset$. Put $V(R'-C') = \{y\}$ and $H = \{x\}$. Then

$$\{C, \langle\{x, y\}\rangle, \langle\{v_1, v_2\}\rangle, \langle\{v_3, v_4\}\rangle, \langle\{v_5\}\rangle\}$$

forms a desired partition. This completes the proof of Case 2. □

This completes the proof of Theorem 1.2. □

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