A Pair of Forbidden Subgraphs and Perfect Matchings

Shinya Fujita*
Ken-ichi Kawarabayashi†‡
Claudio Leonardo Lucchesi§¶
Katsuhiro Ota*
Michael D. Plummer**
Akira Saito††‡‡

Abstract

In this paper, we study the relationship between forbidden subgraphs and the existence of a matching. Let $\mathcal{H}$ be a set of connected graphs, each of which has three or more vertices. A graph G is said to be $\mathcal{H}$-free if no graph in $\mathcal{H}$ is an induced subgraph of G. We completely characterize the set $\mathcal{H}$ such that every connected $\mathcal{H}$-free graph of sufficiently large even order has a perfect matching in the following cases.

(1) Every graph in $\mathcal{H}$ is triangle-free.

*Department of Mathematics, Keio University, 3-14-1 Hiyoshi, Kohoku-Ku, Yokohama 223-8522, Japan

†Graduate School of Information Sciences, Tohoku University, Aramaki aza Aoba 09, Aoba-ku, Sendai, Miyagi 980-8579, Japan

‡Partially supported by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research, by Sumitomo Foundation and by Inoue Research Award for Young Scientists.

§Institute of Computing, UNICAMP, Campinas, BRAZIL

¶Supported by a grant from CNPq and by Pronex/CNPq 664107/1997-4.

**Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37240, USA

††Department of Computer Science, Nihon University, Sakurajosui 3–25–40, Setagaya-Ku, Tokyo 156-8550, JAPAN

‡‡Partially supported by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research (C), 13640138, 2002 and 15540140, 2003–2004.

(2) $\mathcal{H}$ consists of two graphs (i.e. a pair of forbidden subgraphs).

A matching M in a graph of odd order is said to be a near-perfect matching if every vertex of G but one is incident with an edge of M. We also characterize $\mathcal{H}$ such that every $\mathcal{H}$-free graph of sufficiently large odd order has a near-perfect matching in the above cases.

keywords: perfect matching, near-perfect matching, forbidden subgraph

1 Introduction

In this paper, we consider the relationship between matchings of graphs and forbidden subgraphs. Let $\mathcal{H}$ be a set of connected graphs. Then a graph G is said to be *$\mathcal{H}$-free* if no member of $\mathcal{H}$ is an induced subgraph of G. For a singleton set $\mathcal{H} = \{H\}$, we say that G is *H-free* instead of $\{H\}$-free. A $K_{1,3}$-free graph is often called a claw-free graph.

Las Vergnas [7] and Sumner [10] independently proved the existence of a perfect matching in a connected claw-free graph of even order.

Theorem A ([7, 10]) *Every connected claw-free graph of even order has a perfect matching.*

A matching M of a graph G of odd order is called a *near-perfect matching* if it has $\frac{1}{2}(|V(G)| - 1)$ edges, or equivalently, every vertex of G except for one is an endvertex of an edge in M. As the counterpart of Theorem A, Jünger, Pulleyblank and Reinelt [6] proved the existence of a near-perfect matching in a connected claw-free graph of odd order.

Theorem B ([6]) *Every connected claw-free graph of odd order has a near-perfect matching.*

In [9], the fourth and the fifth authors investigated converses of the above theorems. They considered the statement "every connected H-free graph of even (resp. odd) order has a perfect (resp. near-perfect) matching", and proved that $K_{1,2}$ and $K_{1,3}$ are the only graphs having this property.

Theorem C ([9]) *Let H be a connected graph of order at least three.*

(1) *If there exists a positive integer* n_0 *such that every connected* H*-free graph* G *with* $|V(G)| > n_0$ *and* $|V(G)| \equiv 0 \pmod 2$ *has a perfect matching, then* $H = K_{1,2}$ *or* $H = K_{1,3}$.

(2) *If there exists a positive integer* n_0 *such that every connected* H*-free graph* G *with* $|V(G)| > n_0$ *and* $|V(G)| \equiv 1 \pmod 2$ *has a near-perfect matching, then* $H = K_{1,2}$ *or* $H = K_{1,3}$.

As an extension of the above theorem, in this paper we forbid two or more connected graphs and investigate what forbidden sets of this type force the existence of a perfect matching and that of a near-perfect matching when the graph is sufficiently large. In particular, we completely characterize those sets consisting of two forbidden subgraphs (i.e. a pair of forbidden subgraphs).

We note that there are similar studies on hamiltonian cycles. Bedrossian [1] and Faudree and Gould [4] determined all the pairs $\{H_1, H_2\}$ of connected graphs such that every $\{H_1, H_2\}$-free 2-connected graph of sufficiently large order is hamiltonian. It may be interesting to compare the results in [1] and [4] with Theorems 7 and 8 in this paper.

For graph-theoretic terminology and notation not defined in this paper, we refer the reader to [3]. Given two or more graphs G_1, $G_2, \dots, G_m$ on mutually disjoint vertex sets, we define the *join* $G_1 + G_2 + \cdots + G_m$ of G_1, $G_2, \dots, G_m$ by the graph with vertex set $\bigcup_{i=1}^{m} V(G)$ and edge set $\bigcup_{i=1}^{m} E(G) \cup \{xy\colon x \in V(G_i), y \in V(G_{i+1}), 1 \le i \le m-1\}$. Also, for a positive integer n, we denote by nG the graph consisting of n disjoint copies of G. We denote the maximum degree and the diameter of G by $\Delta(G)$ and $\mathrm{diam}(G)$, respectively. For $S \subset V(G)$, the subgraph of G induced by S is denoted by $G[S]$. A component of a graph is called an odd (resp. even) component if its order is odd (resp. even). The number of components of G is denoted by $c(G)$, and the number of odd components of G is denoted by $c_o(G)$. In this paper, when no possibility of confusion arises, we often identify a subgraph of G by its vertex set. For example, if a vertex x belongs to a component C of G, we often write $x \in C$ instead of $x \in V(C)$. The cycle of order m is denoted by C_m and the path of order m is denoted by P_m. We often call C_3 a triangle. Let $W = x_0x_1x_2 \dots x_{n-1}$ be a walk in G. For $0 \le i \le j \le n-1$, we denote the subwalk $x_ix_{i+1} \dots x_{j-1}x_j$ by $x_i\overrightarrow{W}x_j$, and its reverse $x_jx_{j-1} \dots x_{i+1}x_i$ by

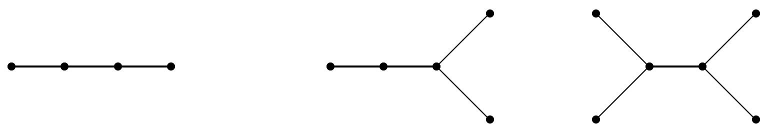

Figure 1: Trees of diameter three and maximum degree at most three

$x_j \overleftarrow{W} x_i$. Moreover, we denote by x_{i+1} and x_{i-1} by $x^{+(W)}$, $x^{-(W)}$, respectively. A path which starts at a vertex u and ends at a vertex v is called a uv-path.

The trees of diameter three and maximum degree at most three appear in this paper. One of them is P_4. The others, depicted in Figure 1, are denoted by *chair* and T_6. The deficiency of a graph, denoted by $\mathrm{def}(G)$, is defined by $\mathrm{def}(G) = |V(G)| - 2|M|$, where $M \subset V(G)$ is a maximum matching of G. Thus, a graph G has a perfect (resp. near-perfect) matching if and only if $\mathrm{def}(G) = 0$ (resp. $\mathrm{def}(G) = 1$). Note that by definition, $\mathrm{def}(G) \equiv |V(G)| \pmod 2$.

Concerning the deficiency of a graph, Berge [2] has proved the following theorem, which we use in this paper. It is sometimes called The Tutte-Berge Formula.

Theorem D ([2], see also [8, Theorem 3.1.14]) *For a graph G,*

$$\mathrm{def}(G) = \max_{S \subset V(G)} (c_o(G - S) - |S|).$$

2 Poset Structure of Forbidden Subgraphs

In this section, in order to give a clear picture of the effect of forbidden subgraphs, we define a certain partially ordered set, or a poset.

First of all, in order to avoid set-theoretic ambiguity, when we consider labelled finite graphs, their vertices are always taken from the set of integers. Then the binary relation of being isomorphic is an equivalence relation in the set of all the labelled finite graphs. In this paper the set of all the (finite unlabelled) graphs is the quotient set by this equivalence relation. Let $\mathcal{G}$ be the set of all connected graphs with three or more vertices. For G, $H \in \mathcal{G}$, if H is an induced subgraph of G, we write $H \prec G$. If H is not induced subgraph of G, we write $H \not\prec G$. Thus, for $\mathcal{H} \subset \mathcal{G}$, a graph G is $\mathcal{H}$-free if

$H \not\prec G$ for any $H \in \mathcal{H}$. Note that $\prec$ is a partial order in $\mathcal{G}$. Let

$$\boldsymbol{H_0} = \{\mathcal{H} \subset \mathcal{G}\colon \text{there exists a positive integer } n_0 \text{ such that every connected } \mathcal{H}\text{-free graph } G \text{ with } |G| \equiv 0 \pmod 2 \text{ and } |G| > n_0 \text{ has a perfect matching}\},$$

and

$$\boldsymbol{H_1} = \{\mathcal{H} \subset \mathcal{G}\colon \text{there exists a positive integer } n_0 \text{ such that every connected } \mathcal{H}\text{-free graph } G \text{ with } |G| \equiv 1 \pmod 2 \text{ and } |G| > n_0 \text{ has a near-perfect matching}\}.$$

The purpose of this paper is to investigate $\boldsymbol{H_0}$ and $\boldsymbol{H_1}$.

First, we remove redundant members in a set of forbidden subgraphs. The following lemma is trivial, but worth noting.

Lemma 1 *Let $\mathcal{H}_1$, $\mathcal{H}_2 \subset \mathcal{G}$. If $\mathcal{H}_1 \subset \mathcal{H}_2$, then every $\mathcal{H}_2$-free graph is $\mathcal{H}_1$-free. In particular, if $\mathcal{H}_1 \in \boldsymbol{H_i}$, then $\mathcal{H}_2 \in \boldsymbol{H_i}$ ($i = 0$, 1).*

By this lemma, $\boldsymbol{H_i}$ is determined by its inclusion-minimal elements ($i = 0$, 1). Let $\boldsymbol{H'_i}$ be the set of inclusion-minimal elements of $\boldsymbol{H_i}$:

$$\boldsymbol{H'_i} = \left\{\mathcal{H} \in \boldsymbol{H_i}\colon \mathcal{H}' \not\subset \mathcal{H} \text{ for any } \mathcal{H}' \in \boldsymbol{H_i} - \{\mathcal{H}\}\right\} \quad (i = 0,\ 1).$$

We claim that if $\mathcal{H} \in \boldsymbol{H'_i}$, then no pair of members in $\mathcal{H}$ are comparable with respect to $\prec$.

Lemma 2

(1) *Let $\mathcal{H} \subset \mathcal{G}$ and let H_1, $H_2 \in \mathcal{H}$. If $H_1 \neq H_2$ and $H_1 \prec H_2$, then every $(\mathcal{H} - \{H_2\})$-free graph is $\mathcal{H}$-free.*

(2) *Let $i \in \{0, 1\}$ and let $\mathcal{H} \in \boldsymbol{H'_i}$. Then $H_1 \not\prec H_2$ for each pair of distinct graphs H_1, H_2 in $\mathcal{H}$.*

Proof. (1) Assume that there exists an $(\mathcal{H} - \{H_2\})$-free graph G which is not $\mathcal{H}$-free. Then $H_2 \prec G$, and since $H_1 \prec H_2$, we have $H_1 \prec G$. Since $H_1 \neq H_2$, $H_1 \in \mathcal{H} - \{H_2\}$. This contradicts the assumption that G is $(\mathcal{H} - \{H_2\})$-free.
(2) If $H_1 \prec H_2$ for some pair of distinct graphs H_1, H_2 in $\mathcal{H}$, then by (1) every $(\mathcal{H} - \{H_2\})$-free graph is $\mathcal{H}$-free, which implies $\mathcal{H} - \{H_2\} \in \boldsymbol{H_i}$. This contradicts the minimality of $\mathcal{H}$. $\square$

Next, we introduce a binary relation $\leq$ in $\boldsymbol{H}_i'$. Let $\mathcal{H}_1$, $\mathcal{H}_2 \in \boldsymbol{H}_i'$. Then we write $\mathcal{H}_1 \leq \mathcal{H}_2$ if for each $H \in \mathcal{H}_2$ there exists a graph $H' \in \mathcal{H}_1$ with $H' \prec H$.

Lemma 3

(1) The relation $\leq$ is a partial order in $\boldsymbol{H}_i'$ $(i = 0, 1)$.

(2) Let $\mathcal{H}_1$, $\mathcal{H}_2 \subset \mathcal{G}$. Suppose $\mathcal{H}_1 \leq \mathcal{H}_2$. Then every $\mathcal{H}_1$-free graph is $\mathcal{H}_2$-free. In particular, $\mathcal{H}_2 \in \boldsymbol{H}_i$ implies $\mathcal{H}_1 \in \boldsymbol{H}_i$ $(i = 0, 1)$.

Proof. (1) By the definition of $\leq$, it is obviously reflexive. Suppose $\mathcal{H}_1 \leq \mathcal{H}_2$ and $\mathcal{H}_2 \leq \mathcal{H}_3$ for $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3 \in \boldsymbol{H}_i'$. Let $H \in \mathcal{H}_3$. Then since $\mathcal{H}_2 \leq \mathcal{H}_3$, $H' \prec H$ for some $H' \in \mathcal{H}_2$, and since $\mathcal{H}_1 \leq \mathcal{H}_2$, $H'' \prec H'$ for some $H'' \in \mathcal{H}_1$. Hence $H'' \prec H$. Therefore, $\mathcal{H}_1 \leq \mathcal{H}_3$ and $\leq$ is transitive.

Suppose $\mathcal{H}_1 \leq \mathcal{H}_2$ and $\mathcal{H}_2 \leq \mathcal{H}_1$. Let $H \in \mathcal{H}_1$. Then since $\mathcal{H}_2 \leq \mathcal{H}_1$, $H' \prec H$ for some $H' \in \mathcal{H}_2$, and since $\mathcal{H}_1 \leq \mathcal{H}_2$, $H'' \prec H$ for some $H'' \in \mathcal{H}_1$. Therefore, $H'' \prec H$. By Lemma 2 (2), this implies $H'' = H$. Then $H = H'$, which yields $H \in \mathcal{H}_2$. Therefore, we have $\mathcal{H}_1 \subset \mathcal{H}_2$. By symmetry, we also have $\mathcal{H}_2 \subset \mathcal{H}_1$ and hence $\mathcal{H}_1 = \mathcal{H}_2$. Therefore, $\leq$ is anti-symmetric.

(2) Assume G is $\mathcal{H}_1$-free but not $\mathcal{H}_2$-free. Then $H \prec G$ for some $H \in \mathcal{H}_2$. Since $\mathcal{H}_1 \leq \mathcal{H}_2$, $H' \prec H$ for some $H' \in \mathcal{H}_1$ and this implies $H' \prec G$. This contradicts the assumption that G is $\mathcal{H}_1$-free. $\square$

By Lemma 3, $\boldsymbol{H}_i'$ is determined by the maximal elements with respect to $\leq$.

Using this partial order, we restate Theorems A, B and C in the following way.

Theorem E *Let $\mathcal{H} \subset \mathcal{G}$ and suppose $|\mathcal{H}| = 1$. Then*

(1) $\mathcal{H} \in \boldsymbol{H}_0'$ if and only if $\mathcal{H} \leq \{K_{1,3}\}$, and

(2) $\mathcal{H} \in \boldsymbol{H}_1'$ if and only if $\mathcal{H} \leq \{K_{1,3}\}$.

In other words, $\{K_{1,3}\}$ is the only maximal element in $\boldsymbol{H}_i'$ that consists of exactly one forbidden subgraph $(i = 1, 2)$.

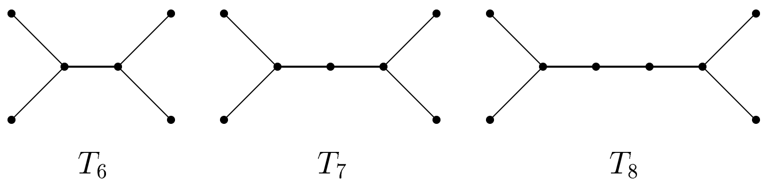

Figure 2: T_6, T_7 and T_8

3 Forbidding Trees

In the arguments below, we characterize families in $\boldsymbol{H}'_i$ in several cases in which the forbidden subgraphs are certain types of trees. Thus, we first investigate the families in $\boldsymbol{H}'_i$ which consist only of trees.

We have defined the tree T_6 in Section 1. It is a unique tree having degree sequence $(3,3,1,1,1,1)$. For $n \geq 7$, let T_n be the tree obtained from T_6 by replacing the edge joining the two vertices of degree three with the path P_{n-4}. Note that the order of T_n is n. (Figure 2.) Let $\mathcal{T}_d = \{K_{1,d+3}\} \cup \{T_n : n \geq 6\}$.

First, we prove that every connected $\mathcal{T}_d$-free graph has deficiency at most d.

Theorem 4 *A connected $\mathcal{T}_d$-free graph G with $|G| \equiv d \pmod 2$ satisfies $\mathrm{def}(G) \leq d$.*

Proof. Assume $\mathrm{def}(G) > d$. By the parity of $|G|$, $\mathrm{def}(G) \geq d+2$, and by Theorem D $c_o(G-S) \geq |S|+d+2$ for some $S \subset V(G)$. Choose such S so that S is inclusion-minimal. Note that $S \neq \emptyset$ since G is connected.

For $x \in S$, let $\mathcal{C}(x)$ be the set of all the components of $G-S$ that contain a neighbor of x. Also, for $x \in S$, let $S' = S - \{x\}$. By the minimality of S, $c_o(G-S') \leq |S'| + d$. Since $c_o(G-S') \geq c_o(G-S) - |\mathcal{C}(x)|$, we have

$$|S| - 1 + d = |S'| + d \geq c_o(G-S') \geq c_o(G-S) - |\mathcal{C}(x)| \geq |S| + d + 2 - |\mathcal{C}(x)|.$$

Therefore, we have $|\mathcal{C}(x)| \geq 3$ for each $x \in S$.

Since G is connected, $\left|\bigcup_{x\in S} \mathcal{C}(x)\right| = c(G-S) \geq c_o(G-S) \geq |S| + d + 2$, and $S \cup \left(\bigcup_{x\in S} \bigcup_{C\in\mathcal{C}(x)} V(C)\right) = V(G)$. Let T be an inclusion-minimal subset of S which satisfies conditions (1) and (2) below, where $G_T = G\left[T \cup \left(\bigcup_{x\in T} \bigcup_{C\in\mathcal{C}(x)} V(C)\right)\right]$.

(1) G_T is connected, and

(2) $\left|\bigcup_{x\in T}\mathcal{C}(x)\right| \ge |T| + d + 2$.

By the condition (2), $T \ne \emptyset$. Take two vertices x, $y \in T$ so that a longest xy-path P in G_T is as long as possible.

If $|T| = 1$, then $x = y$ and the condition (2) implies $|\mathcal{C}(x)| \ge d+3$. But this implies that G has an induced $K_{1,d+3}$. This is a contradiction. Thus, we have $|T| \ge 2$.

For each $v \in T$, define $\mathcal{C}^*(v)$ by $\mathcal{C}^*(v) = \mathcal{C}(v) - \bigcup_{u\in T-\{v\}}\mathcal{C}(u)$. In other words, $\mathcal{C}^*(v)$ is the set of components C in $\mathcal{C}(v)$ that satisfy $N_G(C) \cap T = \{v\}$.

We next claim that $G_{T-\{x\}}$ is connected. Note $G_{T-\{x\}} = G_T - \left(\{x\} \cup \left(\bigcup_{C\in\mathcal{C}^*(x)} V(C)\right)\right)$. Let D be the component of $G_{T-\{x\}}$ that contains y. Let $v \in T - \{x\}$. Since G_T is connected, there exists a vy-path Q in G_T. For each $C \in \mathcal{C}^*(x)$, since $N_G(C) \cap T = \{x\}$ and $y \in T - \{x\}$, $V(Q) \cap C = \emptyset$. Thus, if $x \notin V(Q)$, Q is also a path in $G_{T-\{x\}}$ and hence $v \in D$. Suppose $x \in V(Q)$. Let $x' = x^{-(Q)}$. If $x' \in T$, then since $x' \in N_G(x)$ and P is a longest path joining two vertices in T, we have $x' \in V(P)$, and $v\overrightarrow{Q}x'\overrightarrow{P}y$ is a walk in $G_{T-\{x\}}$. This again implies $v \in D$. If $x' \notin T$, then $x' \in C$ for some $C \in \mathcal{C}(x) - \mathcal{C}^*(x)$, and $C \in \mathcal{C}(z)$ for some $z \in T - \{x\}$. Let $u \in N_G(z) \cap C$ and let R be a ux'-path in C. Then $v\overrightarrow{Q}x'\overleftarrow{R}uz$ is a walk in $G_{T-\{x\}}$ and v and z belong to the same component of $G_{T-\{x\}}$. If $(V(R)\cup\{z\}) \cap (V(P) - \{x\}) = \emptyset$, then $zu\overrightarrow{R}x'x\overrightarrow{P}y$ is a path in G_T, which is longer than P, contradicting the choice of P. Hence $(V(R)\cup\{z\}) \cap (V(P) - \{x\}) \ne \emptyset$, which implies $z \in D$ and hence $v \in D$. Now we have $v \in D$ in every case, and hence $T - \{x\} \subset D$.

Let $v \in V(G_{T-\{x\}}) - T$. Then $v \in C$ for some $z \in T - \{x\}$ and $C \in \mathcal{C}(z)$. Since $z \in D$ and $N_G(z) \cap C \ne \emptyset$, we have $v \in D$. Therefore, $V(G_{T-\{x\}}) = D$ and the claim follows.

By the above claim and the choice of T, $\left|\bigcup_{v\in T-\{x\}}\mathcal{C}(v)\right| \le |T-\{x\}|+d+1 = |T|+d$. However, $\left|\bigcup_{v\in T-\{x\}}\mathcal{C}(v)\right| = \left|\bigcup_{v\in T}\mathcal{C}(v)\right| - |\mathcal{C}^*(x)|$, and hence $|T| + d \ge \left|\bigcup_{v\in T}\mathcal{C}(v)\right| - |\mathcal{C}^*(x)| \ge |T| + d + 2 - |\mathcal{C}^*(x)|$, or $|\mathcal{C}^*(x)| \ge 2$. By symmetry, we also have $|\mathcal{C}^*(y)| \ge 2$.

Let C_x and D_x be two distinct components in $\mathcal{C}^*(x)$. Similarly, let C_y and D_y be two distinct components in $\mathcal{C}^*(y)$. Let $c_x \in N_G(x) \cap C_x$, $d_x \in N_G(x) \cap D_x$, $c_y \in N_G(y) \cap C_y$ and $d_y \in N_G(y) \cap D_y$. Let P' be a shortest xy-path in G_T. Then $V(P') \cap C_x = V(P') \cap D_x = V(P') \cap C_y = V(P') \cap D_y = \emptyset$. Again by the definition of C_x, C_y, D_x and D_y, $N_G(\{c_x, c_y, d_x, d_y\}) \cap \left(V(P') - \{x, y\}\right) = \emptyset$. Therefore, $P' \cup \{c_x, c_y, d_x, d_y\}$ induces

T_n for some n, $n \geq 6$. This is a final contradiction, and the theorem follows. $\square$

By Theorem 4, we see that $\mathcal{T}_0 \in \boldsymbol{H_0}$ and $\mathcal{T}_1 \in \boldsymbol{H_1}$. It is clear that no pair of graphs in $\mathcal{T}_1$ are comparable with respect to $\prec$. However, $\mathcal{T}_0$ has the minimum element $K_{1,3}$. Thus by Lemmas 2 and 1, the class of $\mathcal{T}_0$-free graphs coincides with the class of claw-free graphs.

4 Forbidding Triangle-Free Graphs

In this section, we show that if no forbidden subgraphs contain a triangle, then the problem on the existence of a (near-)perfect matching becomes quite simple.

Theorem 5 *Let $\mathcal{H} \in \boldsymbol{H_0'}$. If every graph in $\mathcal{H}$ is triangle-free, then $\mathcal{H} \leq \{K_{1,3}\}$.*

Proof. Since $\mathcal{H} \in \boldsymbol{H_0'}$, there exists a positive integer n_0 such that every $\mathcal{H}$-free graph G with $|G| \equiv 0 \pmod 2$ and $|G| > n_0$ has a perfect matching. Let n be an even integer greater than $\max\{n_0, 3\}$, and let F_1 denote the graph obtained from K_{n-2} by adding two new vertices, each of which has degree one in F_1 and both are adjacent to the same vertex in K_{n-2}. Then F_1 does not have a perfect matching, and hence $H \prec F_1$ for some $H \in \mathcal{H}$. However, every connected triangle-free induced subgraph of F_1 is an induced subgraph of $K_{1,3}$. Therefore, $H \prec K_{1,3}$. This implies $\mathcal{H} \leq \{K_{1,3}\}$. $\square$

Theorem 6 *Let $\mathcal{H} \in \boldsymbol{H_1'}$. If each graph in $\mathcal{H}$ is triangle-free, then $\mathcal{H} \leq \mathcal{T}_1$.*

Proof. Since $\mathcal{H} \in \boldsymbol{H_1'}$, there exists a positive integer n_0 such that every connected $\mathcal{H}$-free graph G with $|G| > n_0$ and $|G| \equiv 1 \pmod 2$ has a near-perfect matching. Let k be an integer with $k \geq 6$, and let $V_1, \ldots, V_{k-4}$ be mutually disjoint nonempty sets of vertices of order $n_1, \ldots n_{k-4}$, respectively. Let G_i be the complete graph on V_i $(1 \leq i \leq k-4)$. We fix a vertex u_0 in V_1 and a vertex v_0 in V_{k-4}. Introduce four new vertices u_1, u_2, v_1, v_2. Let F_2 be the graph obtained from $G_1 + G_2 + \cdots + G_{k-4}$ by adding four edges u_1u_0, u_2u_0, v_1v_0 and v_2v_0. The order of F_2 is $\sum_{i=1}^{k-4} n_i + 4$. We adjust $n_1, \ldots, n_{k-4}$ so that $\sum_{i=1}^{k-4} n_i + 4$ is an odd integer greater than $\max\{n_0, 3\}$. Since F_2 has no near-perfect matching, $H \prec F_2$ for some $H \in \mathcal{H}$. However, every connected triangle-free induced subgraph of F_2 is an induced subgraph of T_k. Therefore, for each $k \geq 6$, there exists a graph $H \in \mathcal{H}$ with $H \prec T_k$.

Next, let n be an odd integer greater than $\max\{n_0, 3\}$, and let F_3 be the graph obtained from K_{n-3} by adding three vertices that have degree one in F_3 and are adjacent with the same vertex in K_{n-3}. Then F_3 has no near-perfect matching. This implies $H \prec F_3$ for some $H \in \mathcal{H}$. However, every connected triangle-free induced subgraph of F_3 is an induced subgraph of $K_{1,4}$. Therefore, $H \prec K_{1,4}$. Therefore, we have $\mathcal{H} \leq \mathcal{T}_1$. $\square$

By Theorems 4, 5 and 6, we see that $\{K_{1,3}\}$ determines all the triangle-free forbidden graphs that force the existence of a perfect matching, and that $\mathcal{T}_1$ determines all the triangle-free forbidden graphs that force the existence of a near-perfect matching.

5 Forbidding Pairs of Graphs

In this section, we determine all the pairs of forbidden subgraphs that force the existence of a perfect matching and a near-perfect matching. It is equivalent to determine the families in $\boldsymbol{H}'_0$ and $\boldsymbol{H}'_1$ that consist of two graphs. First, we consider $\boldsymbol{H}'_0$.

Theorem 7 *If $\mathcal{H} \in \boldsymbol{H}'_0$ and $|\mathcal{H}| \leq 2$, then $\mathcal{H} \leq \{K_{1,3}\}$.*

Proof. Suppose $\mathcal{H} \in \boldsymbol{H}'_0$ and $|\mathcal{H}| \leq 2$. By Theorem 5, we have only to prove that each graph in $\mathcal{H}$ is triangle-free.

Assume, to the contrary, that $\mathcal{H}$ contains a graph H which has a triangle. By the assumption, there exists a positive integer n_0 such that every connected $\mathcal{H}$-free graph G with $|G| \equiv 0 \pmod 2$ and $|G| > n_0$ has a perfect matching. Let n be an even integer with $n > \max\{n_0, 4\}$. Since $K_{1,n-1}$ has no perfect matching and the only connected induced subgraphs of $K_{1,n-1}$ are stars, $K_{1,k} \in \mathcal{H}$ for some $k > 0$. Since H contains a triangle, $H \neq K_{1,k}$. Therefore, $\mathcal{H} = \{H, K_{1,k}\}$.

Let F_4 be the graph obtained from $K_{1,3}$ by replacing one edge with P_{n-2}. Then F_4 has no perfect matching, and hence F_4 is not $\mathcal{H}$-free. Since $|F_4| = n \equiv 0 \pmod 2$, $|F_4| > n_0$, F_4 has no triangle and H has a triangle, we have $K_{1,k} \prec F_4$. This is possible only if

$k \leq 3$. However, Theorem A implies $\{K_{1,3}\} \in \boldsymbol{H_0}$. This contradicts the minimality of $\mathcal{H}$. $\square$

Theorem 7 says that $\{K_{1,2}\}$ and $\{K_{1,3}\}$ are the only families in $\boldsymbol{H'_0}$ consisting of at most two forbidden subgraphs. In particular, if we forbid a pair of subgraphs to force the existence of a perfect matching, then one of them is always redundant. On the other hand, we now show that there exist a pair of forbidden subgraphs forcing the existence of near-perfect matching such that a neither member of the pair is redundant.

Theorem 8 *If $\mathcal{H} \in \boldsymbol{H'_1}$ and $|\mathcal{H}| \leq 2$, then $\mathcal{H} \leq \{K_{1,4}, \mathit{chair}\}$.*

Proof. First, we claim that each graph in $\mathcal{H}$ is triangle-free. Assume, to the contrary, that $\mathcal{H}$ has a graph H which contains a triangle. Since $\mathcal{H} \in \boldsymbol{H'_1}$, there exists a positive integer n_0 such that every connected $\mathcal{H}$-free graph G with $|G| \equiv 1 \pmod 2$ and $|G| > n_0$ has a near-perfect matching.

Let n be an odd integer greater than $\max\{n_0, 4\}$. Then $K_{1,n-1}$ has no near-perfect matching and hence it is not $\mathcal{H}$-free. This implies that $K_{1,k} \in \mathcal{H}$ for some $k > 0$. Since H has a triangle, $H \neq K_{1,k}$ and hence $\mathcal{H} = \{H, K_{1,k}\}$.

Consider T_n with $n \equiv 1 \pmod 2$ and $n > \max\{n_0, 6\}$. Then T_n has no near-perfect matching and hence T_n is not $\mathcal{H}$-free. On the other hand, since H contains a triangle, $H \not\prec T_n$, and hence $K_{1,k} \prec T_n$, which implies $k \leq 3$. However, by Theorem B, this implies $\{K_{1,k}\} \in \boldsymbol{H'_1}$, which contradicts the minimality of $\mathcal{H}$. Therefore, the claim follows.

By the above claim and Theorem 6, $\mathcal{H} \leq \mathcal{T}_1$. If $|\mathcal{H}| = 1$, then $\mathcal{H} \leq \{K_{1,3}\}$ by Theorem E and since $\{K_{1,3}\} \leq \{K_{1,4}, \mathit{chair}\}$, the theorem follows. Thus, we may assume $|\mathcal{H}| = 2$. Then again by Theorem E and the minimality of $\mathcal{H}$, $K_{1,3} \notin \mathcal{H}$. On the other hand $K_{1,4} \in \mathcal{T}_1$ and $\mathcal{H} \leq \mathcal{T}_1$. These imply $K_{1,4} \in \mathcal{H}$. Since $\{T_6, T_7\} \subset \mathcal{T}_1$, $\mathcal{H} \leq \mathcal{T}_1$, $K_{1,4} \not\prec T_6$ and $K_{1,4} \not\prec T_7$, $\mathcal{H}$ can be written as $\mathcal{H} = \{K_{1,4}, T\}$ for some T satisfying $T \prec T_6$ and $T \prec T_7$. This implies $T \prec \mathit{chair}$. Therefore, $\mathcal{H} \leq \{K_{1,4}, \mathit{chair}\}$. $\square$

Since the only families $\mathcal{H}$ in $\boldsymbol{H'_1}$ with $|\mathcal{H}| = 2$ and $\mathcal{H} \leq \{K_{1,4}, \mathit{chair}\}$ are $\{K_{1,4}, P_4\}$ and $\{K_{1,4}, \mathit{chair}\}$, we have exactly two pairs forcing the existence of a near-perfect matching such that neither member of the pair is redundant.

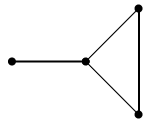

Figure 3: Z_1

6 Forbidding Three Graphs

For a positive integer n, we define $\boldsymbol{G_n}$ by $\boldsymbol{G_n} = \{\mathcal{H} \subset \mathcal{G} \colon |\mathcal{H}| \leq n\}$. Theorems E, 7 and 8 say that $\boldsymbol{G_1} \cap \boldsymbol{H'_0}$, $\boldsymbol{G_1} \cap \boldsymbol{H'_1}$, $\boldsymbol{G_2} \cap \boldsymbol{H'_0}$ and $\boldsymbol{G_2} \cap \boldsymbol{H'_1}$ all have a maximum element with respect to $\leq$. However the situation changes for $\boldsymbol{G_3} \cap \boldsymbol{H'_0}$ and $\boldsymbol{G_3} \cap \boldsymbol{H'_1}$. They have no maximum element. Let Z_1 be the unique graph having the degree sequence $(3, 2, 2, 1)$ (see Figure 3). Gould and Harris [5] studied forbidden subgraphs which force the existence of a hamiltonian path. Let Y_l be the tree obtained from the tree T_{l+2} by removing one endvertex.

Theorem F *For any integer m and l with $m \geq 4$ and $l \geq 4$, there exists a positive integer n_0 such that every connected $\{K_{1,m}, Y_l, Z_1\}$-free graph of order greater than n_0 has a hamiltonian path.*

Note that a graph with a hamiltonian path has a perfect matching (resp. near-perfect matching) if its order is even (resp. odd). Thus, Theorem F implies $\{K_{1,m}, Y_l, Z_1\} \in \boldsymbol{G_3} \cap \boldsymbol{H'_0} \cap \boldsymbol{H'_1}$ for each m and l with $m \geq 4$ and $l \geq 4$. In particular, for each $l \geq 4$, $\{\{K_{1,m}, Y_l, Z_1\} \colon m \geq 4\}$ forms an infinite chain with respect to $\leq$, and hence neither $\boldsymbol{G_3} \cap \boldsymbol{H'_0}$ nor $\boldsymbol{G_3} \cap \boldsymbol{H'_1}$ has a maximum element.

7 Conclusion and Remarks

We have defined $\boldsymbol{G_n}$ in the previous section. We define $\boldsymbol{G_T}$ by

$$\boldsymbol{G_T} = \{\mathcal{H} \subset \mathcal{G} \colon \text{each } H \in \mathcal{H} \text{ is triangle-free}\}.$$

By the results in [9] and this paper, several parts of the posets $\boldsymbol{H'_0}$ and $\boldsymbol{H'_1}$ are determined:

- $\boldsymbol{G_T} \cap \boldsymbol{H'_0} = \{\mathcal{H} \subset \mathcal{G} \colon \mathcal{H} \leq \{K_{1,3}\}\}$
- $\boldsymbol{G_T} \cap \boldsymbol{H'_1} = \{\mathcal{H} \subset \mathcal{G} \colon \mathcal{H} \leq \mathcal{T}_1\}$
- $\boldsymbol{G_1} \cap \boldsymbol{H'_0} = \{\mathcal{H} \subset \mathcal{G} \colon \mathcal{H} \leq \{K_{1,3}\}\}$
- $\boldsymbol{G_1} \cap \boldsymbol{H'_1} = \{\mathcal{H} \subset \mathcal{G} \colon \mathcal{H} \leq \{K_{1,3}\}\}$
- $\boldsymbol{G_2} \cap \boldsymbol{H'_0} = \{\mathcal{H} \subset \mathcal{G} \colon \mathcal{H} \leq \{K_{1,3}\}\}$
- $\boldsymbol{G_2} \cap \boldsymbol{H'_1} = \{\mathcal{H} \subset \mathcal{G} \colon \mathcal{H} \leq \{K_{1,4}, \mathit{chair}\}\}$

Moreover, we have proved that neither $\boldsymbol{G_3} \cap \boldsymbol{H'_0}$ nor $\boldsymbol{G_3} \cap \boldsymbol{H'_1}$ has a maximum element. These results seem to suggest that the structure of $\boldsymbol{G_n} \cap \boldsymbol{H'_i}$ becomes more complicated as n grows ($i = 1$, 2).

As an extension of $\boldsymbol{H_0}$ and $\boldsymbol{H_1}$, we can define $\boldsymbol{H_d}$ as

$$\boldsymbol{H_d} = \{\mathcal{H} \subset \mathcal{G} \colon \text{there exists a positive integer } n_0 \text{ such that every connected } \mathcal{H}\text{-free graph } G \text{ with } |G| \equiv d \pmod 2 \text{ and } |G| > n_0 \text{ satisfy } \operatorname{def}(G) \leq d\}.$$

Lemmas 1, 2 and 3 hold for $\boldsymbol{H_d}$ and hence $\boldsymbol{H_d}$ is determined by the set $\boldsymbol{H'_d}$ of its inclusion-minimal elements, and $\boldsymbol{H'_d}$ is determined by its maximal elements with respect to the partial order $\leq$. Define $\boldsymbol{T}$ by $\boldsymbol{T} = \{\mathcal{H} \subset \mathcal{G} \colon$ each $H \in \mathcal{H}$ is a tree$\}$. Theorem 4 says that $\mathcal{T}_d \in \boldsymbol{T} \cap \boldsymbol{H_d}$ for each $d \geq 0$, and Theorems 5 and 6 imply $\boldsymbol{T} \cap \boldsymbol{H'_d} = \{\mathcal{H} \subset \mathcal{G} \colon \mathcal{H} \leq \mathcal{T}_d\}$ for $d \in \{0, 1\}$. But we do not know whether $\boldsymbol{T} \cap \boldsymbol{H'_d}$ coincides with $\{\mathcal{H} \subset \mathcal{G} \colon \mathcal{H} \leq \mathcal{T}_d\}$ for $d \geq 2$.

In [9], $\boldsymbol{G_1} \cap \boldsymbol{H'_d}$ is characterized for each $d \geq 0$.

Theorem G ([9]) *For each $d \geq 0$, $\mathcal{H} \in \boldsymbol{G_1} \cap \boldsymbol{H'_d}$ if and only if $\mathcal{H} \leq \{K_{1,3}\}$.*

But the structure of $\boldsymbol{G_2} \cap \boldsymbol{H'_d}$ is unknown.

Acknowledgment

We thank Prof. Yoshimi Egawa for his helpful comments.

References

[1] P. Bedrossian, Forbidden subgraphs and minimum degree conditions for hamiltonicity, Thesis, Memphis State University, USA, 1991.

[2] C. Berge, Sur le couplage maximum d'un graphe, *C.R. Acad. Sci. Paris Sér. I Math.* **247** (1958) 258–259.

[3] G. Chartrand and L. Lesniak, *Graphs & Digraphs* (3rd ed.), Wadsworth & Brooks/Cole, Monterey, CA, (1996).

[4] R.J. Faudree and R.J. Gould, Characterizing forbidden pairs for hamiltonian properties, *Discrete Mathematics* **173** (1997) 45–60.

[5] R.J. Gould and J.M. Harris, Forbidden triples of subgraphs and traceability, *Congr. Numer.* **108** (1995) 183–192.

[6] M. Jünger, W.R. Pulleyblank and G. Reinelt, On partitioning the edges of graphs into connected subgraphs, *J. Graph Theory* **9** (1985) 539–549.

[7] M. Las Vergnas, A note on matchings in graphs, *Colloque sur la Théorie des Graphes (Paris, 1974)*, Cahiers Centre Études Rech. Opér. **17** (1975) 257–260.

[8] L. Lovász and M.D. Plummer, *Matching Theory*, Ann. Discrete Math. **29**, North-Holland, Amsterdam, 1986.

[9] M.D. Plummer and A. Saito, Forbidden subgraphs and bounds on the size of a maximum matching, *J. Graph Theory*, **50** (2005) 1–12.

[10] D.P. Sumner, 1-factors and antifactor sets, *J. London Math. Soc.* **13** (1976) 351–359.