# Partition of a Graph into Cycles and Isolated Vertices

Shinya Fujita
Department of Applied Mathematics
Science University of Tokyo
1-3 Kagurazaka, Shinjuku-ku, Tokyo, 162-8601 Japan

**Abstract**

Let $k, r, n$ be integers with $k \geq 2, 0 \leq r \leq k-1$ and $n \geq 10k+3$. We prove that if $G$ is a graph of order $n$ such that the degree sum of any pair of nonadjacent vertices is at least $n-r$, then $G$ contains $k$ vertex-disjoint subgraphs $H_i$, $1 \leq i \leq k$, such that $V(H_1) \cup \ldots \cup V(H_k) = V(G)$ and such that $H_i$ is a cycle or isomorphic to $K_1$ for each $i$ with $1 \leq i \leq r$, and $H_i$ is a cycle for each $i$ with $r+1 \leq i \leq k$.

## 1 Introduction

In this paper, we consider only finite, simple, undirected graphs with no loops and no multiple edges. For a graph $G$, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$, respectively. For a vertex $x$ of a graph $G$, the neighborhood of $x$ in $G$ is denoted by $N_G(x)$, and we let $d_G(x) := |N_G(x)|$. For a noncomplete graph $G$, let $\sigma_2(G) :=\min\{d_G(x) + d_G(y) |\ xy \notin E(G)\}$; if $G$ is a complete graph, let $\sigma_2(G) := \infty$. For an integer $n \geq 1$, we let $K_n$ denote the complete graph of order $n$. In this paper, "disjoint" means "vertex-disjoint".

A sufficient condition for the existence of a specified number of disjoint cycles covering all vertices was given by Brandt et al. in [1]:

**Theorem A**([1]) *Let $k, n$ be integers with $n \geq 4k$. Let $G$ be a graph of order $n$, and suppose that $\sigma_2(G) \geq n$. Then $G$ contains $k$ disjoint cycles $H_i$, $1 \leq i \leq k$, such that $V(H_1) \cup \ldots \cup V(H_k) = V(G)$.*

In [4], Enomoto and Li showed that if we regard $K_1$ and $K_2$ as cycles, then the condition on $\sigma_2(G)$ in Theorem A can be weakened:

**Theorem B**([4]) *Let $k, n$ be positive integers with $n \geq k$. Let $G$ be a graph of order $n$, and suppose that $\sigma_2(G) \geq n-k+1$. Then unless $k = 2$ and $G$ is a cycle of length 5, $G$ contains $k$ disjoint subgraphs $H_i$, $1 \leq i \leq k$, such that $V(H_1) \cup \ldots \cup V(H_k) = V(G)$ and such that for each $1 \leq i \leq k$, $H_i$ is either* *a cycle or isomorphic to* $K_1$ *or* $K_2$*.*

Also, in [7], Hu and Li showed that if the order of $G$ is sufficiently large, then we do not need $K_2$ in Theorem B:

**Theorem C(**[7]**)** *Let* $k, n$ *be positive integers with* $n \geq 10k + 3$*. Let* $G$ *be a graph of order* $n$*, and suppose that* $\sigma_2(G) \geq n - k + 1$*. Then* $G$ *contains* $k$ *disjoint subgraphs* $H_i$*,* $1 \leq i \leq k$*, such that* $V(H_1) \cup \ldots \cup V(H_k) = V(G)$ *and such that for each* $1 \leq i \leq k$*,* $H_i$ *is either a cycle or isomorphic to* $K_1$*.*

Along a slightly different line, Kawarabayashi [8] proved the following refinement of Theorem A:

**Theorem D(**[8]**)** *Let* $k, n$ *be integers with* $k \geq 2$ *and* $n \geq 4k$*. Let* $G$ *be a graph of order* $n$*, and suppose that* $\sigma_2(G) \geq n - 1$*. Then one of the following holds:*

(i) $G$ *contains* $k$ *disjoint cycles* $H_i$*,* $1 \leq i \leq k$*, such that* $V(H_1) \cup \ldots \cup V(H_k) = V(G)$*;*

(ii) $G$ *has a vertex set* $S \subset V(G)$ *with* $|V(S)| = \frac{n-1}{2}$ *such that* $G - S$ *is independent; or*

(iii) $G$ *is isomorphic to the graph obtained from* $K_{n-1}$ *by adding a vertex and join it to precisely one vertex of* $K_{n-1}$ *(i.e.,* $G \cong (K_{n-2} \cup K_1) + K_1$*).*

The purpose of this paper is to "interpolate" Theorem C and Theorems D and A by proving the following theorem, which was conjectured by Enomoto [5]:

**Theorem 1** *Let* $k, r, n$ *be integers with* $2 \leq r \leq k - 2$ *and* $n \geq 7k$*. Let* $G$ *be a graph of order* $n$*, and suppose that* $\sigma_2(G) \geq n - r$*. Then* $G$ *contains* $k$ *disjoint subgraphs* $H_i$*,* $1 \leq i \leq k$*, such that* $V(H_1) \cup \ldots \cup V(H_k) = V(G)$ *and such that* $H_i$ *is a cycle or isomorphic to* $K_1$ *for each* $i$ *with* $1 \leq i \leq r$*, and* $H_i$ *is a cycle for each* $i$ *with* $r + 1 \leq i \leq k$*.*

Combining Theorems A,C and D and Theorem 1, we obtain the following corollary:

**Corollary 2** *Let* $k, r, n$ *be integers with* $k \geq 2, 0 \leq r \leq k-1$ *and* $n \geq 10k+3$*. Let* $G$ *be a graph of order* $n$*, and suppose that* $\sigma_2(G) \geq n-r$*. Then* $G$ *contains* $k$ *disjoint subgraphs* $H_i$*,* $1 \leq i \leq k$*, such that* $V(H_1) \cup \ldots \cup V(H_k) = V(G)$ *and such that* $H_i$ *is a cycle or isomorphic to* $K_1$ *for each* $i$ *with* $1 \leq i \leq r$*, and* $H_i$ *is a cycle for each* $i$ *with* $r + 1 \leq i \leq k$*.*

Our notation is standard except possibly for the following. Let $G$ be a graph. For a subset $L$ of $V(G)$, the subgraph induced by $L$ is denoted by $\langle L \rangle$. For a subset $M$ of $V(G)$, we let $G - M = \langle V(G) - M \rangle$ and, for a subgraph $H$ of $G$, we let $G - H = \langle V(G) - V(H) \rangle$. For subsets $L$ and $M$ of $V(G)$, we let $E(L, M)$ denote the set of edges of $G$ joining a vertex in $L$ and a vertex in $M$. A vertex $x$ is often identified with the set $\{x\}$. Thus if $x \in V(G)$, then $\langle x \rangle$ means $\langle \{x\} \rangle$, $G - x$ means $G - \{x\}$, and $E(x, M)$ means $E(\{x\}, M)$ for $M \subset V(G)$. We say that $G$ is pancyclic if $|V(G)| \geq 3$ and $G$ contains a cycle of length $l$ for each $l$ with $3 \leq l \leq |V(G)|$. For a cycle $C = x_1 x_2 \dots x_{|V(C)|} x_1$ and for a vertex $x = x_i \in V(C)$, we define $x^{+j} = x_{i+j}$ and $x^{-j} = x_{i-j}$ (indices are to be read modulo $|V(C)|$). Also, we let $x^+ = x^{+1}, x^- = x^{-1}$.

We conclude this section by listing known results which we use in the proof of Theorem 1.

**Theorem E**([6]) *Let $n \geq 3$ be an integer. Let $G$ be a 2-connected graph of order $n$, and suppose that max$\{d_G(x), d_G(y)\} \geq \frac{n}{2}$ for any $x, y \in V(G)$ such that $x$ and $y$ are at distance 2 apart. Then $G$ has a hamiltonian cycle.*

**Theorem F**([2]) *Let $k, d, n$ be integers with $k \geq 3, d \geq 4k - 1$ and $n \geq 3k$. Let $G$ be a graph of order $n$, and suppose that $\sigma_2(G) \geq d$. Then $G$ contains $k$ disjoint cycles covering at least min$\{d, n\}$ vertices of $G$.*

The following theorem, announced in [2], asserts that Theorem F holds for $k = 2$ as well.

**Theorem G**([3]) *Let $d, n$ be integers with $d \geq 7$ and $n \geq 6$. Let $G$ be a graph of order $n$, and suppose that $\sigma_2(G) \geq d$. Then $G$ contains two disjoint cycles covering at least min$\{d, n\}$ vertices of $G$.*

## 2 Preparation for the proof of Theorem 1

We start with three lemmas related to Theorem E.

**Lemma 2.1.** *Let $\alpha \geq 3$ be an integer. Let $F$ be a 2-connected graph of order $\alpha$, and suppose that max$\{d_F(x), d_F(y)\} > \frac{\alpha}{2}$ for any $x, y \in V(F)$ with $x \neq y$ and $xy \notin E(F)$. Then $F$ is pancyclic.*

**Proof.** If $\alpha = 3$ or 4, then the assumption of the Lemma implies that $F \cong K_\alpha$. Thus we may assume $\alpha \geq 5$. We first prove that the following claim.

**Claim.** There exists $x \in V(F)$ with $d_F(x) > \frac{\alpha}{2}$ such that $F - x$ contains a cycle $D$ of length $\alpha-1$ or $\alpha-2$.

**Proof.** By Theorem E, $F$ contains a hamiltonian cycle $C$. Take $x \in V(C) = V(G)$ with $d_F(x) > \frac{\alpha}{2}$. If $d_F(x^-) \le \frac{\alpha}{2}$ and $d_F(x^+) \le \frac{\alpha}{2}$, then $x^-x^+ \in E(F)$, and hence $F-x$ contains a cycle of length $\alpha-1$; if $d_F(x^-) > \frac{\alpha}{2}$ and $d_F(x^+) > \frac{\alpha}{2}$, then there exists $y \in V(C)$ such that $y \in N_F(x^-)$ and $y^+ \in N_F(x^+)$ (it is possible that $y = x^+$ or $y^+ = x^-$), and hence $F-x$ contains a cycle of length $\alpha-1$. Thus we may assume $d_F(x^-) \le \frac{\alpha}{2}$ and $d_F(x^+) > \frac{\alpha}{2}$. Arguing similarly with $x$ replaced by $x^+$, we may also assume $d_F(x^{+2}) \le \frac{\alpha}{2}$. But then $x^-x^{+2} \in E(F)$, and hence $F - \{x, x^+\}$ contains a cycle of length $\alpha-2$. $\square$

Returning to the proof of the lemma, let $x, D$ be as in the Claim. If $|V(D)| = \alpha-2$, then $|E(x, V(D))| > \frac{\alpha}{2} - 1 = \frac{|V(D)|}{2}$; if $|V(D)| = \alpha-1$, then $|E(x, V(D))| > \frac{\alpha}{2} > \frac{|V(D)|}{2}$. In either case, $|E(x, V(D))| > \frac{|V(D)|}{2}$. Now let $3 \le l \le \alpha-1$. Then there exists $z \in V(D)$ such that $z \in N_F(x)$ and $z^{+(l-2)} \in N_F(x)$. Thus $\langle \{x\} \cup \{z, z^+, \ldots, z^{+(l-2)}\} \rangle$ contains a cycle of length $l$. $\square\square$

**Lemma 2.2.** *Let $r, \alpha$ be integers with $\alpha \ge r+2 \ge 4$. Let $F$ be a graph of order $\alpha$, and suppose that $F$ is not 2-connected, and max$\{d_F(x), d_F(y)\} \ge \frac{\alpha}{2}$ for any $x, y \in V(F)$ with $x \ne y$ and $xy \notin E(F)$. Then one of the following holds:*

(1) *$F$ contains $r$ disjoint subgraphs $A_1, \ldots, A_r$ such that $V(A_1) \cup \ldots \cup V(A_r) = V(F)$ and such that for each $1 \le j \le r$, $A_j$ is either a cycle or isomorphic to $K_1$;*

(2) *$r = 2$, $F$ is disconnected, and one of the components of $F$ has order 2; or*

(3) *$r = 2$, and there exists $e \in E(F)$ such that one of the components of $F - e$ has order 2.*

**Proof.** If $F$ is connected, then let $B$ be an endblock of $F$ such that $B - c$ contains a vertex $a$ with $d_F(a) \ge \frac{\alpha}{2}$, where $c$ is the cut vertex of $F$ contained in $B$; if $F$ is disconnected, then let $B$ be a component of $F$ such that $B$ contains a vertex $a$ with $d_F(a) \ge \frac{\alpha}{2}$, and take $c \in V(B)$. Then $|V(B)| \ge d_B(a) + 1 = d_F(a) + 1 \ge \frac{\alpha}{2} + 1$. Hence for each $z \in V(F - B)$, $d_F(z) \le |(V(F-B) \cup \{c\}) - \{z\}| \le \frac{\alpha}{2} - 1$. This implies that $F - B$ is a complete graph, and that

$$d_B(x) = d_F(x) \geq \frac{\alpha}{2} > \frac{|V(B)|}{2} \text{ for every } x \in V(B-c). \qquad (2.1)$$

If $|V(F-B)| \leq r-1$, then by (2.1) and Lemma 2.1, $B$ contains a cycle $C$ of length $\alpha-(r-1)$, and hence $\{C\} \cup \{\langle v \rangle | \ v \in V(F-C)\}$ forms a collection of subgraphs having the properties required in (1). Thus we may assume $|V(F-B)| \geq r$. Then $|V(B)| \geq \frac{\alpha}{2}+1 \geq |V(F-B)|+2 \geq r+2$. If $|V(F-B)| \geq 3$, then $F-B$ contains a cycle $C$ of length $|V(F-B)|$ and $B$ contains a cycle $D$ of length $|V(B)|-(r-2)$, and hence $\{C, D\} \cup \{\langle v \rangle | \ v \in V(F-C-D)\}$ forms a collection of subgraphs with the desired properties. Thus we may assume $|V(F-B)| = 2$, which forces $r = 2$. By (2.1), $d_{B-c}(x) \geq \frac{\alpha}{2}-1 = \frac{|V(B-c)|+1}{2}$ for every $x \in V(B-c)$. This in particular implies that $B-c$ is 2-connected. Hence by Theorem E, $B-c$ contains a cycle $C$ of length $|V(B)|-1 = \alpha-3$. Now if $|E(c, V(F-B))| = 2$, then $C$ and $\langle (V(F-B) \cup \{c\} \rangle$ satisfy the properties required in (1). Thus we may assume $|E(c, V(F-B))| \leq 1$, which implies that (2) or (3) holds. □

**Lemma 2.3.** *Let $r, \alpha$ be integers with $\alpha \geq r+2 \geq 4$. Let $F$ be a graph of order $\alpha$, and suppose that $max\{d_F(x), d_F(y)\} > \frac{\alpha}{2}$ for any $x, y \in V(F)$ with $x \neq y$ and $xy \notin E(F)$. In the case where $r = 2$, suppose further that $|V(F)| \leq 6$. Then $F$ contains $r$ disjoint subgraphs $A_1, \ldots, A_r$ such that $V(A_1) \cup \ldots \cup V(A_r) = V(F)$ and such that for each $1 \leq j \leq r$, $A_j$ is either a cycle or isomorphic to $K_1$.*

**Proof.** If $F$ is 2-connected, then by Lemma 2.1, $F$ contains a cycle $C$ of length $\alpha-(r-1)$, and hence $\{C\} \cup \{\langle v \rangle | \ v \in V(F-C)\}$ forms a collection of desired subgraphs. Thus we may assume $F$ is not 2-connected. In view of Lemma 2.2, we may also assume that (2) or (3) of Lemma 2.2 holds. Then $r = 2$ and, with $B$ and $a$ as in the proof of Lemma 2.2, we have $d_F(a) > \frac{\alpha}{2}$, and hence $\alpha = |V(F)| = |V(B)|+2 \geq (d_F(a)+1)+2 > \frac{\alpha}{2}+3$. This contradicts the assumption that we have $\alpha \leq 6$ when $r = 2$. □

Throughout the rest of this paper, let $n, k, r$ be as in Theorem 1, and let $G$ be a counterexample to Theorem 1. Let $L = \{v \in V(G) | \ d_G(v) < \frac{n-r}{2}\}$. Note that $xy \in E(G)$ for any $x, y \in L$ by the assumption that $\sigma_2(G) \geq n-r$. We first prove the following lemma.

**Lemma 2.4.** *In $G$, there exist $k-r$ disjoint cycles $H_1, \ldots, H_{k-r}$ such that $n-3r \leq |\cup_{i=1}^{k-r} V(H_i)| \leq n-r$.*

**Proof.** Take $v_1, \ldots, v_r \in V(G)$, and let $G' = G - \{v_1, \ldots, v_r\}$. Then $\sigma_2(G') \geq n-3r$. Since $k-r \geq 2$ and $n-r > n-3r > 4(k-r)$, it follows from Theorems F and G that $G'$ contains $k-r$ disjoint cycles $H_1, \dots, H_{k-r}$ such that $|\cup_{i=1}^{k-r} V(H_i)| \geq n-3r$. Since $|\cup_{i=1}^{k-r} V(H_i)| \leq |V(G')| = n-r$, $H_1, \dots, H_{k-r}$ are cycles with the desired properties. □

Let $H_1, \dots, H_{k-r}$ be as in Lemma 2.4. We choose $H_1, \dots, H_{k-r}$ so that

(a) $|\cup_{i=1}^{k-r} V(H_i)|$ is maximum (subject to the condition that $|\cup_{i=1}^{k-r} V(H_i)| \leq n-r$) and,

subject to condition (a), so that

(b) $|(\cup_{i=1}^{k-r} V(H_i)) \cap L|$ is maximum
(we make use of (b) only in the proof of Lemma 2.15).

Let $H = \langle \cup_{i=1}^{k-r} V(H_i) \rangle$ and let $\alpha = |V(G-H)|$. If $\alpha = r$, then $\{H_1, \dots, H_{k-r}\} \cup \{\langle v \rangle |\ v \in V(G-H)\}$ forms a collection of subgraphs having the properties required in Theorem 1. Thus we may assume $\alpha \geq r+1$.

We now prove several lemmas which we use in estimating the degree of various vertices.

**Lemma 2.5.** *Let $P = v_1 v_2 \dots v_l (l \geq 1)$ be a path in $G-H$ and let $1 \leq i \leq k-r$, and suppose that $|V(H_i)| \geq l+1$. Suppose that $N_G(v_1) \cap V(H_i) \neq \emptyset$, and let $x \in N_G(v_1) \cap V(H_i)$. Then $E(v_l, \{x^{-l}, x^{+l}\}) = \emptyset$.*

**Proof.** Suppose not. By symmetry, we may assume $v_l x^{+l} \in E(G)$. Then $\langle V(H_i) \cup V(P) - \{x^{+1}, \dots, x^{l-1}\} \rangle$ contains a cycle $C$ of length $|V(H_i)|+1$. Hence by replacing $H_i$ by $C$, we get a contradiction to the maximality of $|\cup_{i=1}^{k-r} V(H_i)|$. □

**Lemma 2.6.** *Let $v \in V(G-H)$, and let $1 \leq i \leq k-r$. Then the following hold.*

(i) *No two vertices in $N_G(v) \cap V(H_i)$ are consecutive on $H_i$.*

(ii) *$|E(v, V(H_i))| \leq |V(H_i)|/2$.*

**Proof.** Applying Lemma 2.5 with $l = 1$, we see that (i) holds, and (ii) follows from (i). □

**Lemma 2.7.** *Let $v \in V(G-H)$. Then $|E(v, V(H))| \leq (n-\alpha)/2$.*

**Proof.** By Lemma 2.6(ii), $|E(v, V(H))| \leq \sum_{i=1}^{k-r} |V(H_i)|/2 = (n-\alpha)/2$. $\square$

**Lemma 2.8.** *Suppose that $\alpha = r+1$. Let $v, v' \in V(G-H)$ with $v \neq v'$, and let $1 \leq i \leq k-r$. Let $a, b \in V(H_i)$ with $a \neq b$, and suppose that $a, b^+ \in N_G(v)$ and $a^+, b \in N_G(v')$. Then $\{a, a^+\} \cap \{b, b^+\} \neq \emptyset$.*

**Proof.** Suppose that $\{a, a^+\} \cap \{b, b^-\} = \emptyset$. Then $\langle V(H_i) \cup \{v, v'\}\rangle$ contains disjoint cycles $C, D$ such that $V(C) \cup V(D) = V(H_i) \cup \{v, v'\}$. Since $\alpha = r+1$, this means that $\{H_1, \ldots, H_{i-1}, C, D, H_{i+1}, \ldots, H_{k-r}\} \cup \{\langle u \rangle | \, u \in V(G-H) - \{v, v'\}\}$ forms a collection of subgraphs with the desired properties. $\square$

**Lemma 2.9.** *Let $vv' \in E(G-H)$, and let $1 \leq i \leq k-r$. Then the following statements hold:*

(i) *If $v$ is adjacent to a vertex $x \in V(H_i)$ and $E(v', \{x^-, x^+\}) \neq \emptyset$, then $\alpha = r+1$.*

(ii) *$|E(\{v, v'\}, V(H_i))| \leq (2|V(H_i)| + 4)/3$.*

(iii) *If $N_{G-H}(v) \cap N_{G-H}(v') \neq \emptyset$, then $|E(\{v, v'\}, V(H_i))| \leq (|V(H_i)|+1)/2$.*

**Proof.** If $vx \in E(G)$, $E(v', \{x^-, x^+\}) \neq \emptyset$ and $\alpha \geq r+2$, then $\langle V(H_i) \cup \{v, v'\}\rangle$ contains a cycle $C$ of length $|V(H_i)| + 2$ and, by replacing $H_i$ by $C$, we get a contradiction to the maximality of $|\cup_{i=1}^{k-r} V(H_i)|$. Thus (i) holds.We proceed to the proof of (ii) and (iii). If $|V(H_i)| = 3$, then by Lemma 2.6(ii), $|E(\{v, v'\}, V(H_i))| \leq 1 + 1 = 2$. Thus we assume that $|V(H_i)| \geq 4$, and define $f(x) = |E(\{v, v'\}, \{x^-, x, x^+\})|$ for each $x \in V(H_i)$ and, if $N_{G-H}(v) \cap N_{G-H}(v') \neq \emptyset$, then we also define $g(x) = |E(\{v, v'\}, \{x^-, x, x^+, x^{+2}\})|$ for each $x \in V(H_i)$.

We first prove (ii). We start with the following claim.

**Claim 1.** Let $z \in V(H_i)$. Then $f(z) \leq 3$. Further if equality holds, then $\alpha = r+1$, and one of the following holds:

(1) $E(v, \{z^-, z, z^+\}) = \{vz^-, vz^+\}$ and $E(v', \{z^-, z, z^+\}) = \{v'z\}$; or
(2) $E(v, \{z^-, z, z^+\}) = \{vz\}$ and $E(v', \{z^-, z, z^+\}) = \{v'z^-, v'z^+\}$.

**Proof.** Supoose that $f(z) \geq 3$. Then $|E(v, \{z^-, z, z^+\})| \geq 2$ or $|E(v', \{z^-, z, z^+\})| \geq 2$. We may assume $|E(v, \{z^-, z, z^+\})| \geq 2$. Then by Lemma 2.6(i), $E(v, \{z^-, z, z^+\}) = \{vz^-, vz^+\}$. Therefore applying Lemma 2.5 with $l = 2$, we obtain $E(v', \{z^-, z\ , z^+\}) = \{v'z\}$, and hence $\alpha = r+1$ by (i). $\square$

Now by way of contradiction, suppose that $|E(\{v, v'\}, V(H_i))| > (2|V(H_i)|+4)/3$. Then since $|E(\{v, v'\}, V(H_i))| = (\sum_{z \in V(H_i)} f(z))/3$, it follows from Claim 1 that $\alpha = r+1$, $|V(H_i)| \geq 5$, and the number of those vertices $z$ of $H_i$ for which $f(z) = 3$ is at least 5. Hence there exist $x, y \in V(H_i)$ with $f(x) = f(y) = 3$ such that $|\{x^-, x, x^+\} \cap \{y^-, y, y^+\}| \leq 1$. By the symmetry of $x$ and $y$, we may assume $\{x^-, x\} \cap \{y^-, y, y^+\} = \emptyset$. By the symmetry of $v$ and $v'$, we may assume (1) of Claim 1 holds for $x$. Now if (1) holds for $y$, we get a contradiction by applying Lemma 2.8 with $a = x^-$ and $b = y$; similarly if (2) holds for $y$, we get a contradiction by applying Lemma 2.8 with $a = x^-$ and $b = y^-$. Thus (ii) is proved.

To prove (iii), suppose that $N_{G-H}(v) \cap N_{G-H}(v') \neq \emptyset$.

**Claim 2.** Let $z \in V(H_i)$. Then $g(z) \leq 3$. Further if equality holds, then $\alpha = r+1$, and one of the following holds:

(1) $E(v, \{z^-, z, z^+, z^{+2}\}) = \{vz^-, vz^+\}$ and $E(v', \{z^-, z, z^+, z^{+2}\}) = \{v'z\}$;
(2) $E(v, \{z^-, z, z^+, z^{+2}\}) = \{vz\}$ and $E(v', \{z^-, z, z^+, z^{+2}\}) = \{v'z^-, v'z^+\}$;
(3) $E(v, \{z^-, z, z^+, z^{+2}\}) = \{vz, vz^{+2}\}$ and $E(v', \{z^-, z, z^+, z^{+2}\}) = \{v'z^+\}$;or
(4) $E(v, \{z^-, z, z^+, z^{+2}\}) = \{vz^+\}$ and $E(v', \{z^-, z, z^+, z^{+2}\}) = \{v'z, v'z^{+2}\}$.

**Proof.** Suppose that $g(z) \geq 3$. Then $|E(v, \{z^-, z, z^+, z^{+2}\})| \geq 2$ or $|E(v', \{z^-, z, z^+, z^{+2}\})| \geq 2$. We may assume $|E(v, \{z^-, z, z^+, z^{+2}\})| \geq 2$. Then by Lemma 2.6(i), $E(v, \{z^-, z, z^+, z^{+2}\}) = \{vz^-, vz^{+2}\}, \{vz^-, vz^+\}$ or $\{vz, vz^{+2}\}$. If $E(v, \{z^-, z\ , z^+, z^{+2}\}) = \{vz^-, vz^{+2}\}$, then applying Lemma 2.5 with $l = 2, 3$, we get $E(v', \{z^-, z, z^+, z^{+2}\}) = \emptyset$, which contradicts the assumption that $g(z) \geq 3$. Thus $E(v, \{z^-, z, z^+, z^{+2}\}) = \{vz^-, vz^+\}$ or $\{vz, vz^{+2}\}$. We may assume $E(v, \{z^-, z, z^+, z^{+2}\}) = \{vz^-, vz^+\}$. Then applying Lemma 2.5 again with $l = 2, 3$, we obtain $E(v', \{z^-, z, z^+, z^{+2}\}) = \{v'z\}$, and hence $\alpha = r+1$ by (i). $\square$

Returning to the proof of (iii), suppose that $|E(\{v, v'\}, V(H_i))| > (|V(H_i)|+1)/2$. Then since $|E(\{v, v'\}, V(H_i))| = (\sum_{z \in V(H_i)} g(z))/4$, it follows from Claim 2 that $\alpha = r+1$ and the number of those vertices $z$ of $H_i$ for which $g(z) = 3$ is at least 3. Take $x \in V(H_i)$ with $g(x) = 3$. By symmetry, we may assume (1) of Claim 2 holds for $x$. Then $E(\{v, v'\}, x^{+2}) = \emptyset$. Applying Claim 2 with $z = x^+$, we also see that $E(\{v, v'\}, x^{+3}) = \emptyset$. Similarly applying Claim 2 with $z = x^{-1}$ and $z = x^{-2}$, we get $E(\{v, v'\}, x^{-2}) = \emptyset$ and $E(\{v, v'\}, x^{-3}) = \emptyset$. Hence again by Claim 2, $g(z) \leq 2$ for each $z \in$

$\{x^{-4}, x^{-3}, x^{-2}, x^{+}, x^{+2}, x^{+3}\}$. Consequently $|V(H_i)| \geq 9$ and there exists $y \in V(H_i) - \{x^{-4}, x^{-3}, x^{-2}, x^{-}, x, x^{+}, x^{+2}, x^{+3}\}$ such that $g(y) = 3$. Then $\{x^{-}, x, x^{+}, x^{+2}\} \cap \{y^{-}, y, y^{+}, y^{+2}\} = \emptyset$. Therefore we get a contradiction by applying Lemma 2.8 with $a = x^{-}$ and $b = y^{-}, y$ or $y^{+}$, which proves (iii).□□

**Lemma 2.10.** *Let $vv' \in E(G - H)$. Then the following hold.*

(i) $|E(\{v, v'\}, V(H))| \leq (2(n - \alpha) + 4(k - r))/3$.

(ii) *If $N_{G-H}(v) \cap N_{G-H}(v') \neq \emptyset$, then* $|E(\{v, v'\}, V(H))| \leq ((n - \alpha) + (k - r))/2$.

**Proof.** By Lemma 2.9(ii), $|E(\{v, v'\}, V(H))| \leq \sum_{i=1}^{k-r}(2|V(H_i)| + 4)/3 = (2(n-\alpha)+4(k-r))/3$ and, if $N_{G-H}(v) \cap N_{G-H}(v) \neq \emptyset$, then by Lemma 2.9(iii), $|E(\{v, v'\}, V(H))| \leq \sum_{i=1}^{k-r}(|V(H_i)| + 1)/2 = ((n - \alpha) + (k - r))/2$. □

**Lemma 2.11.** *Let $v \in V(G - H)$, and let $1 \leq i \leq k - r$. Let $x \in V(H_i)$, and suppose that $N_G(v) \supset \{x, x^{+2}\}$. Then $d_H(x^{+}) \leq (n - \alpha)/2$.*

**Proof.** By the assumption that $N_G(v) \supset \{x, x^{+2}\}$, there exists a cycle $C$ of length $|V(H_i)|$ in $\langle (V(H_i) - \{x^{+}\}) \cup \{v\} \rangle$. Thus arguing similarly as in the proof of Lemma 2.6, we see from the maximality of $|\cup_{j=1}^{k-r} V(H_j)|$ that $|E(x^{+}, V(H_j))| \leq |H_j|/2$ for each $j$ with $1 \leq j \leq k - r$ and $j \neq i$, and $|E(x^{+}, V(C))| \leq |V(C)|/2$, and hence $|E(x^{+}, V(H_i) - \{x^{+}\})| \leq |V(C)|/2 = |V(H_i)|/2$. Consequently, $d_H(x^{+}) \leq \frac{1}{2}\sum_{j=1}^{k-r} |V(H_j)| = (n - \alpha)/2$. □

The following two lemmas are used when we choose an appropriate vertex in $H$ where degree is to be estimated.

**Lemma 2.12.** *Let $v \in V(G - H) - L$. Suppose that either $d_{G-H}(v) \leq \frac{1}{2}\alpha$ or $\alpha \leq r + 2$. Then for some $i$ with $1 \leq i \leq k - r$, there exist three distinct vertices $x, y, z \in V(H_i)$ such that $N_G(v) \supset \{x, x^{+2}, y, y^{+2}, z, z^{+2}\}$(it is possible that $\{x, y, z\} \cap \{x^{+2}, y^{+2}, z^{+2}\} \neq \emptyset$).*

**Proof.** Suppose not. Then it follows from Lemma 2.6(i) that for each $1 \leq i \leq k - r$, we have $|E(v, \{x, x^{+}, x^{+2})| \leq 1$ for every vertex $x \in V(H_i)$ possibly except two. Hence $|E(v, V(H))| = \frac{1}{3}\sum_{i=1}^{k-r}\sum_{x \in V(H_i)} |E(v, \{x^{-}, x, x^{+}\})| \leq \frac{1}{3}(n - \alpha) + \frac{2}{3}(k - r)$. Since $v \notin L$, this implies $\frac{n-\alpha}{3} + \frac{2}{3}(k - r) + d_{G-H}(v) \geq d_G(v) \geq \frac{n-r}{2}$, and hence $n \leq 4k - r - 2\alpha + 6d_{G-H}(v)$. Now if $d_{G-H}(v) \leq \frac{\alpha}{2}$, then from $\alpha \leq 3r$ and $r \leq k - 2$, we obtain $n \leq 4k - r + \alpha \leq 4k + 2r < 6k$, which contradicts the assumption that $n \geq 7k$; if $\alpha \leq r+2$, then from $d_{G-H}(v) \leq |V(G-H)|-1 = \alpha - 1$ and $r \leq k-2$, we obtain $n \leq 4k - r + 4\alpha - 6 \leq 4k + 3r + 2 < 7k$, which again contradicts the assumption that $n \geq 7k$. □

**Lemma 2.13.** *Let $v \in V(G-H) - L$ and $v' \in N_{G-H}(v)$, and suppose that either $d_{G-H}(v) \leq \frac{\alpha}{2}$ or $\alpha \leq r+2$. Then for some $i$ with $1 \leq i \leq k-r$, there exists $x \in V(H_i)$ such that $x, x^{+2} \in N_G(v)$, $v, v' \notin N_G(x^+)$ and $|E(x^+, V(G-H))| \leq \frac{\alpha-2}{2}$.*

**Proof.** Let $i, x, y, z$ be as in Lemma 2.12. Then by Lemma 2.6(ii), $|V(H_i)| \geq 6$. Suppose that some two of $x^+, y^+$ and $z^+$, say $x^+$ and $y^+$, have a common neighbor $u$ in $V(G-H)-\{v\}$. Then $\langle V(H_i) \cup \{v, u\} \rangle$ contains a cycle of length $|V(H_i)|+2$. In view of the maximality of $|\cup_{i=1}^{k-r} V(H_i)|$, this implies $\alpha = r+1$. On the other hand, since $|V(H_i)| \geq 6$,it follows from Lemma 2.6(i) that we have $\{x, x^+\} \cap \{y^+, y^{+2}\} = \emptyset$ or $\{x^+, x^{+2}\} \cap \{y, y^+\} = \emptyset$. Consequently we get a contradiction by applying Lemma 2.8 with $a = x$ and $b = y^+$ or $a = y$ and $b = x^+$. Thus no two of $x^+, y^+$ and $z^+$ have a common neighbor in $V(G-H) - \{v\}$. In particular, at most one of $x^+, y^+$ and $z^+$ is adjacent to $v'$. We may assume $x^+v', y^+v' \notin E(G)$. We may also assume $|E(x^+, V(H_i))| \leq |E(y^+, V(H_i))|$. Then since $x^+v, y^+v \notin E(G)$ by Lemma 2.6(i), $E(x^+, V(G-H)) \leq \frac{|V(G-H-\{v,v'\})|}{2} = \frac{\alpha-2}{2}$. Thus $x$ has the desired properties. □

Finally we prove two lemmas which we need in considering the case where $V(G-H) \subset L$.

**Lemma 2.14.** *Suppose that $\alpha = r+1$ and there exists a triangle $T$ in $G-H$. Let $1 \leq i \leq k-r$ with $|V(H_i)| \geq 4$, and let $x \in V(H_i)$. Then $d_H(x) + d_H(x^+) \leq n - \alpha$.*

**Proof.** Suppose that $d_H(x) + d_H(x^+) > n - \alpha$. Then there exists $j$ such that $|E(x, V(H_j))| + |E(x^+, V(H_j))| > |V(H_j)|$. Assume for the moment that $j = i$. Then there exists $y \in V(H_i)$ such that $xy, x^+y^{+2} \in E(G)$ (it is possible that $y = x^+$ or $y^{+2} = x$). Since $|V(H_i)| \geq 4$, this implies that $\langle V(H_i) - \{y^+\} \rangle$ contains a cycle $C$ of length $|V(H_i)| - 1$, and hence $\{H_1, \ldots, H_{i-1}, C, \{y^+\}, H_{i+1}, \ldots, H_{k-r}, T\} \cup \{\langle v \rangle |\, v \in V(G-H-T)\}$ forms a collection of subgraphs with the desired properties. Thus we may assume $j \neq i$. Then there exists $y \in V(H_j)$ such that $xy, x^+y^{+3} \in E(G)$. (it is possible that $y = y^{+3}$), which implies that $\langle V(H_i) \cup (V(H_j) - \{y^+, y^{+2}\} \rangle$

contains a cycle $C$ of length $|V(H_i)| + |V(H_j)| - 2$. Hence replacing $H_i$ and $H_j$ by $C$ and $T$, we get a contradiction to the maximality of $|\cup_{h=1}^{k-r} V(H_h)|$. $\square$

**Lemma 2.15.** *Suppose that $V(G-H) \subset L$, and let $1 \leq i \leq k-r$.*

(i) *If $z \in V(H_i)$ and $E(z, V(G-H)) \neq \emptyset$, then $E(z^{+2}, V(G-H)) = \emptyset$.*

(ii) *There exists $x \in V(H_i)$ such that $E(x, V(G-H)) = \emptyset$ and $E(x^+, V(G-H)) = \emptyset$.*

**Proof.** Suppose that there exists $z \in V(H_i)$ such that $E(z, V(G-H)) \neq \emptyset$ and $E(z^{+2}, V(G-H)) \neq \emptyset$, and take $v \in N_G(z) \cap V(G-H)$ and $v' \in N_G(z^{+2}) \cap V(G-H)$. If $v \neq v'$, then $\langle (V(H_i) \cup \{v, v'\}) - \{z^+\} \rangle$ contains a cycle $C$ of length $|V(H_i)|+1$, and hence we get a contradiction to the maximality of $|\cup_{j=1}^{k-r} V(H_j)|$ by replacing $H_i$ by $C$. Thus $v = v'$. Then $\langle (V(H_i) \cup \{v\}) - \{z^+\} \rangle$ contains a cycle $C$ of length $|V(H_i)|$. Since $vz^+ \notin E(G)$ by Lemma 2.6(i) and since $v \in L$ by the assumption of the lemma, $z^+ \notin L$ by the assumption that $\sigma_2(G) \geq n-r$. Consequently, replacing $H_i$ by $C$, we get a contradiction to the maximality of $|(\cup_{i=1}^{k-r} V(H_i)) \cap L|$. This proves (i). We now prove (ii). We may assume $E(V(H_i), V(G-H)) \neq \emptyset$. Take $y \in V(H_i)$ with $E(y, V(G-H)) \neq \emptyset$. Then $E(y^{+2}, V(G-H)) = \emptyset$ by (i). If $E(y^+, V(G-H)) = \emptyset$, then $y^+$ has the desired properties. Thus we may assume $E(y^+, V(G-H)) \neq \emptyset$. Then $E(y^{+3}, V(G-H)) = \emptyset$ by (i)( so $|V(H_i)| \geq 4$), and hence $y^{+2}$ has the desired properties. $\square$

## 3 Proof of Theorem 1

We continue with the notation of the preceding section, and complete the proof of Theorem 1. We divide the proof into two cases.

Case 1: $V(G-H) \not\subseteq L$

Subcase 1.1. $r+3 \leq \alpha \leq 3r$.

If $d_{G-H}(z) > \alpha/2$ for all $z \in V(G-H) - L$, then by Lemma 2.3, $G-H$ contains $r$ disjoint subgraphs $A_1, \ldots, A_r$ such that $V(A_1) \cup \ldots \cup V(A_r) = V(G-H)$ and $A_j$ is either a cycle or isomorphic to $K_1$ for each $j$ (note that we have $|V(G-H)| \leq 3r = 6$ in the case where $r = 2$), and they together with $H_1, \ldots, H_{k-r}$ yield subgraphs with the desired properties. Thus we may assume there exists $v \in V(G-H) - L$ such that $d_{G-H}(v) \leq \alpha/2$. We first consider the case where there exists $v' \in N_{G-H}(v)$ such that $N_{G-H}(v) \cap N_{G-H}(v') \neq \emptyset$. By Lemma 2.13, there exists a cycle $H_i$ and there exists $x \in V(H_i)$ such that $x, x^{+2} \in N_G(v)$ and $v, v' \notin N_G(x^+)$. Since $\alpha \geq r+3$, we see from the maximality of $|\sum_{j=1}^{k-r} V(H_j)|$ that $N_G(x^+) \cap N_G(v) \cap V(G-H) = \emptyset$ and $N_G(x^+) \cap N_G(v') \cap V(G-H) = \emptyset$, and hence $|N_G(x^+) \cap V(G-H)| + |N_G(v) \cap V(G-H)| \leq \alpha$ and $|N_G(x^+) \cap V(G-H)| + |N_G(v') \cap V(G-H)| \leq \alpha$. Since $|N_G(x^+) \cap V(H)| \leq (n-\alpha)/2$ by Lemma 2.11 and $|N_G(v) \cap V(H)| + |N_G(v') \cap V(H)| \leq ((n-\alpha)+(k-r))/2$ by Lemma 2.10(ii), this implies $2d_G(x^+) + d_G(v) + d_G(v') \leq 2\alpha + (n-\alpha) + ((n-\alpha)+(k-r))/2 = 3n/2 + k/2 - r/2 + \alpha/2$. On the other hand, since $v, v' \notin N_G(x^+)$, $2d_G(x^+) + d_G(v) + d_G(v') \geq 2n - 2r$ by the assumption that $\sigma_2(G) \geq n-r$. Consequently $2n - 2r \leq 3n/2 + k/2 - r/2 + \alpha/2$, which implies $n \leq k + 3r + \alpha \leq k + 6r < 7k$, a contradiction. We now consider the case where $N_{G-H}(v) \cap N_{G-H}(z) = \emptyset$ for every $z \in N_{G-H}(v)$. In this case, we have $|N_G(v) \cap (L - V(H))| \leq 1$ by the fact that $\langle L - V(H) \rangle$ is a complete graph. Since $d_{G-H}(v) = d_G(v) - |N_G(v) \cap V(H)| \geq (n-r)/2 - (n-\alpha)/2 > 1$ by Lemma 2.7 and the assumption of Subcase 1.1, this implies $N_{G-H-L}(v) \neq \emptyset$. Take $v' \in N_{G-H-L}(v)$. Since $N_{G-H}(v) \cap N_{G-H}(v') = \emptyset$, $|N_G(v) \cap V(G-H)| + |N_G(v') \cap V(G-H)| \leq \alpha$. Since $|N_G(v) \cap V(H)| + |N_G(v') \cap V(H)| \leq (2(n-\alpha) + 4(k-r))/3$ by Lemma 2.10(i), this implies $d_G(v) + d_G(v') \leq \alpha + (2(n-\alpha) + 4(k-r))/3$. On the other hand, we get $d_G(v) + d_G(v') \geq n - r$ from $v, v' \notin L$. Consequently $n - r \leq 2n/3 + 4k/3 - 4r/3 + \alpha/3$, which implies $n \leq 4k - r + \alpha \leq 4k + 2r < 6k$, a contradiction.

Subcase 1.2. $r + 1 \leq \alpha \leq r + 2$.

Let $v \in V(G-H) - L$. By Lemma 2.7,$d_{G-H}(v) = d_G(v) - |N_G(v) \cap V(H)| \geq \frac{n-r}{2} - \frac{n-\alpha}{2} > 0$. Take $v' \in N_{G-H}(v)$. By Lemma 2.13, we can find a cycle $H_i$ for which there exists $x \in V(H_i)$ such that $x, x^{+2} \in N_G(v)$, $v, v' \notin N_G(x^+)$, and $|N_G(x^+) \cap V(G-H)| \leq \frac{\alpha-2}{2}$. If $N_{G-H}(v) \cap N_{G-H}(v') \neq \emptyset$, then by Lemma 2.10(ii) and Lemma 2.11, $2n - 2r \leq 2d_G(x^+) + d_G(v) + d_G(v') \leq 2(\frac{n-\alpha}{2} + \frac{\alpha-2}{2}) + \frac{(n-\alpha)+(k-r)}{2} + 2(\alpha-1)$, which implies $n \leq k + 3r + 3\alpha - 8 \leq k + 6r - 2 < 7k$, a contradiction. Thus we may assume $N_{G-H}(v) \cap N_{G-H}(v') = \emptyset$. Then $|N_G(v) \cap V(G-H)| + |N_G(v') \cap V(G-H)| \leq \alpha$. Hence by Lemma 2.10(i) and Lemma 2.11, $2n - 2r \leq 2d_G(x^+) + d_G(v) + d_G(v') \leq 2(\frac{n-\alpha}{2} + \frac{\alpha-2}{2}) + \frac{2(n-\alpha)+4(k-r)}{3} + \alpha$, which implies $n \leq 4k + 2r + \alpha - 6 \leq 4k + 3r - 4 < 7k$. This is a contradiction, which completes the discussion for Case 1.

Case 2: $V(G-H) \subset L$

In this case, $G - H$ is a complete graph by the definition of $L$. If $\alpha \geq r + 2$, then $G - H$ contains a cycle $C$ of length $\alpha - (r-1) \geq 3$, and hence

$\{H_1, \ldots, H_{k-r}, C\} \cup \{\langle v \rangle |\ v \in V(G-H-C)\}$ forms a collection of desired subgraphs of $G$. Thus we may assume $\alpha = r+1$. Since $|V(H)| = n-(r+1) > 3k$, there exists $H_i$ with $|V(H_i)| \geq 4$. By Lemma 2.15(ii), there exists $x \in V(H_i)$ such that $N_G(x) \subset V(H)$ and $N_G(x^+) \subset V(H)$. Take $v, v' \in V(G-H)$. Note that $\{v, v'\}$ is contained in a triangle of $G-H$ because $|V(G-H)| = r+1 \geq 3$. Hence by Lemma 2.10(i) and Lemma 2.14, $2n-2r \leq d_G(v) + d_G(v') + d_G(x) + d_G(x^+) = (|N_G(v) \cap V(H)| + |N_G(v') \cap V(H)|) + (|N_G(v) \cap V(G-H)| + |N_G(v') \cap V(G-H)|) + (|N_G(x) \cap V(H)| + |N_G(x^+) \cap V(H)|) \leq \frac{(n-r-1)+(k-r)}{2} + 2r + (n-r-1)$. Therefore $n \leq k+4r-3 < 5k$, which is a contradiction.

This completes the proof of Theorem 1.

## Acknowledgment

I would like to thank Professor Hikoe Enomoto, who suggested to me the problem discussed in this paper.

## References

[1] S.Brandt, G.Chen, R.J.Gould and L.Lesniak, Degree conditions for 2-factors, J.Graph Theory **24**(1997) 165-173.

[2] Y.Egawa, M.Hagita, K.Kawarabayashi,and H.Wang, Covering vertices of a graph by $k$ disjoint cycles, to appear in Discrete Math.

[3] Y.Egawa, S.Fujita, K.Kawarabayashi,and H.Wang, Existence of two disjoint long cycles in graphs, preprint.

[4] H.Enomoto, H.Li, Partition of a graph into cycles and degenerated cycles, preprint.

[5] H.Enomoto, private communication.

[6] G.H.Fan, New sufficient condition for cycles in graphs, J.Combin. Theory Ser. **B 37**(1984), 221-227.

[7] Z.Hu and H.Li, Weak cycle partition involving degree sum conditions, preprint.

[8] K.Kawarabayashi, Degree sum conditions and graphs which are not covered by $k$ cycles, preprint.