Edge-Dominating Cycles in Graphs

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Abstract

A set S of vertices in a graph G is said to be an edge-dominating set if every edge in G is incident with a vertex in S. A cycle in G is said to be a dominating cycle if its vertex set is an edge-dominating set. Nash-Williams [7] has proved that every longest cycle in a 2-connected graph of order n and minimum degree at least $\frac{1}{3}(n+2)$ is a dominating cycle. In this paper, we prove that for a prescribed positive integer k, under the same minimum degree condition, if n is sufficiently large and if we take k disjoint cycles so that they contain as many vertices as possible, then these cycles form an edge-dominating set. Nash-Williams' theorem corresponds to the case of $k = 1$ of this result.
keywords: dominating cycle, edge-dominating, hamiltonian cycle

1 Introduction

A set of vertices S in a graph G is said to be an edge-dominating set if every edge in G is incident with a vertex in S. Equivalently, S is called an edge-dominating set if $V(G) - S$ is either an empty set or an independent set of G. An edge-dominating set is also called a vertex cover in some textbooks, for example in [3]. A cycle C in a graph G is said to be a dominating cycle if $V(C)$ is an edge-dominating set of G. A dominating cycle is also called a D-cycle or D_2-cycle. By definition, a hamiltonian cycle is a dominating cycle. But not every dominating cycle is a hamiltonian cycle. Thus, the dominating cycles form a broader class than the hamiltonian cycles.

The relationship between cycle-related properties and the minimum degree of a graph has long been studied in graph theory. Dirac's Theorem [5] is one of the oldest results in this topic.

Theorem A (Dirac [5]) *A graph of order $n \geq 3$ and minimum degree at least $\frac{1}{2}n$ has a hamiltonian cycle.*

The bound $\frac{1}{2}n$ of the minimum degree is sharp. In fact, the complete bipartite graph $K_{m,m+1}$ has order $n = 2m + 1$ and minimum degree $m = \frac{1}{2}(n-1)$, but it does not have a hamiltonian cycle.

Dirac's Theorem has been extended in many directions. One of them is to replace the existence of a hamiltonian cycle with that of a dominating cycle. Since the class of dominating cycles is broader than that of hamiltonian cycles, one may expect that a weaker minimum degree condition guarantees the existence of a dominating cycle. Actually, Nash-Williams [7] has proved the following.

Theorem B (Nash-Williams [7]) *Let G be a 2-connected graph of order n and minimum degree at least $\frac{1}{3}(n+2)$. Then each longest cycle in G is a dominating cycle.*

Though the conclusion of the above theorem is stronger than just the existence of a dominating cycle, the bound $\frac{1}{3}(n+2)$ is best-possible even if we concern the existence of a dominating cycle. For $m \geq 2$, let G be the graph obtained by joining K_m and $(m+1)K_2$. Then the minimum degree of G is $m+1 = \frac{1}{3}(|G|+1)$. However, G has no dominating cycle.

Another extension of Dirac's Theorem discusses the existence of a 2-factor with a specified number of components. A hamiltonian cycle can be interpreted as a 2-factor with one component. This interpretation leads us to a minimum degree condition for a graph to have a 2-factor with k components, where k is a prescribed positive integer. This direction of research has been carried out by Brandt et al. [2].

Theorem C ([2]) *Let k be a positive integer, and let G be a graph of order n. If $n \geq 4k$ and $\deg_G(x) + \deg_G(y) \geq n$ for every pair of nonadjacent vertices x, y in G, then G has a 2-factor with k components.*

The purpose of this paper is to combine the above two directions of research. In the same way as Brandt et al. have extended a hamiltonian cycle into a 2-factor with a specified number of components, we extend Nash-Williams' Theorem, and give a minimum degree condition for a graph to have a specified number of disjoint cycles which form an edge-dominating set.

Let $\mathcal{C} = \{C_1, \ldots, C_k\}$ be a set of k disjoint cycles. Then we define $\bigcup \mathcal{C}$ and $n(\mathcal{C})$ by $\bigcup \mathcal{C} = \bigcup_{C \in \mathcal{C}} V(C)$ and $n(\mathcal{C}) = |\bigcup \mathcal{C}|$. We say that $\mathcal{C}$ is *maximum* if there does not exist another set of k disjoint cycles $\mathcal{C}'$ with $n(\mathcal{C}') > n(\mathcal{C})$. The following is the main theorem of this paper.

Theorem 1 *Let k be a positive integer, and let G be a 2-connected graph of order $n \geq 44$ and minimum degree at least $\frac{1}{3}(n+2)$. Then for every maximum set $\mathcal{C}$ of k disjoint cycles, $\bigcup \mathcal{C}$ is an edge-dominating set of G.*

Note that the bound $\frac{1}{3}(n+2)$ is best-possible. The sharpness example $G = K_m + (m+1)K_2$ of Nash-Williams' Theorem has k disjoint cycles if $m \geq k$, but does not have an edge-dominating set which induces a subgraph with a 2-factor. Note also that Theorem 1 does not guarantee the existence of k disjoint cycles. If a graph G in Theorem 1 does not have k disjoint cycles, the conclusion holds in a vacuous way. For a minimum degree condition for the existence of k disjoint cycles, we refer the reader to the following theorem.

Theorem D (Corrádi and Hajnal [4]) *A graph of order at least $3k$ and minimum degree at least $2k$ has k disjoint cycles.*

Later, Enomoto [6] has extended this theorem. For a noncomplete graph G, we define $\sigma_2(G)$ by

$$\sigma_2(G) = \min\{\deg_G(x) + \deg_G(y)\colon x, y \in V(G), xy \notin E(G), \text{ and } x \neq y\}.$$

If G is a complete graph, we define $\sigma_2(G) = +\infty$. Enomoto [6] has proved a relationship between $\sigma_2(G)$ and the existence of disjoint cycles.

Theorem E (Enomoto [6]) *A graph G with $|G| \geq 3k$ and $\sigma_2(G) \geq 4k-1$, then G has k disjoint cycles.*

We use this theorem in the proof of one lemma.

In the next section, we prepare several lemmas. Then in Section 3, we give a proof of Theorem 1. We also make a slight improvement to this theorem. In Section 4, we give concluding remarks.

For standard graph-theoretic terminology not explained in this paper, we refer the reader to [3]. Let $W = w_0w_1\ldots w_l$ be a walk in a graph G. We call l the length of W and denote it by $l(W)$. We call w_0 and w_l the endvertices of W. For $0 \leq i \leq j \leq l$, we denote the subwalk $w_iw_{i+1}\ldots w_j$ by $w_i\overrightarrow{W}w_j$, and its reverse $w_jw_{j-1}\ldots w_i$ by $w_j\overleftarrow{W}w_i$. We define the successor w_i^+ and the predecessor w_i^- by $w_i^+ = w_{i+1}$ and $w_i^- = w_{i-1}$. We also define w_i^{++} by $w_i^{++} = w_{i+2}$. If $A \subset V(W)$, then we define A^+ and A^- by $A^+ = \{w^+\colon w \in A\}$ and $A^- = \{w^-\colon w \in A\}$. If $\mathcal{C}$ is a set of disjoint cycles, then for each $x \in \bigcup\mathcal{C}$, there exists a unique cycle $C \in \mathcal{C}$ such that $x \in V(C)$. Then x^+ is defined to be the successor of x in C. For $u, v \in V(G)$, a path which starts at u and ends at v is called a uv-path. A graph is said to be trivial if its order is one.

Let G be a graph, x be a vertex in G and H be a subgraph of G. Note that possibly $x \notin V(H)$. Then we define $N_H(x)$ to be the set of the neighbors of x contained in $V(H)$. We define the degree $\deg_H(x)$ of x in H by $\deg_H(x) = |N_H(x)|$. We denote by $\alpha(G)$ be the independence number of G.

Let T be a tree. Then for any pair of vertices u, v of T, there exists a unique uv-path in T. We denote this path by uTv. A vertex of degree one in a tree T is called an endvertex of T.

2 Lemmas

In this section, we prepare several lemmas, which we frequently use in the proof of the main theorem. The next two lemmas easily follow from the maximality of k disjoint cycles and standard arguments.

Lemma 2 *Let $\mathcal{C}$ be a maximum set of k disjoint cycles in a graph G, and let H be a component of $G - \bigcup\mathcal{C}$. Then $N_{\bigcup\mathcal{C}}(H) \cap N_{\bigcup\mathcal{C}}(H)^+ = \emptyset$.*

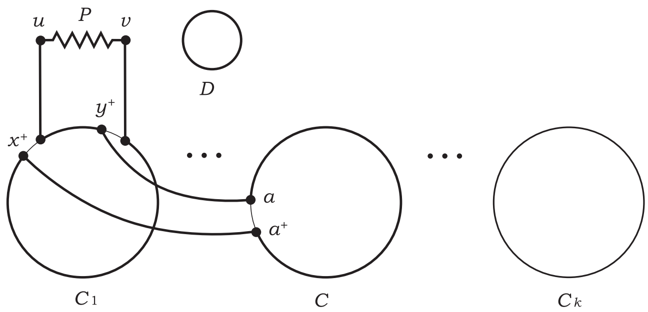

Figure 1: $\mathcal{C}$ in the proof of Lemma 4

Lemma 3 *Let $\mathcal{C}$ be a maximum set of k disjoint cycles in a graph G, and let H be a component of $G - \bigcup\mathcal{C}$. Let $C \in \mathcal{C}$ and suppose C has a pair of distinct neighbors x, y of H. Then*

(1) $x^+y^+ \notin E(G)$,

(2) $N_G(x^+)^- \cap N_G(y^+) \cap V(x^+\overrightarrow{C}y) = \emptyset$,

(3) $N_G(x^+) \cap N_G(y^+)^- \cap V(y^+\overrightarrow{C}x) = \emptyset$, *and*

(4) $N_{G-\bigcup\mathcal{C}}(x^+) \cap N_{G-\bigcup\mathcal{C}}(y^+) = \emptyset$.

The next three lemmas deal with two or more cycles in a maximum set of disjoint cycles.

Lemma 4 *Let $\mathcal{C}$ be a maximum set of k disjoint cycles. Let H be a component of $G - \bigcup\mathcal{C}$ and let $C_1 \in \mathcal{C}$. Suppose there exists a pair of edges ux and vy in G with $\{u,v\} \subset V(H)$, $\{x,y\} \subset V(C_1)$ and $x \neq y$. If H has a uv-path P such that $H - V(P)$ has a cycle, then for each $C \in \mathcal{C} - \{C_1\}$, $N_C(x^+)^- \cap N_C(y^+) = \emptyset$.*

Proof. Let D be a cycle in $H - V(P)$. Assume $N_C(x^+)^- \cap N_C(y^+) \neq \emptyset$, and let $a \in N_C(x^+)^- \cap N_C(y^+)$. Let $C' = u\overrightarrow{P}vy\overleftarrow{C_1}x^+a^+\overrightarrow{C}ay^+\overrightarrow{C_1}xu$ and $\mathcal{C}' = (\mathcal{C} - \{C_1, C\}) \cup \{C', D\}$ (see Figure 1). Then $\mathcal{C}'$ is a set of k disjoint cycles with $n(\mathcal{C}') > n(\mathcal{C})$. This contradicts the maximality of $\mathcal{C}$. □

Lemma 5 *Let G be a graph and let $\mathcal{C}$ be a maximum set of k disjoint cycles. Let H be a component of $G - \bigcup\mathcal{C}$. Let C_1 and C_2 be a pair of distinct cycles in $\mathcal{C}$, and suppose that there exists a pair of edges ux, vy in G with $\{u,v\} \subset V(H)$, $x \in V(C_1)$ and $y \in V(C_2)$ (possibly $u = v$). Further, suppose H has a uv-path P such that $H - V(P)$ has a cycle. Then*

(1) $N_{C_1}(x^+)^- \cap N_{C_1}(y^+) = \emptyset$. *In particular,* $x^+y^+ \notin E(G)$, *and*

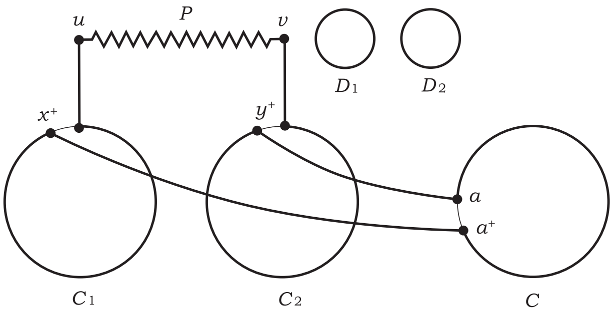

Figure 2: $\mathcal{C}$ in the proof of Lemma 6

(2) $N_{G-\bigcup\mathcal{C}}(x^+) \cap N_{G-\bigcup\mathcal{C}}(y^+) = \emptyset$.

Proof. Let D be a cycle in $H - V(P)$.
(1) Assume $N_{C_1}(x^+)^- \cap N_{C_1}(y^+) \neq \emptyset$, and let $a \in N_{C_1}(x^+)^- \cap N_{C_1}(y^+)$. Note that $a \neq x$. Let $C' = u\overrightarrow{P}vy\overleftarrow{C_2}y^+a\overleftarrow{C_1}x^+a^+\overrightarrow{C_1}xu$, and let $\mathcal{C}' = (\mathcal{C} - \{C_1, C_2\}) \cup \{C', D\}$. Then $\mathcal{C}'$ is a set of k disjoint cycles with $n(\mathcal{C}') > n(\mathcal{C})$. This is a contradiction.
(2) Assume $N_{G-\bigcup\mathcal{C}}(x^+) \cap N_{G-\bigcup\mathcal{C}}(y^+) \neq \emptyset$, and let $a \in N_{G-\bigcup\mathcal{C}}(x^+) \cap N_{G-\bigcup\mathcal{C}}(y^+)$. By Lemma 2, $a \notin V(H)$. Let $C' = u\overrightarrow{P}vy\overleftarrow{C_2}y^+ax^+\overrightarrow{C_1}xu$ and $\mathcal{C}' = (\mathcal{C} - \{C_1, C\}) \cup \{C', D\}$. Then $\mathcal{C}'$ is a set of k disjoint cycles with $n(\mathcal{C}') > n(\mathcal{C})$. This is a contradiction. □

Lemma 6 *Let G be a graph, and let $\mathcal{C}$ be a maximum set of k disjoint cycles ($k \geq 3$). Let H be a component of $G - \bigcup\mathcal{C}$, and let C_1 and C_2 be a pair of distinct cycles in $\mathcal{C}$. Suppose there exists a pair of edges ux, vy with $\{u, v\} \subset V(H)$, $x \in V(C_1)$ and $y \in V(C_2)$ (possibly $u = v$), and H has a uv-path P such that $H - V(P)$ has a pair of disjoint cycles. Then for each $C \in \mathcal{C} - \{C_1, C_2\}$, $N_C(x^+)^- \cap N_C(y^+) = \emptyset$.*

Proof. Let D_1 and D_2 be a pair of disjoint cycles in $H - V(P)$. Assume $N_C(x^+)^- \cap N_C(y^+) \neq \emptyset$, and let $a \in N_C(x^+)^- \cap N_C(y^+)$. Let $C' = u\overrightarrow{P}vy\overleftarrow{C_2}y^+a\overleftarrow{C}a^+x^+\overrightarrow{C_1}xu$ and $\mathcal{C}' = (\mathcal{C} - \{C_1, C_2, C\}) \cup \{C', D_1, D_2\}$ (see Figure 2). Then $\mathcal{C}'$ is a set of k disjoint cycles with $n(\mathcal{C}') > n(\mathcal{C})$, a contradiction. □

We prepare two more lemmas. The first one is a simple observation.

Lemma 7 *Let G be a connected graph.*

(1) Let u and v be vertices in G and let P be a shortest uv-path in G. Then for each $w \in V(G) - V(P)$, $\deg_{G-V(P)}(w) \geq \deg_G(w) - 3$.

(2) Suppose G has a cycle and let C be a shortest cycle. Then for each $w \in V(G) - V(C)$, $\deg_{G-V(C)}(w) \geq \deg_G(w) - 3$. Moreover, if the equality holds, then $|C| = 3$ and $V(C) \subset N_G(w)$.

Proof. (1) If $\deg_{G-V(P)}(w) \leq \deg_G(w) - 4$, then $\deg_P(w) \geq 4$. Choose a, $b \in N_P(w)$ so that $a\overrightarrow{P}b$ is as long as possible. Then since $|V(a\overrightarrow{P}b)| \geq 4$, $u\overrightarrow{P}awb\overrightarrow{P}v$ is shorter than P. This is a contradiction.
(2) The proof of $\deg_{G-V(C)}(w) \geq \deg_G(w) - 3$ is the same as that of (1). Suppose the equality holds. Let $N_C(w) = \{u_1, u_2, u_3\}$. We may assume that u_1, u_2 and u_3 appear in the consecutive order along C. Then since $u_1wu_3\overrightarrow{C}u_1$ is not shorter than C, we have $u_3 = u_2^+$ and $u_2 = u_1^+$. We can apply the same argument to $u_2wu_1\overrightarrow{C}u_2$ to obtain $u_1 = u_3^+$. Therefore, $C = u_1u_2u_3u_1$. □

Lemma 8 *Let G be a graph of order at least two. If G has no pair of disjoint cycles, then at least one of the following holds.*

(a) *G has a pair of distinct vertices u, v and uv-path P such that* $\deg_G(u) + \deg_G(v) \leq 6$ *and* $l(P) \geq 2$.

(b) *G has a pair of distinct vertices u, v with* $\deg_G(u) + \deg_G(v) \leq 2$.

(c) $G \simeq K_5$

(d) $G \simeq K_5 \cup K_1$

Proof. First, we prove the lemma in the case that G is connected. We assume that G neither satisfies (a) nor (b), and prove $G \simeq K_5$.

If $|G| \geq 6$, then by Theorem E, G has a pair of nonadjacent vertices u, v with $\deg_G(u) + \deg_G(v) \leq 6$. Since G is connected, u and v can be joined by a path of length at least two, and hence (a) holds, a contradiction. Therefore, $|G| \leq 5$. If G is not complete, G has a pair of nonadjacent vertices u, v. Then $\max\{\deg_G(u), \deg_G(v)\} \leq |G| - 2 \leq 3$ and again since G is connected, u and v can be joined by a path of length at least two. Therefore, (a) follows. This is again a contradiction, and hence G is a complete graph. Since K_4 and K_3 satisfy (a) and K_2 satisfies (b), we have $G \simeq K_5$.

Next, suppose G is disconnected. If G has two or more acyclic components, or G has an acyclic component of order at least two, then G has a pair of distinct endvertices, and (b) holds. On the other hand, by the assumption, G does not have a pair of components which contain a cycle. Therefore, G has exactly two components, one of which contains a cycle and the other is a trivial component. Let H be the component containing a cycle. Since H does not have a pair of disjoint cycles, does not satisfy (a) or (b), and it is connected, we have $H \simeq K_5$. Therefore, we have $G \simeq K_5 \cup K_1$ and (d) holds. □

3 Proof of the Main Theorem

Now we prove the main theorem.

Proof of Theorem 1. If G has no set of k disjoint cycles, then the theorem vacuously holds. Thus, we may assume that G has a set of k disjoint cycles. Let $\mathcal{C}$ be a maximum set of k disjoint cycles, and assume that $\bigcup \mathcal{C}$ is not an edge-dominating set. Let H be a maximum component of $G - \bigcup \mathcal{C}$. Then $|H| \geq 2$. Since G is 2-connected, there exists a pair of independent edges ux, vy in G with $\{u, v\} \subset V(H)$ and $\{x, y\} \subset \bigcup \mathcal{C}$. Let P be a shortest uv-path in H.

Claim 1 *$H - V(P)$ does not have a pair of disjoint cycles.*

Proof. Assume, to the contrary, that $H - V(P)$ has a pair of disjoint cycles D_1 and D_2. We consider two cases.

Case 1: x and y lie in the same cycle in $\mathcal{C}$.

Let C_1 be a member of $\mathcal{C}$ with $\{x, y\} \subset V(C_1)$. Let

$$\begin{aligned}
&X_1 = N_{C_1}(x^+)^- \cap V(x^+\overrightarrow{C_1}y), \quad X_2 = N_{C_1}(x^+) \cap V(y^+\overrightarrow{C_1}x),\\
&X_3 = \bigcup_{C \in \mathcal{C} - \{C_1\}} N_C(x^+)^-, \quad X_4 = N_{G - \bigcup \mathcal{C}}(x^+),\\
&Y_1 = N_{C_1}(y^+) \cap V(x^+\overrightarrow{C_1}y), \quad Y_2 = N_{C_1}(y^+)^- \cap V(y^+\overrightarrow{C_1}x),\\
&Y_3 = \bigcup_{C \in \mathcal{C} - \{C_1\}} N_C(y^+), \quad Y_4 = N_{G - \bigcup \mathcal{C}}(y^+),\\
&X = \bigcup_{i=1}^{4} X_i \text{ and } Y = \bigcup_{i=1}^{4} Y_i.
\end{aligned}$$

Since X_1, X_2, X_3 and X_4 are pairwise disjoint and $y^+ \notin N_{C_1}(x^+)$ by Lemma 3 (1),

$$\begin{aligned}
|X| &= |X_1| + |X_2| + |X_3| + |X_4|\\
&= |N_{C_1}(x^+)^- \cap V(x^+\overrightarrow{C_1}y)| + |N_{C_1}(x^+) \cap V(y^+\overrightarrow{C_1}x)| + \sum_{C \in \mathcal{C} - \{C_1\}} |N_C(x^+)^-| + |N_{G - \bigcup \mathcal{C}}(x^+)|\\
&= |N_{C_1}(x^+) \cap V(x^{++}\overrightarrow{C_1}y^+)| + |N_{C_1}(x^+) \cap V(y^+\overrightarrow{C_1}x)| + \sum_{C \in \mathcal{C} - \{C_1\}} |N_C(x^+)| + |N_{G - \bigcup \mathcal{C}}(x^+)|\\
&= \deg_{C_1}(x^+) + \sum_{C \in \mathcal{C} - \{C_1\}} \deg_C(x^+) + \deg_{G - \bigcup \mathcal{C}}(x^+) = \deg_G(x^+).
\end{aligned}$$

Similarly, we have $|Y| = \deg_G(y^+)$.

By Lemma 3 (2), (3), (4) and Lemma 4, $X \cap Y = \emptyset$.

Assume $\{a, a^+\} \subset X \cup Y$ for some $a \in N_{\bigcup \mathcal{C}}(u) - \{x, y\}$.

- If $a \in V(x^+\overrightarrow{C_1}y^-)$, then by Lemma 3 (1), we have $a^+ \in X_1$ and $a \in Y_1$. Note that by Lemma 2, $a^+ \neq y$ and hence $a^{++} \in N_G(x^+) \cap V(x^+\overrightarrow{C_1}y)$. Let $C' = u\overrightarrow{P}vy\overleftarrow{C_1}a^{++}x^+\overrightarrow{C_1}ay^+\overrightarrow{C_1}xu$ and $\mathcal{C}' = (\mathcal{C} - \{C_1\}) \cup \{C'\}$.

- If $a \in V(y^+\overrightarrow{C_1}x^-)$, then again by Lemma 3 (1) and Lemma 2, we have $a \in N_G(x^+)$ and $a^{++} \in N_G(y^+) \cap V(y^+\overrightarrow{C_1}x)$. Let $C' = u\overrightarrow{P}vy\overleftarrow{C_1}x^+a\overleftarrow{C_1}y^+a^{++}\overrightarrow{C_1}xu$ and $\mathcal{C}' = (\mathcal{C} - \{C_1\}) \cup \{C'\}$.

- If $a \in \bigcup_{C \in \mathcal{C}-\{C_1\}} V(C)$, let $a \in V(C_2)$, $C_2 \in \mathcal{C} - \{C_1\}$. Then by Lemma 5 (1), $a^+ \in X_3$ and $a \in Y_3$. Let $C' = u\overrightarrow{P}vy\overleftarrow{C_1}x^+a^{++}\overrightarrow{C_2}ay^+\overrightarrow{C_1}xu$ and $\mathcal{C}' = (\mathcal{C} - \{C_1, C_2\}) \cup \{C', D_1\}$.

Then in each case, since $u \neq v$, $\mathcal{C}'$ is a set of k disjoint cycles with $n(\mathcal{C}') > n(\mathcal{C})$. This is a contradiction. Therefore, for each $a \in N_{\bigcup \mathcal{C}}(u) - \{x, y\}$, we have $\{a, a^+\} \not\subset X \cup Y$ and we can take $f(a) \in \{a, a^+\}$ such that $f(a) \notin X \cup Y$. By Lemma 2, $f \colon N_{\bigcup \mathcal{C}}(u) - \{x, y\} \to \bigcup \mathcal{C}$ is an injection. Let $Z = \{f(a) \colon a \in N_{\bigcup \mathcal{C}}(u) - \{x, y\}\} \cup N_H(u)$. Then $|Z| \geq \deg_G(u) - 2$, and by the definition of f and Lemma 2, we have $Z \cap (X \cup Y) = \emptyset$ and $u \notin X \cup Y \cup Z$. Therefore,

$$\begin{aligned} n - 1 &\geq |X \cup Y \cup Z| = |X| + |Y| + |Z| \geq \deg_G(u) - 2 + \deg_G(x^+) + \deg_G(y^+) \\ &\geq \left(\frac{n+2}{3}\right) \cdot 3 - 2 = n. \end{aligned}$$

This is a contradiction, and the claim follows in this case.

Case 2: x and y belong to different cycles in $\mathcal{C}$.

Let $x \in V(C_1)$ and $y \in V(C_2)$ for C_1, $C_2 \in \mathcal{C}$, $C_1 \neq C_2$. Let

$$\begin{aligned} &X_1 = N_{C_1}(x^+)^-, \quad X_2 = N_{C_2}(x^+), \quad X_3 = \bigcup_{C \in \mathcal{C} - \{C_1, C_2\}} N_C(x^+)^-, \quad X_4 = N_{G - \bigcup \mathcal{C}}(x^+), \\ &Y_1 = N_{C_1}(y^+), \quad Y_2 = N_{C_2}(y^+)^-, \quad Y_3 = \bigcup_{C \in \mathcal{C} - \{C_1, C_2\}} N_C(y^+), \quad Y_4 = N_{G - \bigcup \mathcal{C}}(y^+), \\ &X = X_1 \cup X_2 \cup X_3 \cup X_4, \quad \text{and } Y = Y_1 \cup Y_2 \cup Y_3 \cup Y_4. \end{aligned}$$

Note that X_1, X_2, X_3 and X_4 are pairwise disjoint, and

$$\begin{aligned} |X| &= |X_1| + |X_2| + |X_3| + |X_4| \\ &= |N_{C_1}(x^+)^-| + |N_{C_2}(x^+)| + \sum_{C \in \mathcal{C} - \{C_1, C_2\}} |N_C(x^+)^-| + |N_{G - \bigcup \mathcal{C}}(x^+)| \\ &= |N_{C_1}(x^+)| + |N_{C_2}(x^+)| + \sum_{C \in \mathcal{C} - \{C_1, C_2\}} |N_C(x^+)| + |N_{G - \bigcup \mathcal{C}}(x^+)| = \deg_G(x^+). \end{aligned}$$

Similarly, we have $|Y| = \deg_G(y^+)$. By Lemma 5 (1), (2) and Lemma 6, $X \cap Y = \emptyset$.

Assume $\{a, a^+\} \subset X \cup Y$ for some $a \in N_{\bigcup \mathcal{C}}(u) - \{x, y\}$.

- If $a \in V(C_1)$, then by Lemma 3 (1) and Lemma 5 (1), $a \in Y_1$ and $a^+ \in X_1$. By Lemma 2, $a^+ \neq x$, and hence $a^{++} \in N_G(x^+) \cap V(a\overrightarrow{C_1}x)$. Let $C' = u\overrightarrow{P}vy\overleftarrow{C_2}y^+a\overleftarrow{C_1}x^+a^{++}\overrightarrow{C_1}xu$ and $\mathcal{C}' = (\mathcal{C} - \{C_1, C_2\}) \cup \{C', D_1\}$.

- If $a \in V(C_2)$, then again by Lemma 3 (1), Lemma 5 (1) and Lemma 2, $a \in X_2$, $a^+ \in Y_2$ and $a^{++} \in V(a\overrightarrow{C_2}y)$. Let $C' = u\overrightarrow{P}vy\overleftarrow{C_2}a^{++}y^+\overrightarrow{C_2}ax^+\overrightarrow{C_1}xu$ and $\mathcal{C}' = (\mathcal{C} - \{C_1, C_2\}) \cup \{C', D_1\}$.

- If $a \notin V(C_1) \cup V(C_2)$, let $a \in V(C_3)$, $C_3 \in \mathcal{C} - \{C_1, C_2\}$. By Lemma 5 (1), $a^+ \in X_3$ and $a \in Y_3$. Let $C' = u\overrightarrow{P}vy\overleftarrow{C_2}y^+a\overleftarrow{C_3}a^{++}x^+\overrightarrow{C_1}xu$ and $\mathcal{C}' = (\mathcal{C} - \{C_1, C_2, C_3\}) \cup \{C', D_1, D_2\}$.

Then in each case, $\mathcal{C}'$ is a set of k disjoint cycles, and since $u \neq v$, $n(\mathcal{C}') > n(\mathcal{C})$. This is a contradiction. Therefore, for each $a \in N_{\bigcup \mathcal{C}}(u) - \{x, y\}$, we have $\{a, a^+\} \not\subset X \cup Y$ and we can take $f(a) \in \{a, a^+\}$ such that $f(a) \notin X \cup Y$. Let $Z = \{f(a) \colon a \in N_{\bigcup \mathcal{C}}(u) - \{x, y\}\} \cup N_H(u)$. Then f is an injection, and hence $|Z| \geq \deg_G(u) - 2$. By the definition of f and Lemma 2, we have $Z \cap (X \cup Y) = \emptyset$ and $u \notin X \cup Y \cup Z$. Therefore, we have

$$
\begin{aligned}
n - 1 &\geq |X \cup Y \cup Z| = |X| + |Y| + |Z| \\
&= \deg_G(x^+) + \deg_G(y^+) + \deg_G(u) - 2 \geq \frac{1}{3}(n+2) \cdot 3 - 2 = n.
\end{aligned}
$$

This is a contradiction, and Claim 1 follows. □

Claim 2 *H has a pair of distinct vertices w_1 and w_2 such that either*

(1) $\deg_H(w_1) + \deg_H(w_2) \leq 12$ *and w_1 and w_2 are joined by a path of length at least two, or*

(2) $\deg_H(w_1) + \deg_H(w_2) \leq 6$.

Proof. Assume, to the contrary, that no pair of distinct vertices satisfies (1) or (2). By Claim 1, $H - V(P)$ has no pair of disjoint cycles. Therefore, by Lemma 8,

(a) there exists a pair of distinct vertices w_1, w_2 in $H - V(P)$ such that $\deg_{H-V(P)}(w_1) + \deg_{H-V(P)}(w_2) \leq 6$ and w_1 and w_2 are joined by a path of length at least two.

(b) there exists a pair of distinct vertices w_1 and w_2 in $H - V(P)$ with $\deg_{H-V(P)}(w_1) + \deg_{H-V(P)}(w_2) \leq 2$,

(c) $H - V(P) \simeq K_5$, or

(d) $H - V(P) \simeq K_5 \cup K_1$.

If (a) occurs, then by Lemma 7 (1), $\deg_H(w_i) \leq \deg_{H-V(P)}(w_i) + 3$ ($i = 1, 2$), and (1) follows. This is a contradiction. If (c) occurs, $\deg_H(u) \leq \deg_P(u) + 5 = 6$, $\deg_H(v) \leq 6$ and hence $\deg_H(u) + \deg_H(v) \leq 12$. Therefore, if $l(P) \geq 2$, (1) follows and we have a contradiction. Thus, $P = uv$. Moreover, if $N_{H-V(P)}(u) \neq \emptyset$ and $N_{H-V(P)}(v) \neq \emptyset$, then (1) again holds, a contradiction. Therefore, we may assume $N_{H-V(P)}(v) = \emptyset$, which implies $\deg_H(v) = 1$. Now let $w \in V(H) - V(P)$. Then since $wv \notin E(G)$, we have $\deg_H(w) \leq 5$ and hence $\deg_H(v) + \deg_H(w) \leq 6$. Thus, (2) follows, a contradiction.

Suppose (d) occurs. Let $V(H) - V(P) = \{v_0, v_1, \ldots, v_5\}$, where v_0 is an isolated vertex in $H - V(P)$ and $\{v_1, \ldots, v_5\}$ induces K_5. By Lemma 7 (2), $\deg_H(v_0) \leq 3$ and

$\deg_H(u) \le 7$, and hence $\deg_H(v_0) + \deg_H(u) \le 10$. Since (1) does not hold, v_0 and u are not joined by a path of length at least two. This implies $N_P(v_0) = \{u\}$. However, in this case $v_0 v \notin E(G)$ and $\deg_H(v) \le 6$. Therefore, (1) follows for v and v_0, a contradiction.

Finally, suppose (b) occurs. By Lemma 7 (1), $\deg_H(w_1) + \deg_H(w_2) \le 2 + 3 \times 2 = 8$. If $N_P(w_1) \neq \emptyset$ and $N_P(w_2) \neq \emptyset$, then (1) holds, a contradiction. Hence, we may assume $N_P(w_2) = \emptyset$. This implies $\deg_H(w_1) + \deg_H(w_2) \le 2 + 3 = 5$, and hence (2) follows. This is a contradiction, and the claim follows. $\square$

Suppose (1) of Claim 2 occurs. Let $X_1 = N_{\bigcup\mathcal{C}}(w_1)$, $X_2 = N_{\bigcup\mathcal{C}}(w_1)^+$, $X_3 = N_{\bigcup\mathcal{C}}(w_2)^-$ and $X_4 = N_{\bigcup\mathcal{C}}(w_2)^{--}$. By Lemma 2, $X_1 \cap X_2 = X_1 \cap X_3 = X_3 \cap X_4 = \emptyset$. Assume $X_2 \cap X_4 \neq \emptyset$. Let $a \in X_2 \cap X_4$ and let P_1 be a w_1w_2-path of length at least two. Let C be the member of $\mathcal{C}$ with $a \in V(C_1)$. Let $C' = w_1 \overrightarrow{P_1} w_2 a^{++} \overrightarrow{C_1} a^- w_1$ and $\mathcal{C}' = (\mathcal{C} - \{C_1\}) \cup \{C'\}$. Then $\mathcal{C}'$ is a set of k disjoint cycles with $n(\mathcal{C}') > n(\mathcal{C})$ since $l(P) \ge 2$. This is a contradiction, and hence $X_2 \cap X_4 = \emptyset$. Similarly, we have $X_1 \cap X_4 = X_2 \cap X_3 = \emptyset$. Let $\deg_H(w_1) + \deg_H(w_2) = t$. Since $|X_1| = |X_2| = \deg_{\bigcup\mathcal{C}}(w_1) = \deg_G(w_1) - \deg_H(w_1)$ and $|X_3| = |X_4| = \deg_G(w_2) - \deg_H(w_2)$. Then

$$n - |H| \ge |\bigcup \mathcal{C}| \ge \left| \bigcup_{i=1}^{4} X_i \right| = \sum_{i=1}^{4} |X_i| \ge 2 \left(2 \cdot \frac{n+2}{3} - t \right) = \frac{4}{3}(n+2) - 2t.$$

This implies $n + 3|H| \le 6t - 8$. We may assume $\deg_H(w_1) \le \deg_H(w_2)$. Then $\deg_H(w_2) \ge \frac{1}{2}t$, and $|H| \ge \deg_H(w_2) + 1 \ge \frac{1}{2}t + 1$ and $6t - 8 \ge n + 3|H| \ge n + \frac{3}{2}t + 3$. Therefore, $n \le \frac{9}{2}t - 11$. Since $t \le 12$, $n \le 43$. This contradicts the assumption.

Next, suppose (1) of Claim 2 does not hold and $|H| \ge 3$. Then H has a path Q of length two. Let u_1 and u_2 be the endvertices of Q. Since (1) of Claim 2 does not hold, $\deg_H(u_1) + \deg_H(u_2) \ge 13$. This implies that $|H| \ge \max\{\deg_H(u_1), \deg_H(u_2)\} + 1 \ge 8$.

By Claim 2, G has a pair of distinct vertices w_1 and w_2 with $\deg_H(w_1) + \deg_H(w_2) \le 6$. By symmetry, we may assume $\deg_H(w_1) \le \deg_H(w_2)$, which implies $\deg_H(w_1) \le 3$. Let P_2 be a w_1w_2-path in H, and let $X_1 = N_{\bigcup\mathcal{C}}(w_1)$, $X_2 = N_{\bigcup\mathcal{C}}(w_1)^+$ and $X_3 = N_{\bigcup\mathcal{C}}(w_2)^-$. Then since $l(P_2) \ge 1$, and by Lemma 2, X_1, X_2 and X_3 are pairwise disjoint. Therefore,

$$\begin{aligned} |X_1| + |X_3| &= \deg_{\bigcup\mathcal{C}}(w_1) + \deg_{\bigcup\mathcal{C}}(w_2) \\ &= \deg_G(w_1) - \deg_H(w_1) + \deg_G(w_2) - \deg_H(w_2) \\ &\ge 2 \cdot \frac{n+2}{3} - 6 = \frac{2n-14}{3} \end{aligned}$$

and $|X_2| = \deg_{\bigcup\mathcal{C}}(w_1) \ge \frac{n+2}{3} - 3 = \frac{n-7}{3}$, we have $n - |H| \ge |\bigcup\mathcal{C}| \ge |X_1| + |X_2| + |X_3| \ge n - 7$, which implies $|H| \le 7$. This is a contradiction.

Finally, assume $|H| = 2$. Let $V(H) = \{w_1, w_2\}$ and let $Z_1 = N_{\bigcup\mathcal{C}}(w_1)$, $Z_2 = N_{\bigcup\mathcal{C}}(w_1)^+$ and $Z_3 = N_{\bigcup\mathcal{C}}(w_2)^-$. Then Z_1, Z_2 and Z_3 are pairwise disjoint. Since

$|H| = 2$, $|N_{\bigcup \mathcal{C}}(w_i)| = \deg_G(w_i) - 1 \geq \frac{1}{3}(n-1)$ $(i = 1, 2)$, which implies $|Z_j| \geq \frac{1}{3}(n-1)$ $(1 \leq j \leq 3)$. Then

$$n - 2 \geq n - |H| \geq |\bigcup \mathcal{C}| \geq |Z_1| + |Z_2| + |Z_3| \geq n - 1.$$

This is a final contradiction, and the theorem follows. □

In Theorem 1, we assume that the graph in consideration is 2-connected. This assumption is necessary for $k = 1$. Let $m \geq 22$ and $G = K_m + K_1 + K_m$. Then $|G| = 2m + 1 \geq 44$ and $\delta(G) = m \geq \frac{1}{3}(|G| + 2)$. However, G has no dominating cycle. However, this assumption is not necessary for $k \geq 2$.

Theorem 9 *Let $k \geq 2$ and let G be a graph of order $n \geq 44$ and minimum degree at least $\frac{1}{3}(n+2)$. Then for every maximum set of k disjoint cycles $\mathcal{C}$, $\bigcup \mathcal{C}$ is an edge-dominating set of G.*

Proof. Assume, to the contrary, that $\bigcup \mathcal{C}$ is not an edge-dominating set for some maximum set of k disjoint cycles, and let H be a nontrivial component of $G - \bigcup \mathcal{C}$. Note that in the proof of Theorem 1, we use the 2-connectedness only to guarantee the existence of a pair of independent edges between H and $\bigcup \mathcal{C}$. Therefore, we have only to prove the existence of such edges without using 2-connectedness. Assume no pair of independent edges exists between H and $\bigcup \mathcal{C}$. Let P be a longest path in H, and let u and v be the endvertices of P. Since $|H| \geq 2$, $|P| \geq 2$ and hence $u \neq v$. Since there does not exist a pair of independent edges between H and $\bigcup \mathcal{C}$, $\deg_{\bigcup \mathcal{C}}(u) \leq 1$ or $\deg_{\bigcup \mathcal{C}}(v) \leq 1$. By symmetry we may assume $\deg_{\bigcup \mathcal{C}}(u) \leq 1$. Then $\deg_H(u) \geq \frac{1}{3}(n-1)$. Since $N_H(u) \subset V(P)$, H has a cycle of order at least $\frac{1}{3}(n-1) + 1 = \frac{1}{3}(n+2)$. Then by the maximality of $\mathcal{C}$, $|C| \geq \frac{1}{3}(n+2)$ for each $C \in \mathcal{C}$. Since $k \geq 2$, this implies $n(\mathcal{C}) \geq \frac{2}{3}(n+2)$, and hence $n \geq n(\mathcal{C}) + |H| \geq \frac{2}{3}(n+2) + \frac{1}{3}(n+2) = n + 2$. This is a contradiction. □

4 Concluding Remarks

In Theorems 1 and 9, we assume that the order of a graph G is at least 44. But this constant comes from the proof and we believe that it is just a technical condition.

Nash-Williams' Theorem was later extended by Bondy [1]. For a graph G and $k \leq \alpha(G)$, define $\sigma_k(G)$ by

$$\sigma_k(G) = \min \left\{ \sum_{s \in S} \deg_G(x) \colon S \text{ is an independent set of order } k \text{ in } G \right\}.$$

For $k > \alpha(G)$, $\sigma_k(G)$ is defined as $\sigma_k(G) = +\infty$.

Theorem F (Bondy [1]) *Let G be a 2-connected graph of order n satisfying $\sigma_3(G) \geq n+2$. Then every longest cycle in G is a dominating cycle.*

We believe that Theorem 1 admits the same extension. From these observations, we believe the following holds.

Conjecture 10 *Let G be a 2-connected graph of order n and let k be a positive integer. If $\sigma_3(G) \geq n+2$, then for every maximum set of k disjoint cycles, $\bigcup \mathcal{C}$ is an edge-dominating set of G.*

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