Forbidden Pairs for Vertex-Disjoint Claws

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Abstract

Let $k \geq 4$, and let H_1, H_2 be connected graphs with $|V(H_i)| \geq 3$ for $i = 1, 2$. A graph G is said to be $\{H_1, \dots, H_l\}$-free if none of $H_1, \dots, H_l$ is an induced subgraph of G. We prove that if there exists a positive integer n_0 such that every $\{H_1, H_2\}$-free graph G with $|V(G)| \geq n_0$ and $\delta(G) \geq 3$ contains k vertex-disjoint claws, then $\{H_1, H_2\} \cap \{K_{1,t} \mid t \geq 2\} \neq \emptyset$. Also, we prove that every $K_{1,r}$-free graph of sufficiently large order with minimum degree at least t contains k vertex-disjoint copies of $K_{1,t}$.

1 Introduction

In this paper, we consider only finite, simple, undirected graphs with no loops and no multiple edges. For a graph G, we denote by $V(G)$, $E(G)$ and $\delta(G)$ the vertex set, the edge set and the minimum degree of G, respectively. For a vertex x of a graph G, the neighborhood of x in G is denoted by $N_G(x)$, and we let $d_G(x) := |N_G(x)|$. Let $H_1, \dots, H_l$ be connected graphs. A graph G is said to be $\{H_1, \dots, H_l\}$-free if none of $H_1, \dots, H_l$ is an induced subgraph of G. When $l = 1$, we write H_1-free briefly. A graph $K_{1,3}$ is called *claw*, and a $K_{1,3}$-free graph is called a *claw-free* graph.

Our notation is standard except possibly for the following. Let G be a graph. For a subset L of $V(G)$, the subgraph induced by L is denoted by $\langle L \rangle$. For a subset M of $V(G)$, we let $G - M = \langle V(G) - M \rangle$. For subsets L and M of $V(G)$ with $L \cap M = \emptyset$, we let $E(L, M)$ denote the set of edges of G joining a vertex in L and a vertex in M. A vertex x is often identified with the set $\{x\}$. Thus if $x \in V(G)$, then $G - x$ means $G - \{x\}$, and $E(x, M)$ means $E(\{x\}, M)$ for $M \subset V(G - x)$.

As for the existence of vertex-disjoint claws in general graphs, Egawa and Ota proved the following theorem.

Theorem 1 ([1]). *Let G be a graph of order at least $4k+6$ with $\delta(G) \geq k+2$. Then G contains k vertex-disjoint claws.*

Recently, the author obtained a similar result concerning claw-free graphs.

Theorem 2 ([2]). *Let G be a claw-free graph of order at least $7k-6$ with $\delta(G) \geq 3$. Then G contains k vertex-disjoint claws.*

It is an interesting problem for us to consider degree conditions and forbidden subgraphs for the existence of vertex-disjoint claws in graphs. In this paper, we are concerned with the relationship between the existence of vertex-disjoint claws and a class of graphs which is described in terms of forbidden subgraphs. Now we propose the following problem.

Problem. *Consider the statement "every $\{H_1, \dots, H_l\}$-free graph of sufficiently large order with minimum degree at least three contains k vertex-disjoint claws", and determine $\{H_1, \dots, H_l\}$ that makes the statement true.*

Along this line, the author obtained the following results.

Theorem 3. *Let $k \geq 4$, and let H_1, H_2 be connected graphs with $|V(H_i)| \geq 3$ for $i = 1, 2$. If there exists a positive integer n_0 such that every $\{H_1, H_2\}$-free graph G with $|V(G)| \geq n_0$ and $\delta(G) \geq 3$ contains k vertex-disjoint claws, then $\{H_1, H_2\} \cap \{K_{1,t} \mid t \geq 2\} \neq \emptyset$.*

Theorem 4. *Let r, t be an integer with $r \geq 2, t \geq 2$. Let G be a $K_{1,r}$-free graph of order at least $(t+1)(k-1)\{t(r-1)+1\}+1$ with $\delta(G) \geq t$. Then G contains k vertex-disjoint copies of $K_{1,t}$.*

Applying Theorem 4 to $t = 3$, Theorems 3 and 4 mean that if we forbid a pair of subgraphs to force the existence of vertex-disjoint claws, one of them must be a star and the other becomes redundant. In Theorem 4, the bound on $|V(G)|$ is not best possible. As for this part, we propose the following conjecture:

Conjecture. *Let k, r, t be integers with $k \geq 2$, $r \geq 3$ and $t \geq 2$.If G is a $K_{1,r}$-free graph of order at least $(k-1)\{t(r-1)+1\}+1$ with $\delta(G) \geq t$, then G contains k vertex-disjoint copies of $K_{1,t}$.*

If the conjecture is true, the bound on $|V(G)|$ is best possible. To see this, let $B_i = K_t$ for each $1 \leq i \leq r-1$, and consider $G = \cup_{i=1}^{k-1} A_i$ where $A_i = K_1 + \cup_{j=1}^{r-1} B_j$ for each $1 \leq i \leq k-1$. Then G is a $K_{1,r}$-free graph of order $(k-1)\{t(r-1)+1\}$ with $\delta(G) \geq t$. It is easy to check that G does not contain k vertex-disjoint copies of $K_{1,t}$. Theorem 2 implies that the above conjecture is true for $r = t = 3$.

2 Proof of Theorem 3

We define two graphs G_1, G_2 as follows:

(1) $G_1 = K_{3,m}$ where $m + 3 \geq n_0$.

(2) $G_2 = K_3 + \overline{K_m}$ where $m + 3 \geq n_0$.

Furthermore, we construct a graph G_3 as follows:

Let B be a graph with $V(B) = \{b_1, b_2, b_3\}$ such that $B \cong \overline{K_3}$, and let $C = c_1c_2 \dots c_{3l}c_1$ be a cycle with no chord where l is an integer with $3l \geq n_0 - 3$. Define G_3 by
$V(G_3) = V(B) \cup V(C)$ and $E(G_3) = E(C) \cup \left(\bigcup_{j=1}^{3}\{b_jc_i \mid i \equiv j-1(\bmod 3), 1 \leq i \leq 3l\}\right)$.

For each i with $1 \leq i \leq 3$, G_i is a graph of order at least n_0 with $\delta(G_i) \geq 3$. It is easy to see that $G_i (1 \leq i \leq 3)$ does not contain k vertex-disjoint claws.

By contradiction, suppose that neither H_1 nor H_2 is a star. By the assumption of Theorem 3, each $G_i (1 \leq i \leq 3)$ must contain either H_1 or H_2 as an induced subgraph. Without loss of generality, we may assume that G_1 contains H_1 as an induced subgraph. Since H_1 is not a star and $|H_1| \geq 3$, it follows that $H_1 \cong K_{p,q}$ where $2 \leq p \leq q$. This implies that H_1 contains C_4 as an induced subgraph. We see from the structure of G_2 that G_2 does not contain C_4 as an induced subgraph. This means that G_2 contains H_2 as an induced subgraph. Since H_2 is not a star and $|H_2| \geq 3$, it follows that H_2 contains C_3 as an induced subgraph. We see from the structure of G_3 that G_3 is a graph of girth 6, which means that none of C_3, C_4 is an induced subgraph of G_3. Hence none of H_1 and H_2 is an induced subgraph of G_3. This is a contradiction. This completes the proof of Theorem 3. □

3 Proof of Theorem 4

We take vertex-disjoint subgraphs $C_1, C_2, \dots, C_s$ such that C_i contains $K_{1,t}$ as a spanning subgraph for each $1 \leq i \leq s$ so that s is maximum. Let $C = \langle \cup_{i=1}^{s} V(C_i) \rangle$ and $H = G - C$. Then it follows from the maximality of s that $d_H(v) \leq t-1$ for each $v \in V(H)$. We may assume that $s \leq k-1$. Hence $|V(H)| \geq (t+1)(k-1)\{t(r-1)+1\}+1-s(t+1) \geq (t+1)(k-1)t(r-1)+1$. Now we prove the following claim.

Claim 3.1. *There exists a subgraph X in H such that $X = \overline{K}_{(t+1)(k-1)(r-1)+1}$.*

Proof. We take $(t+1)(k-1)(r-1)+1$ vertices $x_1, \dots, x_{(t+1)(k-1)(r-1)+1}$ in H as follows:

(i) Let $H_1 = H$, and take $x_1 \in V(H_1)$.

(ii) For $i = 2, \dots, (t+1)(k-1)(r-1)+1$, let $H_i = H_{i-1} - N_G(x_{i-1})$, and take $x_i \in V(H_i)$.

Since $d_H(v) \leq t-1$ for each $v \in V(H)$, note that $|V(H_i)| - |V(H_{i-1})| \leq t$ for each $2 \leq i \leq (t+1)(k-1)(r-1)+1$ and hence $H_1 \supset H_2 \supset \dots \supset H_{(t+1)(k-1)(r-1)+1} \neq \emptyset$ because $|V(H_{(t+1)(k-1)(r-1)+1})| \geq |V(H)| - t(t+1)(k-1)(r-1) \geq 1$. Thus we can take $(t+1)(k-1)(r-1)+1$ vertices $x_1, \dots, x_{(t+1)(k-1)(r-1)+1}$. Then by the choice of these vertices,

$$\langle \{x_1, \dots, x_{(t+1)(k-1)(r-1)+1}\} \rangle \cong \overline{K}_{(t+1)(k-1)(r-1)+1}.$$

Thus the claim holds. □

By Claim 3.1, there exists a subgraph X in H such that $X = \overline{K}_{(t+1)(k-1)(r-1)+1}$. Since $\delta(G) \geq t$, note that $E(x, V(C)) \neq \emptyset$ for every $x \in V(X)$. Hence there exists C_j with $1 \leq j \leq s$ such that $|\{x \in V(X) \mid E(x, V(C_j)) \neq \emptyset\}| \geq (t+1)(r-1)+1$ because $s \leq k-1$ and $|V(X)| = (t+1)(k-1)(r-1)+1$. Since $|V(C_j)| = t+1$, there exists a vertex $y \in V(C_j)$ such that $|E(y, V(X))| \geq r$. Then $\langle \{y\} \cup (N_G(y) \cap V(H)) \rangle$ contains $K_{1,r}$ as an induced subgraph. This contradicts the assumption that G is $K_{1,r}$-free. This completes the proof of Theorem 4. □

References

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[2] S.Fujita, Vertex-disjoint copies of $K_1 + (K_1 \cup K_2)$ in claw-free graphs, submitted.