High connectivity keeping connected subgraph

Shinya Fujita*
Department of Mathematics
Gunma National College of Technology
580 Toriba-machi, Maebashi, Gunma 371-8530, Japan
E-mail: shinyaa@mti.biglobe.ne.jp

and

Ken-ichi Kawarabayashi†
National Institute of Informatics,
2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan
E-mail: k_keniti@nii.ac.jp

Abstract

It was proved by Mader that, for every integer l, every k-connected graph of sufficiently large order contains a vertex set X of order precisely l such that $G - X$ is $(k-2)$-connected. This is no longer true if we require X to be connected, even for $l = 3$.

Motivated by this fact, we are trying to find an "obstruction" for k-connected graphs without such a connected subgraph. It turns out that the obstruction is an essentially 3-connected subgraph W such that $G - W$ is still highly connected. More precisely, our main result says the following.

For $k \geq 7$ and every k-connected graph G, either there exists a connected subgraph W of order 4 in G such that $G - W$ is $(k-2)$-connected, or else G contains an "essentially" 3-connected subgraph W, i.e., a subdivision of a 3-connected graph, such that $G - W$ is still highly connected, actually, $(k-6)$-connected.

This result can be compared to Mader's result [6] which says that every k-connected graph G of sufficiently large order ($k \geq 4$) has a connected subgraph H of order exactly four such that $G - H$ is $(k-3)$-connected.

*This work is supported by the JSPS Research Fellowships for Young Scientists.
†Research partly supported by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research, by C&C Foundation, by Inamori Foundation and by Kayamori Foundation.

1 Introduction

The following type of problems is well-studied by many researchers, see [3, 5, 8].

Problem. Given positive integers k, l, d with $k \geq d$, is it true that every k-connected graph of sufficiently large order has a vertex set X of order exactly l such that $G - X$ is $(k-d)$-connected ?

In connection to the above question, (k, l)-critical graphs, which we will define later, are studied by many researchers. A k-connected graph G is said to be (k, l)-*critical* if for any vertex set W of order l with $k \geq l$, $G - W$ is $(k-l)$-connected. The main question concerning (k, l)-critical graphs is whether or not there are only finitely many (k, l)-critical graphs, see the survey by Mader [8].

A best possible result for the above problem would be that we can find a vertex set W of prescribed order such that $G - W$ is $(k-1)$-connected, but this is not true as Mader has pointed out in [7]. On the other hand, Mader has proved (in the same paper) that every k-connected graph G of sufficiently large order contains a vertex set S of prescribed order such that $G - S$ is $(k-2)$-connected. A natural question which arose from the above result is, "could it be true if we require S to be connected ?". Unfortunately, this is no longer true. In [7], Mader has pointed out that for every $k \geq 18$, there are infinitely many k-connected graphs such that for any connected subgraph W of order exactly 3, $G - W$ is not $(k-2)$-connected. So we cannot even hope for the case $k = 3$. Let us observe that Mader [2] has proved that there are only finitely many $(k, 3)$-critical graphs.

On the other hand, Mader [6] has proved that every k-connected graph G of sufficiently large order has a connected subgraph H of order exactly four such that $G - H$ is $(k-3)$-connected. Again the connectivity is best possible.

Our motivation comes from the above result. More precisely, we are lead to the following question: When can a k-connected graph G have a connected subgraph W of order exactly four such that $G - W$ is $(k-2)$-connected ? Which graphs would be obstructions ? It turns out that the obstructions are nearly 3-connected, i.e., a subdivision of a 3-connected graph. In addition, these obstructions are, in a sense, non-separating subgraphs. More precisely, for any obstruction W, $G - W$ is $(k-6)$-connected. To state our main result, we need some definition and notation, but before that, let us give one more motivation.

Thomassen [10] conjectured that every $(a+b+1)$-connected graph can be decomposed into two parts A and B in such a way that A is a-connected and B is b-connected. It was shown by Thomassen himself [10] that if $b \leq 2$, then the conjecture is true. Even the case $b = 3$ is not settled yet, and the conjecture is wide open for $a, b \geq 3$. We would like to prove this conjecture when $b = 3$, but we failed. However since our obstructions are essentially 3-connected, and otherwise, we could delete a connected subgraph of order 4 in such a way that the connectivity of the resulting graph does not decrease more than 2, so our result may be the first step for the conjecture when $b = 3$.

At this moment, we should mention the 3-connected case. It was conjectured in [9] that for every positive integer k, every 3-connected graph G of sufficiently large order has a connected subgraph W of order precisely k such that $G - W$ is 2-connected. This conjecture is verified for $k = 2$ in [11], for $k = 3$ in [9] and for $k = 4$ in [1]. The cases $k \geq 5$ are open. So as we see here, when a given graph G is 3-connected, a stronger conclusion may be true.

We need some definition and notation to state our main result.

Let C_4 be a quadrilateral, i.e., a cycle of length 4. Let C_5 be a pentagon, i.e., a cycle of length 5. Also let W_4 be a graph which is isomorphic to $K_1 + C_4$ and let W_5 be a graph which is isomorphic to $K_1 + C_5$, where $A + B$ means the graph obtained from the disjoint union of the graphs A and B by adding the edges between them. A subdivision of K_4 of order 5 can be regarded as a graph which can be obtained from W_4 by the deletion of one edge, which is adjacent to the central vertex of W_4. So, in the following argument, we call this graph "W_4^-". Similarly, we define "W_5^-", which is a graph obtained from W_5 by the deletion of one edge, which is adjacent to the central vertex of W_5 (W_5^- can be also regarded as a subdivision of W_4).

Our main theorems is the following.

Theorem 1. *Let k be an integer with $k \geq 7$. Suppose that G is k-connected. Then G contains a subgraph M which satisfies one of the following:*

(i) $M \cong K_4$.

(ii) $M \cong W_4^-$.

(iii) $M \cong W_5^-$.

(iv) M is a connected graph such that $G - M$ is $(k-2)$-connected and $|M| = 4$.

As we pointed out, the graph M described in (i)-(iii) in the above theorem is, in a sense, non-separating subgraphs since it is small. Furthermore, since W_5^- consists of 6 vertices and is a subdivision of a 3-connected graph, we get the following corollary.

Corollary 1. *Let k be an integer with $k \geq 7$. Suppose that G is k-connected. Then G contain a subgraph M which satisfies one of the following:*

(i) M is a subdivision of a 3-connected graph such that $G - M$ is $(k-6)$-connected.

(ii) M is a connected graph such that $G - M$ is $(k-2)$-connected and $|M| = 4$.

All graphs considered in this paper are finite, undirected, and without loops or multiple edges. For a graph G, $V(G)$, $E(G)$ and $\delta(G)$ denote the set of vertices and the set of edges and the minimum degree of G, respectively. For a given graph G and $v \in V(G)$, we write $N_G(x)$ for the neighbourhood of $V(G)$ and $d_G(x) = |N_G(x)|$. For a subset S of $V(G)$, the subgraph induced by S is denoted by $\langle S \rangle$. With a slight abuse of notation, for a subgraph H of G and a vertex $v \in V(G)$, $N_H(v) = N_G(v) \cap V(H)$ and $d_H(v) = |N_H(v)|$. Also, for an edge $e = xy \in E(G)$, let $V(e) = \{x, y\}$. In addition, for a subgraph H of G and a subset S of $V(G)$, $N_G(S) = \bigcup_{v \in S} N_G(v)$, and when $S \cap V(H) = \emptyset$, $N_H(S) = \bigcup_{v \in S} N_H(v)$. Also let $E(A, B)$ be the number of edges between vertex sets A and B for $A \cap B = \emptyset$ where A and B are vertex subsets or subgraphs in G. In this paper, when there is no fear of confusion, a vertex subset X and the subgraph $\langle X \rangle$ are often identified.

For an edge $e \in E(G)$, let G/e be the graph obtained from G by contracting e (and replacing each of the resulting pairs of double edges by a single edge). Let $k \geq 2$ be an integer. An edge e of a k-connected graph is said to be *k-contractible* if G/e is still k-connected. Let *k-cutset* be a cutset consisting of k vertices.

2 Basic definitions for the proof of Theorem 1

Let k be an integer with $k \geq 7$. By contradiction, suppose G is a counterexample in Theorem 1. In G, from the assumption that G is a counterexample in Theorem 1, note that for every connected subgraph M of order 4, there exists a $(k+1)$-cutset which contains M. Also, we can assume that G has no subgraphs isomorphic to one of the graphs described in (i)-(iii) in Theorem 1.

In the following argument, we define some special subsets and subgraphs for our sake of the proof of Theorem 1.

Let $D' = \{\ e \in E(G) \mid\ e$ is not contained in any triangle in $G\ \}$ and let $E' = \{\ e \in E(G) \mid\ e$ is k-contractible and $e \in D'\ \}$.
Also, let $F' = \{\ v \in V(G) \mid\ E(v, V(G-v)) \cap D' = \emptyset\ \ \}$.
We define some special subgraphs in G as follows:

- For a quadrilateral $C = v_1v_2v_3v_4v_1$, if $\{v_1v_2, v_3v_4\} \subset E'$ or $\{v_1v_4, v_2v_3\} \subset E'$, then we say that C is a *good* C_4.
- For an induced subgraph $Z \cong P_4$, i.e., a path of length 3, if every edge of Z is in E', then we say that Z is a *good* P_4.
- For an induced subgraph $Z \cong K_4^-$, if every triangle in Z is not contained in any k-cutset in G, we say that Z is a *good* K_4^-.
- For an induced subgraph $Z \cong K_1 + (K_1 \cup K_2)$, say, $V(Z) = \{a, b, c, d\}$ and $E(Z) = \{ab, bc, cd, db\}$, if Z satisfies the following properties (i) and (ii), then we say that Z is a *good* $K_1 + (K_1 \cup K_2)$:
  - (i) $ab \in E'$.
  - (ii) The triangle $\langle\{b, c, d\}\rangle$ is not contained in any k-cutset in G.
- For a triangle T, if $|V(T) \cap F'| \geq 2$ and T is not contained in any K_4^- and moreover, T is not contained in any k-cutset, then we say that T is a *good* triangle.

In the following argument, those *good* subgraphs which are defined above are often called *good elements*.

Let $L = \{\ S \subset G \mid S$ is a minimum cutset in G and every component of $G - S$ has at least three vertices $\}$.

We define a family $\mathcal{F}$ of subgraphs in G as follows:

$$\mathcal{F} := \{Z \subset G \mid Z \text{ is a good element and } S \in L \text{ holds for every } (k+1)\text{-cutset } S \text{ which contains } Z\ \}.$$

To the end, we need several sets as follows:

$J_1' = \{\ S \subset G \mid S$ is a k-cutset in G such that S contains an edge in $D'\ \}$;

$J_1 = J_1' \cap L$;

$J_2 = \{ \ S \subset G \mid S$ is a k-cutset which contains a triangle $\}$;

$J_3' = \{ \ S \subset G \mid S$ is a $(k+1)$-cutset such that S contains a good element$\}$;

$J_3 = \{ \ S \in J_3' \mid S$ contains a good element Z such that $Z \in \mathcal{F} \ \}$.

Put $K = \cup_{1 \le i \le 3} J_i$. Notice that for each $S \in J_3$, S is not contained in any k-cutset because every good element contains a k-contractible edge. Later, it is shown that $K \neq \emptyset$ in G (see Lemma 4.7).

We conclude this section by proving the following lemma, which is not hard to see.

Lemma 2.1. *Let S be a cutset with $|S| \le k+1$ in G. If S contains a subgraph Z which satisfies one of the followings, then $S \in L$.*

(i) *Z is a triangle and $S \in J_2$.*

(ii) *Z is a good K_4^-.*

(iii) *Z is a good triangle.*

(iv) *Z is a good C_4 and $|E(Z) \cap D'| = 3$ and for the only edge $e \in E(Z) - D'$, $E(V(e), V(G - Z)) \cap D' = \emptyset$.*

(v) *Z is a good C_4 and $E(Z) \subset D'$ and there exists a vertex $v \in V(Z)$ such that $E(v, V(G - Z)) \cap D' = \emptyset$.*

Proof. Suppose not. Then there exists a component H of $G - S$ such that $|H| \le 2$. We can easily find K_4 or W_4^- or W_5^- in $\langle V(H) \cup V(S) \rangle$, or, we get a contradiction to the definition of good elements. The argument is somewhat tedious but it is easy to check. So, here we prove the case where Z satisfies (v) only, and we omit the proofs in other cases (i)-(iv) and leave them to readers.

Let Z, v be as in (v). Since every edge in $E(Z)$ is not contained in any triangle, it is easy to see that $|H| = 2$. Put $V(H) = \{p, q\}$. Since G does not contain W_4^- nor K_4, and G is k-connected, it follows that $|E(p, V(Z))| = |E(q, V(Z))| = 2$, $\langle V(H) \cup V(G - Z) \rangle \cong K_2 + (k-3)K_1$ and $E(V(Z), V(S - Z)) = \emptyset$. Also, since Z is a good C_4, this implies that every vertex in $V(Z)$ is adjacent to $V(H)$. Hence we may assume that $vp \in E(G)$. Then, by the above observation, it is easy to see that vp is not contained in any triangle. However, this contradicts the assumption that $E(v, V(G - Z)) \cap D' = \emptyset$.

■

3 Overview of our main proof

Our proof is to play a game on K, and reach a contradiction. To play this game, we need to show that K is not empty. This will be proved in Lemma 8. But this is not quite enough. We need to show that all the elements we can play in K have to be in $\mathcal{F}$. It follows from Lemma 2.1 that if Z is a subgraph in G such that Z is a good K_4^- or a good triangle, then $Z \in \mathcal{F}$. However, if Z is one of the other good elements, we can not assure $Z \in \mathcal{F}$. In this case, we need to find a new good element which belongs to $\mathcal{F}$, close to Z. In order to do that, we must extensively investigate the structure between a cutset which contain a good element, and the components separated by the cutset. The details will be discussed in the next section. This section is most tricky, and technical (even lengthly). But we do not see a significant shortcut for our purpose, unfortunately.

After Section 4, we begin to play our main game. The idea is that we choose one of the elements Q in K such that a component H in $G - Q$ has smallest cardinality. In order to win, we need to show that $|H|$ is not small. Furthermore, we need to show that each edge in H has a nice property. This will be done in Section 5. Finally, in Section 6, we give our main proof. If Q is in $J_1 \cup J_2$, then the situation is much simpler. It seems that our main challenge in the proof is to deal with the situation that $J_1 \cup J_2$ is empty, and there are many elements in J_3 in H. In fact, to define J_3 in H is one of our challenge. Once we can do that, we are trying to investigate a property of J_3 in H to complete our proof.

4 Preparation for the proof of Theorem 1

As we shall see in the later sections, it is important to show the existence of a cutset S such that every component of $G - S$ has at least three vertices. Using a connected subgraph of order at least three in the component of $G - S$, we can show that the size of the component is large. This will be proved in the next section (see Lemma 5.3). In order to show this, first we need to consider the case where there exists a small component in $G - S$. Actually, we need to deal with the case that the component has exactly two vertices. In this section, we prove several lemmas concerning the structure on such small components. These lemmas will enable us to find a cutset in K whose existence is desired in the later argument of the proof. At the end of this section, we show that $K \neq \emptyset$. This will be the first significant step for the proof of this theorem.

Lemma 4.1. *Let S be a cutset with $|S| \leq k+1$ in G. Suppose that either $S \in J_1' - L$ or $S \in J_3' - L$ holds, and let H be a component of $G - S$ such that $|H| \leq 2$. Moreover, if $S \in J_1' - L$, then, let e be an edge in $D' - E'$ in S, and let $X = V(S) - V(e)$, and if $S \in J_3' - L$, then, let R be a good element in S, and let $X = V(S) - V(R)$. Then $|H| = 2$ holds and the following statements hold:*

- *(i) If $S \in J_1' - L$, then $\langle V(e) \cup V(H) \rangle \cong C_4$, $E(V(e), V(H)) \subset D'$, $\langle V(H) \cup X \rangle \cong K_2 + (k-2)K_1$ and $E(V(e), X) = \emptyset$ (and hence it follows that $|E(y, V(G-y)) \cap D'| \geq 2$ for each $y \in V(e)$ and $|E(z, V(G-z)) \cap D'| = 1$ holds for each $z \in V(H)$).*
- *(ii) If $S \in J_3' - L$ and R is a good C_4, then $\langle V(H) \cup V(R) \rangle$ is isomorphic to a 3-regular 3-connected graph of order 6, and $\langle V(H) \cup X \rangle \cong K_2 + (k-3)K_1$ and $E(V(R), X) = \emptyset$. Moreover, if $|E(R) \cap D'| \geq 3$, then $\langle V(H) \cup V(R) \rangle$ does not contain a triangle.*
- *(iii) If $S \in J_3' - L$ and R is a good $K_1 + (K_1 \cup K_2)$ with $V(R) = \{a, b, c, d\}$ and $E(R) = \{ab, bc, cd, bd\}$, then $\langle V(H) \cup \{c, d\} \rangle \cong C_4$, $|E(a, V(H))| = 2$, $\langle V(H) \cup X \rangle \cong K_2 + (k-3)K_1$, and $E(\{a, c, d\}, X) = \emptyset$ (and hence it follows that $\langle V(H) \cup V(R) \rangle$ is a 3-regular 3-connected graph with two vertex-disjoint triangles, and $E(V(H), V(G-H)) \cap D' = E(V(H), \{c, d\})$.*

(iv) If $S \in J_3' - L$ and R is a good P_4, say, $R = abcd$, then there exists a vertex $u \in V(H)$ such that $\langle \{a,b,c,u\} \rangle \cong C_4$ and $\langle \{b,c,d\} \cup V(H-u) \rangle \cong C_4$ holds. Moreover, $\langle V(H) \cup X \rangle \cong K_2 + (k-3)K_1$, $E(V(R), X) = \emptyset$ and $E(V(H), V(G-H)) \cap D' = E(V(H), V(R))$ hold.

Proof. It is easy to check that $|H| = 2$. We can easily find K_4 or W_4^- or W_5^- in $\langle V(H) \cup V(S) \rangle$, or, we get a contradiction to the definition of good elements. We here prove (iv) only. As for (i)-(iii), we omit the details of the proofs and leave them to readers. (Arguing similarly in the proof of (iv), it is easy to check them.)

Put $V(H) = \{u, v\}$. Since every edge in R is not contained in any k-cutset, we see that $E(z, V(H)) \neq \emptyset$ for any $z \in V(R)$. Also, every edge in R is not contained in any triangle, we see that $|E(y, V(R))| \leq 2$ holds for any $y \in V(H)$. Since G is k-connected, this forces $|E(y, V(R))| = 2$ for any $y \in V(H)$, and thus $\langle V(H) \cup X \rangle \cong K_2 + (k-3)K_1$. This together with the fact that $E(z, V(H)) \neq \emptyset$ for any $z \in V(R)$ determines the structures $\langle \{a,b,c,u\} \rangle \cong C_4$ and $\langle \{b,c,d\} \cup V(H-u) \rangle \cong C_4$, and hence $E(V(R), X) = \emptyset$ holds because G does not contain W_4^-. In view of the above observation in $\langle V(H) \cup V(R) \rangle$ and $\langle V(H) \cup X \rangle$, we have $E(V(H), V(G-H)) \cap D' = E(V(H), V(R))$. ∎

Lemma 4.2. *In G, let S, R, H, X be subgraphs which satisfy the condition exactly as in Lemma 4.1(iii), that is, set R the good $K_1 + (K_1 \cup K_2)$ with $V(R) = \{a,b,c,d\}$ and $E(R) = \{ab, bc, cd, bd\}$. Moreover, set $\langle V(H) \cup \{c,d\} \rangle \cong C_4$, $|E(a, V(H))| = 2$, $\langle V(H) \cup X \rangle \cong K_2 + (k-3)K_1$, and $E(\{a,c,d\}, X) = \emptyset$. Suppose that $\langle V(H) \cup \{a,b\} \rangle$ is a good $K_1 + (K_1 \cup K_2)$ which is not in $\mathcal{F}$. Let $S' \in J_3' - L$ such that S' contains $\langle V(H) \cup \{a,b\} \rangle$, and let H' be a component of $G - S'$ with $|H'| \leq 2$. Then $H' = \langle \{c,d\} \rangle$ holds and $E(\{c,d\}, V(G) - \{c,d\}) \cap D' = E(\{c,d\}, V(H))$. Moreover, $E(V(H'), V(G-H')) \cap D' = E(\{c,d\}, V(H))$ holds.*

Proof. Applying Lemm 4.1(iii) to $\langle V(H) \cup \{a,b\} \rangle$ as R, the assertion immediately follows because $E(V(H), V(G-H)) \cap D' = E(V(H), \{c,d\})$. ∎

Lemma 4.3. *Let $S \in J_1' - L$ and let H be a component of $G - S$ such that $|H| = 2$. Let e be an edge in S such that $e \in D' - E'$. Then, for each edge $e' \in E(V(H), V(e))$, one of the followings holds:*

(i) $e' \in E'$.

(ii) There exists a cutset $S' \in J_1$ such that S' contains e'.

Proof. Suppose that (i) does not hold. Put $e = ab$ and $X = V(S) - V(e)$. In view of Lemma 4.1(i), it follows that $\langle V(H) \cup V(e) \rangle \cong C_4$, $\langle V(H) \cup X \rangle \cong K_2 + (k-2)K_1$, and $E(V(e), X) = \emptyset$. Take $e' \in E(V(H), V(e))$. We may assume that $e' = va$ where $v \in V(H)$. By the assumption that $e' \notin E'$, there exists a cutset $S' \in J_1'$ which contains e'. By the above observation, note that $|E(v, V(G-v) \cap D'| = 1$. In view of Lemma 4.1(i), this forces $S' \in J_1$. Thus (ii) holds. ∎

Lemma 4.4. *Let $e \in D' - E'$, and let S be a cutset in $J_1' - L$ such that S contains e. Then one of the followings holds:*

(i) There exists a subgraph Z in G such that $e \in E(Z)$ and Z is a good C_4. Moreover, $Z \in \mathcal{F}$ holds.

(ii) There exists a k-cutset $S' \in J_1$ such that $S' \cap V(e) \neq \emptyset$.

Proof. Suppose that (ii) does not holds. Since $S \in J_1' - L$, there exists a component H in $G - S$ such that $|H| \leq 2$. By Lemma 4.1, $|H| = 2$. Put $V(H) = \{a, b\}, V(e) = \{c, d\}$ and $X = V(S) - V(e)$. Again, in view of Lemma 4.1(i), we may assume that $bc, da \in D'$, $\langle\{a, b, c, d\}\rangle \cong C_4$ and $\langle V(H) \cup X\rangle \cong K_2 + (k-2)K_1$ and $E(V(e), X) = \emptyset$. In view of Lemma 4.3, $bc, da \in E'$ holds because the assumption (ii) is not assumed to hold. Hence $\langle\{a, b, c, d\}\rangle$ is a good C_4. Note that $E(\{a, b\}, V(G) - \{a, b, c, d\}) \cap D' = \emptyset$ because $\langle V(H) \cup X\rangle \cong K_2 + (k-2)K_1$. This together with $cd, bc, da \in D'$ shows that $\langle\{a, b, c, d\}\rangle$ is a good C_4 which satisfies the property in Lemma 2.1(iv). Hence (i) holds as $Z = \langle\{a, b, c, d\}\rangle$.

∎

Lemma 4.5. *Let S be a cutset in $J_3' - L$ such that S contains a good P_4, and let H be a component of $G - S$ with $|H| = 2$. Put $V(H) = \{u, v\}$. Set $R = abcd$ the good P_4 in S. Suppose that u is a vertex which satisfies the condition in Lemma 4.1(iv), and thus $ua, uc \in E(G)$. Then one of the followings holds:*

(i) $\langle\{a, b, c, u\}\rangle$ is a good C_4 and $\langle\{a, b, c, u\}\rangle \in \mathcal{F}$.

(ii) There exists a cutset $S \in J_1$ such that S contains uc.

Proof. Suppose that (ii) does not hold. First we show that $\langle\{a, b, c, u\}\rangle$ is a good C_4. To show this, we may assume that $uc \notin E'$ because $ab \in E'$. In view of Lemma 4.1(iv), $E(u, V(G)) \cap E' \subset \{ua\}$ and $uc \in D' - E'$. By the assumption that (ii) does not hold, there exists a cutset $S^* \in J_1' - L$ which contains uc and there exists a component H^* of $G - S^*$ with $|H^*| \leq 2$. In view of Lemma 4.1(i), we see that $a \in V(H^*)$ because $E(u, V(G - u)) \cap D' = \{ua, uc\}$. However, then this would imply $|E(a, V(G - a) \cap D'| \geq 2$, and hence this contradicts Lemma 4.1(i). Thus we have $uc \in E'$ and hence $\langle\{a, b, c, u\}\rangle$ is a good C_4. Note that $\langle\{a, b, c, u\}\rangle$ satisfies the condition Lemma 2.1(v) as $Z = \langle\{a, b, c, u\}\rangle$ and $v = u$. Hence $\langle\{a, b, c, u\}\rangle \in \mathcal{F}$, and thus (i) holds.

∎

Lemma 4.6. *Let $S \in J_3' - L$ and let H be a component of $G - S$ such that $|H| = 2$. Set R the good $K_1 + (K_1 \cup K_2)$ in S with $V(R) = \{a, b, c, d\}$ and $E(R) = \{ab, bc, cd, bd\}$. Then one of the following holds:*

(i) There exists a cutset $S \in J_1$ such that S contains an edge $e \in E(c, V(H))$.

(ii) There exists a good element R' with $\{a, b, c\} \subset V(R')$ such that R' is a good C_4 and $R' \in \mathcal{F}$.

(By the symmetry of the roles of c and d, the above statements also hold if we replace c by d.)

Proof. Suppose that (i) does not hold. Put $V(H) = \{u, v\}$ and $X = V(S - R)$. In view of Lemma 4.1(iii), we have $E(\{a, c, d\}, X) = \emptyset$ and we may assume that $E(V(H), V(R)) = \{ua, ud, va, vc\}$ and $\langle V(H) \cup X\rangle \cong K_2 + (k-3)K_1$. Observe that $E(v, V(G - v)) \cap D' = \{vc\}$. Since we assumed that (i) does not holds, this together with Lemma 4.1(i) implies $vc \in E'$. Then $\langle\{a, b, c, v\}\rangle$ is a good C_4. Let $R' = \langle\{a, b, c, v\}\rangle$.

We claim that $R' \in \mathcal{F}$. Suppose not. Then there exists a cutset $S' \in J'_3 - L$ such that $S' \supset R'$ and there exists a component H' with $|H'| \leq 2$. In view of Lemma 4.1(ii), it follows that $|H'| = 2, \langle V(H') \cup V(S' - R') \rangle \cong K_2 + (k-3)K_1$ and $\langle V(H') \cup V(R') \rangle$ is a 3-regular 3-connected graph. Put $V(H') = \{u', v'\}$. Again, let us observe $E(v, V(G - v)) \cap D' = \{vc\}$. Then we can assume that $E(V(H'), V(R')) = \{u'v, u'a, v'b, v'c\}$. Observe that u is an only vertex which is adjacent to both a and u. This means that $u' = u$. Note that H' is contained in many triangles, which implies $v' \neq d$ because $E(u, V(G - u)) \cap D' = \{ud\}$. Since $N_G(u) = \{a, d, v\} \cup X$, this forces $v' \in X$. However, since $E(c, X) = \emptyset$, this is a contradiction. Thus $R' \in \mathcal{F}$ holds as claimed. Hence (ii) holds.

∎

We now prove our main result in this section, which says that $K \neq \emptyset$ in G.

Lemma 4.7. *$K \neq \emptyset$ in G.*

Proof. For a contradiction, assume $K = \emptyset$. Suppose that G contains a triangle which is contained in a k-cutset. Then $J_2 \neq \emptyset$ holds, and thus $K \neq \emptyset$ holds. Hence we may assume that no triangles in G are contained in any k-cutset. If there exists a K_4^-, then it is a good K_4^- because no triangles in K_4^- are contained in any k-cutset. This means $J_3 \neq \emptyset$, and hence $K \neq \emptyset$. Hence we may assume that there is no K_4^- in G. Assume for a while that there exists a triangle T. If T is a good triangle, then it follows that $J_3 \neq \emptyset$, and hence $K \neq \emptyset$. So we may assume that T is not a good triangle. Then there exists an edge e such that $e \in D'$ and $|V(e) \cap V(T)| = 1$. Suppose that $e \notin E'$. We may assume that there exists a k-cutset S with $S \notin J_1$ such that S contains e. Then in view of Lemma 4.4, $J_3 \neq \emptyset$ or $J_1 \neq \emptyset$ holds (according as (i) or (ii) in Lemma 4.4). Hence we may assume that $e \in E'$. Then $\langle V(e) \cup V(T) \rangle$ is a good $K_1 + (K_1 \cup K_2)$. We may assume that $\langle V(e) \cup V(T) \rangle \notin \mathcal{F}$. Then, in view of Lemmas 4.1 and 4.6, $J_1 \neq \emptyset$ or $J_3 \neq \emptyset$ holds (according as (i) or (ii) in Lemma 4.6).

Thus we may assume that there is no triangle in G. Take an edge $e \in E(G)$. If $e \notin E'$, then in view of Lemma 4.4, $J_3 \neq \emptyset$ or $J_1 \neq \emptyset$ holds (according as (i) or (ii) in Lemma 4.4). Thus we may assume that $E(G) \subset E'$. Take a path P of order 4. Then P is a good P_4. If $P \in \mathcal{F}$, then $J_3 \neq \emptyset$ holds and hence $K \neq \emptyset$. So we may assume that $P \notin \mathcal{F}$. Then in view of Lemma 4.5, we see that $J_3 \neq \emptyset$ or $J_1 \neq \emptyset$ holds (according as (i) or (ii) in Lemma 4.5). Thus $K \neq \emptyset$ holds.

∎

Let $Q \in K$, and let H be a component of $G - Q$. Throughout the rest of the proof, we assume that

$$Q \text{ and } H \text{ are chosen so that } |H| \text{ is minimum.}$$

5 Key lemmas

We now get our main proof started. In this section, we investigate the properties on $E(H)$. Also, we show that $|H|$ is large, which means that every component formed by any cutset in K has many vertices. These results will be essential tools for the main part of the proof in the next section.

Lemma 5.1. *Suppose $e = pq$ is an edge in $E' \cap E(H)$ with $E(p, V(H-q)) = \emptyset$. If $E(p, V(G-p)) \cap D' = \{e\}$, then one of the followings holds:*

(i) There exist a cutset S with $S \in J_1 \cup J_2$ such that $V(S) \cap V(H) \neq \emptyset$.

(ii) There exists a good element R such that $V(e) \subset R$ and $R \in \mathcal{F}$.

Proof. By contradiction, suppose that neither (i) nor (ii) holds. Since G is k-connected, note that $|E(p, V(Q))| \geq k-1$ because $E(p, V(H-q)) = \emptyset$. Then there exist two vertices $r, s \in Q$ such that $\langle \{p, r, s\} \rangle \cong K_3$ because $E(p, V(G)) \cap D' = \{e\}$ and $E(p, V(H-q)) = \emptyset$. By the assumption that (i) does not hold, it follows that $\langle \{p, r, s\} \rangle$ is not contained in any k-cutset. In particular, this implies that $\langle \{p, q, r, s\} \rangle$ is a good $K_1 + (K_1 \cup K_2)$. By the assumption that (ii) does not hold, there exist a cutset $Q' \in J_3' - L$ which contains $\langle \{p, q, r, s\} \rangle$ and a component H' of $G - Q'$ with $|H'| \leq 2$. By Lemma 4.1, $|H'| = 2$ and put $V(H') = \{u, v\}$. Again in view of Lemma 4.1(iii), we may assume that $E(\{u, v\}, \{p, q, r, s\}) = \{uq, vq, ur, vs\}$ and $\langle \{u, v, z\} \rangle \cong K_3$ holds for any vertex $z \in Q' - \{p, q, r, s\}$. By the assumption that (i) does not hold, it follows that $\langle \{q, u, v\} \rangle$ is not contained in any k-cutset. Hence $\langle \{p, q, u, v\} \rangle$ is a good $K_1 + (K_1 \cup K_2)$. If $\{u, v\} \cap V(H) \neq \emptyset$, then applying Lemma 4.6 to $\langle \{p, q, u, v\} \rangle$ as $a = p, b = q, \{c, d\} = \{u, v\}$, we obtain either (i) or (ii), a contradiction. Thus we have $\{u, v\} \subset Q$. By the assumption that (ii) does not hold, we have $\langle \{p, q, u, v\} \rangle \notin \mathcal{F}$. Hence there exist a cutset $Q^* \in J_3' - L$ which contains $\langle \{p, q, u, v\} \rangle$ and a component H^* of $G - Q^*$ with $|H^*| \leq 2$. Then, applying Lemma 4.2 to $\langle \{p, q, r, s\} \rangle$ as $a = q, b = p, \{c, d\} = \{r, s\}, H = H'$, we see that $|H^*| = 2$ and $V(H^*) = \{r, s\}$ and moreover,

$$\langle V(H^*) \cup \{z\} \rangle \cong K_3 \text{ holds for any vertex } z \in Q^* - \{p, q, u, v\}. \tag{1}$$

Since $|E(p, V(Q))| \geq k-1$ and $pu, pv \notin E(G)$, we see that $Q \in J_3$ and

$$\text{every vertex in } Q - \{u, v\} \text{ is adjacent to } p. \tag{2}$$

By (1) and (2), we see that $E(\{r, s\}, V(Q) - \{u, v, r, s\}) = \emptyset$ because G does not contain K_4. By the assumption that (ii) does not hold, applying Lemma 4.6 to $\langle \{p, q, r, s\} \rangle$ as $V(H) = \{u, v\}$, we have $ur \notin E'$. Hence we see that $\langle \{r, s, u, v\} \rangle$ does not contain any good element. Since $Q \in J_3$, it follows that $Q - \{r, s, u, v\}$ contains a good element. Since every vertex in $Q - \{r, s, u, v\}$ is adjacent to p, it follows from the definitions of good elements that $Q - \{r, s, u, v\}$ can not contain any good element. This is a contradiction. ∎

We prove the following key lemma, which will be further improved in the later argument.

Lemma 5.2. *Suppose $e = a'b'$ is an edge in $E(H)$. Then one of the followings holds:*

(i) There exist a cutset S with $S \in J_1 \cup J_2$ such that $V(S) \cap V(e) \neq \emptyset$.

(ii) There exists a good element R such that $V(e) \subset R$ and $R \in \mathcal{F}$.

Proof. If there exists a triangle T such that $V(e) \cap V(T) \neq \emptyset$ and T is contained in a k-cutset S, then $S \in J_2$ and (i) holds. Hence we may assume that for every triangle T in G,

$$\text{if } V(e) \cap V(T) \neq \emptyset, \text{ then } T \text{ is not contained in any } k\text{-cutset.} \tag{3}$$

We divide the proof into two cases.

Case 1: e is contained in a triangle T.

If there exists a subgraph R such that $R \cong K_4^-$ and R contains e, then by (3), R is a good K_4^- and $R \in \mathcal{F}$ holds by Lemma 2.1(i). Then (ii) holds. So we may assume that

$$T \text{ is not contained in any } K_4^- \text{ in } G. \tag{4}$$

If T is a good triangle, then, in view of Lemma 2.1, (ii) holds. Hence we may assume that T is not a good triangle. Put $T = a'b'c^*a'$. By the definition of a good triangle, we may assume that there exists a vertex $d' \in V(G - T - c^*)$ such that $b'd' \in D'$. Suppose that $b'd' \in D' - E'$. Then, in view of Lemma 4.3, putting $e = b'd'$, we get (i), or $E(b', V(G - b')) \cap E' \neq \emptyset$. So we may assume that $b'd' \in E'$. Note that $\langle \{a', b', c^*, d'\} \rangle$ is a good $K_1 + (K_1 \cup K_2)$. If $\langle \{a', b', c^*, d'\} \rangle \in \mathcal{F}$, then (ii) holds as $R = \langle \{a', b', c^*, d'\} \rangle$. So we may assume that there exists a $(k+1)$-cutset S^* such that S^* contains $\langle \{a', b', c^*, d'\} \rangle$ and there exists a component H^* of $G - S^*$ with $|H^*| \leq 2$. By Lemma 4.1, $|H^*| = 2$. Applying Lemma 4.6 to R as $a = d', b = b', c = a', d = c^*$, we obtain one of the followings:

(a) There exists a cutset $S \in J_1$ such that S contains an edge e^* with $a' \in V(e^*)$.

(b) There exists a good element R' with $\{d', b', a'\} \subset V(R')$ such that R' is a good C_4 and $R' \in \mathcal{F}$.

If (a) holds, then (i) holds, and if (b) holds, then (ii) holds. This completes the proof of Case 1.

Case 2: e is not contained in any triangle.

If $e \in D' - E'$, then in view of Lemma 4.4, (i) or (ii) holds.

Hence we may assume that $e \in E'$. First suppose that there exists a triangle T such that $V(T) \cap V(e) \neq \emptyset$ and $|V(T) \cap V(H)| \geq 2$. By the definition of E', we see that $|V(T) \cap V(e)| = 1$. By symmetry, we may assume that $b' \in V(T) \cap V(e)$ and take $c' \in (V(T) - V(e)) \cap V(H)$. Put $R = \langle V(e) \cup V(T) \rangle$. By (3), we see that R is a good $K_1 + (K_1 \cup K_2)$. Then applying Lemma 4.6 to R as $a = a', b = b', c = c'$ (and d is the rest of the vertex in T), we obtain one of the followings:

(a) There exists a cutset $S \in J_1$ such that S contains an edge e^* with $c' \in V(e^*)$.

(b) There exists a good element R' with $\{a', b', c'\} \subset V(R')$ such that R' is a good C_4 and $R' \in \mathcal{F}$.

If (a) holds, then (i) holds, and if (b) holds, then (ii) holds.

Thus we may assume that

$$\text{for any edge } e' \in E(V(e), V(H) - V(e)),\ e' \in D'. \tag{5}$$

Suppose that there exists an edge $e' \in E(V(e), V(H) - V(e))$ with $e' \notin E'$. Then there exists a k-cutset S which contains e'. If $S \in J_1$, then (i) holds. So we may assume that $S \in J_1' - L$. In view of Lemma 4.4, we may assume that there exists a subgraph Z such that $e' \in E(Z)$, Z is a good C_4 and $Z \in \mathcal{F}$ (since otherwise (i) holds). If Z also contains e, then (ii) holds. Hence we may assume that Z does not contain e. By the assumption that $e' \notin E'$, this implies that there exists an edge $e^* \in E'$ with $e \neq e^*$ such that $V(e^*) \cap V(e') \cap V(e) \neq \emptyset$. We may assume that for any vertex $y \in V(e)$,

$$\text{if } E(y, V(H) - V(e)) \neq \emptyset, \text{ then } E(y, V(G) - V(e)) \cap E' \neq \emptyset. \tag{6}$$

By (6), we may assume that $E(b', V(G - b')) \cap E' \neq \emptyset$ because H is a connected component with $|H| \geq 3$.

Assume for a while that $E(a', V(G) - V(e)) \cap E' = \emptyset$. By (6), this implies $E(a', V(H)) = \{a'b'\}$. Suppose $E(a', V(Q)) \cap (D' - E') \neq \emptyset$. Take an edge $e^* \in E(a', V(Q)) \cap (D' - E')$. We may assume that there exists a cutset $S^* \in J_1' - L$ such that S^* contains e^* (since otherwise (i) holds). Then applying Lemma 4.4 to e^*, it follows that there exists a subgraph Z such that $e^* \in E(Z)$ and Z is a good C_4 in $\mathcal{F}$ (since otherwise (i) holds). Since $E(a', V(G) - V(e)) \cap E' = \emptyset$, this forces $e \in E(Z)$, which means that (ii) holds. Hence we may assume that $E(a', V(G) - V(e)) \cap D' = \emptyset$. Note that this together with $E(a', V(H)) = \{a'b'\}$ satisfies the assumption of Lemma 5.1 as $p = a'$. Hence by Lemma 5.1, we have either (i) or (ii).

Thus we may assume that $E(a', V(G) - V(e)) \cap E' \neq \emptyset$. This together with $E(b', V(G - b')) \cap E' \neq \emptyset$ shows that there exists a path P of order 4 such that $P = ua'b'v$ and P is a good P_4. If $P \in \mathcal{F}$, then (ii) holds. Thus we may assume that there exists a cutset S in $J_3' - L$ such that S contains P. Then, applying Lemmas 4.1(iv) and 4.5 as $b = a', c = b'$, we have either (i) or (ii). This completes the proof.

■

Now we prove that $|H|$ is large. Since $Q \in K$, recall that H has at least three vertices.

Lemma 5.3. $|H| \geq k - 1$.

Proof. Suppose that H contains a subgraph Z with $V(Z) = \{a, b, c, d\}$, $E(Z) = \{ab, ac, ad, bc, bd\}$, that is, H contains K_4^-. Since G does not contain K_4 nor W_4^-, $cd \notin E(G)$ and $N_G(c) \cap N_G(d) = \{a, b\}$. Since G is k-connected, it follows that $|H \cup Q| \geq |N_G(c) \cup N_G(d)| + 2 = |N_G(c)| + |N_G(d)| - |N_G(c) \cap N_G(d)| + 2 \geq 2k$. Hence $|H| \geq 2k - |Q| \geq 2k - (k+1) = k - 1$. Thus, in the following argument, we may assume that H does not contain K_4^-. Suppose that H contains a subgraph Z with $V(Z) = \{a, b, c, d\}, E(Z) = \{ab, bc, cd, da\}$, that is, H contains C_4. Note that $ac, bd \notin E(G)$ because we have assumed that H does not contain K_4^-. Since G does not contain W_4^-, $|E(v, V(Z))| \leq 2$ for any $v \in V(G - Z)$. Since $|Q| \leq k + 1$, this implies that $|E(V(Z), V(H - Z))| = |E(V(Z), V(G - Z))| - |E(V(Z), V(Q))| \geq 4(k-2) - 2(k+1) = 2k - 10$. Since $|E(v, V(Z))| \leq 2$ for any $v \in V(H - Z)$, this means $|H - Z| \geq k - 5$, and hence $|H| \geq k - 1$. Thus we may assume that H does not contain C_4. Suppose that H contains a triangle $T = abca$.

We claim that $E(V(T), V(H - T)) \neq \emptyset$. Since G does not contain K_4, note that for any vertex $v \in V(Q)$, $|E(v, V(T))| \leq 2$. Hence it is easy to check that $E(V(T), V(H - T)) \neq \emptyset$ if $k = 7, 8$. So, suppose that $k \geq 9$. Then, it follows that $|E(V(T), V(H - T))| = |E(V(T), V(G - T))| - |E(V(T), V(Q))| \geq 3(k-2) - 2(k+1) \geq k - 8 > 0$. Thus $E(V(T), V(H - T)) \neq \emptyset$ holds as claimed. We may assume that there exists a vertex $d \in V(H - T)$ such that $da \in E(G)$. By the assumption that H does not contain K_4^-, note that $\langle \{a, b, c, d\} \rangle \cong K_1 + (K_1 \cup K_2)$. Note that each vertex in $V(Q)$ has at most two neighbors in T. If there are two vertices in $V(Q)$ that have at least three neighbors in the graph $\langle \{a, b, c, d\} \rangle$, then clearly we can find either W_4^- or W_5^-. Therefore, arguing similarly as above, we see that $|E(\{a,b,c,d\}, V(H)-\{a,b,c,d\})| \geq 4k-8-2(k+1)-1 = 2k-11$. Since H does not contain C_4 nor K_4^-, note that at most one vertex in $H-\{a,b,c,d\}$ has at least two neighbors in the graph $\{a,b,c,d\}$ and each of other vertices in $H-\{a,b,c,d\}$ has at most one neighbor in the graph $\{a,b,c,d\}$. Hence it follows from $|E(\{a,b,c,d\}, V(H)-\{a,b,c,d\})| \geq 2k-11$ that $|V(H)-\{a,b,c,d\}| \geq 2k-10$ and hence $|H| \geq 2k-6 \geq k-1$. Thus we may assume that H does not contain K_3 nor C_4.

Suppose that H contains a subgraph Z which is isomorphic to C_5. Put $Z = v_1v_2v_3v_4v_5v_1$. By the assumption that H does not contain K_3 nor C_4, note that $\langle V(Z)\rangle \cong C_5$. Since G does not contain W_5^-, note that $|E(v, V(Z))| \leq 3$ for any $v \in V(G-Z)$. Hence it follows that $|E(V(Z), V(H-Z))| = |E(V(Z), V(G-Z))| - |E(V(Z), V(Q))| \geq 5(k-2)-3(k+1) = 2k-13$. This implies that $|H| \geq 2k-13+5 \geq k-1$ because H does not contain K_3 nor C_4. Hence we may assume that H does not contain a cycle of order less than 6. Suppose that H contains a subgraph Z which is isomorphic to P_4. Put $Z = abcd$. Since H does not contain a cycle of order less than 6, note that $\langle V(Z)\rangle \cong P_4$ and $N_G(a) \cap N_G(d) \cap V(H) = \emptyset$. Since G does not contain W_4^-, it is easy to check that $|\{v \in V(G-Z) || E(v, V(Z))| \geq 3\}| \leq 2$ and the equality holds only if there is no vertex $v \in V(G-Z)$ such that $|E(v, V(Z))| = 4$. This implies that $|E(V(Q), V(Z))| \leq 2(k+1)+2 = 2k+4$. Consequently, $|E(V(Z), V(H-Z))| = |E(V(Z), V(G-Z))| - |E(V(Z), V(Q))| \geq 4k-6-(2k+4) = 2k-10$. This implies $|H| \geq 2k-10+4 = 2k-6 \geq k-1$. Thus we may assume that

$$H \text{ does not contain a cycle of order less than 6 nor } P_4. \tag{7}$$

Hence this together with $|H| \geq 3$ means that H is isomorphic to a star $K_{1,t}$ where t is some integer with $t \geq 2$. Put $V(H) = \{a, b_1, \ldots, b_t\}$ and $E(H) = \{ab_i | \ 1 \leq i \leq t\}$. By (7),

$$|E(b_i, V(Q))| = |E(b_i, V(G-b_i)) - \{b_ia\}| \geq k-1 \text{ for } 1 \leq i \leq t. \tag{8}$$

Since $|Q| \leq k+1$, it follows that $|N_G(b_1) \cap N_G(b_2) \cap V(Q)| \geq k-3 > 2$. Since G does not contain W_4^-, this implies $E(a, N_G(b_1) \cap N_G(b_2) \cap V(Q)) = \emptyset$. This means that $|E(a, V(Q)) \leq 4$. Hence we have $|E(a, V(H))| \geq |E(a, V(G-a))| - |E(a, V(Q))| \geq k-4 \geq 3$. Hence $t \geq 3$. Take $b_3 \in V(H)$. Put $B = \{b_1, b_2, \ldots, b_t\}$ and $M = \{v \in V(Q) | \ |E(v, B)| \leq 1\}$. We claim that $|M| \leq 2$. By (8), it follows that $t|Q-M| \geq |E(V(Q)-M, B)| \geq t(k-1) - |E(M, B)| \geq t(k-1) - |M|$, and hence $|Q-M| \geq k-1-|M|/t$. Consequently, $|M| \leq k+1-|Q-M| \leq k+1-(k-1-|M|/t) = 2+|M|/t$ and hence $|M| \leq 2t/(t-1)$. If $t \geq 4$, the claim holds because $|M|$ is an integer. Suppose that $t = 3$ and $|M| = 3$. Since G does not contain W_4^-, arguing similarly as above, we see that $E(a, Q-M) = \emptyset$. Then, $|E(a, V(G-a))| \leq 3 + |E(a, M)| \leq 3+3 = 6$. Since $k \geq 7$, this is a contradiction. Thus $|M| \leq 2$ holds. Then, arguing similarly as above, we see that $|E(a, V(Q))| \leq |M| \leq 2$. Consequently, $|E(a, V(H))| \geq k - |E(a, V(Q))| \geq k-2$. This shows $|H| \geq k-1$.

∎

6 Main proof of Theorem 1

Using results obtained in the previous section, we can easily find a convenient cutset which will lead us to a final contradiction in this argument.

We start with the definition of such a *good cutset*. In view of Lemma 5.3, note that for any cutset $Q' \in K$, $|H'| \geq k-1$ for any component H' of $G - Q'$. For a cutset $Q' \in K$, if Q' satisfies one of the followings, then we call Q' a *good cutset*.

(i) $Q' \in J_1$ and Q' contains an edge $e \in D' - E'$ such that $V(e) \cap H \cap Q' \neq \emptyset$.

(ii) $Q' \in J_2$ and Q' contains an triangle T such that $V(T) \cap H \cap Q' \neq \emptyset$.

(iii) $Q' \in J_3$ and Q' contains a good element Z such that $V(Z) \cap H \cap Q' \neq \emptyset$.

In fact, in the later argument, we will see that every good cutset belongs to J_3, which means that the above definitions (i) and (ii) will be meaningless (see after Lemma 6.4). However, at the moment, we need to consider the possibility that Q' satisfies (i) or (ii).

In the rest of the proof, if there exists a good cutset $Q' \in K$, we use the following notation:

Let H' and W' be components of $G - Q'$ such that $H' \neq \emptyset, W' \neq \emptyset, H' \cap W' = \emptyset$ and $G - Q' = \langle V(H') \cup V(W') \rangle$. Let H_1, H_2 and H_3 denote $H \cap H', H \cap Q'$ and $H \cap W'$, respectively. Also, let W_1, W_2 and W_3 denote $W \cap H', W \cap Q'$ and $W \cap W'$, respectively. Let Q_1, Q_2 and Q_3 denote $Q \cap H', Q \cap Q'$ and $Q \cap W'$, respectively.

In view of Lemma 5.3, note that $|H| \geq k-1, |H'| \geq k-1, |W| \geq k-1, |W'| \geq k-1$. Also, note that $H_2 \neq \emptyset$.

Lemma 6.1. *Suppose that there exists a good cutset $Q' \in K$. Then the following statements hold:*

(i) *If $H_1 \neq \emptyset$ and $H_2 \cup Q_1 \cup Q_2$ is a cutset with $|H_2 \cup Q_1 \cup Q_2| \leq k+1$, then $|H_2 \cup Q_1 \cup Q_2| = k+1$ and $Q' \in J_1 \cup J_2$.*

(ii) *If $H_3 \neq \emptyset$ and $H_2 \cup Q_2 \cup Q_3$ is a cutset with $|H_2 \cup Q_2 \cup Q_3| \leq k+1$, then $|H_2 \cup Q_2 \cup Q_3| = k+1$ and $Q' \in J_1 \cup J_2$.*

Proof. Otherwise, replacing Q by $H_2 \cup Q_1 \cup Q_2$ or $H_2 \cup Q_2 \cup Q_3$ (according as (i) or (ii)), we get a contradiction to the minimality of $|H|$. Note that the good element Z in Q' is in $H_2 \cup Q_2$. ∎

Lemma 6.2. *Suppose that there exists a good cutset $Q' \in K$. Then the following two statements hold:*

(i) *If $H_1 \neq \emptyset$, then $W_3 \neq \emptyset$.*

(ii) *If $H_3 \neq \emptyset$, then $W_1 \neq \emptyset$.*

Proof. To prove (i), suppose that $H_1 \neq \emptyset$ and $W_3 = \emptyset$. Then by the minimality of $|H|$, we see that $|H_1 \cup H_2| \leq |Q_3|$. Since $H_1 \neq \emptyset$, we have $|H_2| < |Q_3|$.

We claim that $|H_2 \cup Q_1 \cup Q_2| \geq k+1$. Suppose that $|H_2 \cup Q_1 \cup Q_2| = k$. Then by Lemma 6.1(i), $H_2 \cup Q_1 \cup Q_2$ is a good cutset. This contradicts the minimality of $|H|$. Hence $|H_2 \cup Q_1 \cup Q_2| \geq k+1$, as claimed. Consequently, $|H_2| + (k+1) - |Q_3| \geq |H_2| + |Q - Q_3| \geq k+1$, and hence $|H_2| \geq |Q_3|$. This is a contradiction. Thus (i) was proved. We can similarly prove (ii). ∎

Lemma 6.3. *Suppose that there exists a good cutset $Q' \in K$. Then $Q, Q' \in J_3$ holds. Further, for any good element C in Q, either $C \cap H' = \emptyset$ or $C \cap W' = \emptyset$ holds.*

Proof. First we shall prove $Q \in J_3$. By contradiction, assume for a while that $Q \in J_1 \cup J_2$. We claim that $H_1 = H_3 = \emptyset$. Suppose that $H_1 \neq \emptyset$. Then by the minimality of $|H|$, it follows from Lemma 6.1 that $|H_2 \cup Q_1 \cup Q_2| \geq k+1$ or $|H_2 \cup Q_1 \cup Q_2| \geq k+2$ according as $Q' \in J_1 \cup J_2$ or $Q' \in J_3$. This implies $|Q_2 \cup Q_3 \cup W_2| \leq k-1$. Hence $W_3 = \emptyset$, which contradicts Lemma 6.2(i). Thus we have $H_1 = \emptyset$. We can similarly obtain $H_3 = \emptyset$ from Lemma 6.2(ii). Thus $H_1 = H_3 = \emptyset$, as claimed. Suppose that $Q' \in J_1 \cup J_2$. Now we have $H_1 = H_3 = \emptyset$ and $|H_2| = |H| \geq k-1$. Suppose that $W_1 \neq \emptyset$. Since $|W_2 \cup Q_2| \leq 2$, we have $|Q_1| \geq k-2$ and hence $|Q_3| \leq 3$. This implies $W_3 = \emptyset$. Then $|W'| = |Q_3| \leq 3 < k-1$. This is a contradiction. Thus $W_1 = \emptyset$. Arguing similarly, we have $W_3 = \emptyset$. Since either $|H| = |H_2| \leq k/2$ or $|W| = |W_2| \leq k/2$ holds, this is a contradiction. Hence we may assume $Q' \in J_3$. Then note that $H_1 = H_3 = \emptyset$ and hence it follows that $|H_2| = |H| \geq k-1$. Since $|Q_1| \leq k/2$ or $|Q_3| \leq k/2$, this together with $|Q_2 \cup W_2| \leq 2$ implies that $W_1 = \emptyset$ or $W_3 = \emptyset$. By symmetry, we may assume $W_1 = \emptyset$. Then by the minimality of $|H|$, we have $|Q_1| \geq k-1$, and hence $|Q_2 \cup Q_3| \leq 1$. Then $|Q_2 \cup Q_3 \cup W_2| \leq 3$, which means $W_3 = \emptyset$. Then by the minimality of $|H|$, $|Q_3| \geq k-1 > 1$. This is a contradiction. Hence $Q \in J_3$ holds.

We shall now prove $Q' \in J_3$. Suppose that $Q' \in J_1 \cup J_2$. Since Q contains a good element, by the symmetry of the roles of H' and W', we may assume that $Q_1 \cup Q_2$ is not contained in any k-cutset.

First suppose that $H_1 \neq \emptyset$. Since $Q_1 \cup Q_2$ is not contained in any k-cutset, we have $|H_2 \cup Q_1 \cup Q_2| \geq k+1$. Suppose that $|H_2 \cup Q_1 \cup Q_2| = k+1$. If $Q_1 \cup Q_2$ contains a good element, then we get a contradiction to the minimality of $|H|$. Hence it follows that $C \cap Q_3 \neq \emptyset$. Note that then $Q_2 \cup Q_3$ is not contained in any k-cutset. Since now $|Q_2 \cup Q_3 \cup W_2| \leq k$, this implies $W_3 = \emptyset$, which contradicts Lemma 6.2(i). Thus we have $|H_2 \cup Q_1 \cup Q_2| \geq k+2$. Hence $|Q_2 \cup Q_3 \cup W_2| < k$. This implies $W_3 = \emptyset$, which contradicts Lemma 6.2(i). Thus we have $H_1 = \emptyset$. Next suppose that $H_3 \neq \emptyset$. By Lemma 6.2(ii), we may assume that $W_1 \neq \emptyset$. Since $Q_1 \cup Q_2$ is not contained in any k-cutset, $|Q_1 \cup Q_2 \cup W_2| \geq k+1$. This forces $|H_2 \cup Q_2 \cup Q_3| = k$, which contradicts Lemma 6.1(ii).

Thus we have $H_1 = H_3 = \emptyset$. Then $|H_2| \geq k-1$, and note that $|Q_2 \cup W_2| \leq 2$. Suppose that $W_1 \neq \emptyset$. Since $Q_1 \cup Q_2$ is not contained in any k-cutset, we have $|W_2 \cup Q_1 \cup Q_2| \geq k+1$. This implies that $|Q_1| \geq k-1$, and hence $|Q_2 \cup Q_3 \cup W_2| \leq 2$. This forces $W_3 = \emptyset$. Then by the minimality of $|H|$, we have $|Q_3| \geq k-1$. Then $|Q| \geq |Q_1| + |Q_3| > k$, a contradiction. Thus we have $W_1 = \emptyset$. Consequently $|Q_1| = |H'| \geq k-1$. This together with $H_1 = H_3 = \emptyset$ shows $|Q_2 \cup Q_3 \cup W_2| \leq 4$. This forces $W_3 = \emptyset$. Then by the minimality of $|H|$, we have $|H_2| \geq k-1$ and $|W_2| \geq k-1$. Since Q' is a k-cutset, this is a contradiction. Hence $Q' \in J_3$ holds.

Finally we shall prove that for any good element C in Q, either $H' \cap C = \emptyset$ or $W' \cap C = \emptyset$ holds. Suppose that $H' \cap C \neq \emptyset$ and $W' \cap C \neq \emptyset$. Note that by the definition of good elements, $Q_1 \cup Q_2$ is not contained in any k-cutset and also $Q_2 \cup Q_3$ is not contained in any k-cutset. Now we show $H_1 = H_3 = \emptyset$. Suppose that $H_1 \neq \emptyset$. Then, in view of Lemma 6.1(ii), note that $|Q_1 \cup Q_2 \cup H_2| \geq k+1$ and the equality holds only if $Q' \in J_1 \cup J_2$. This implies that $|Q_2 \cup Q_3 \cup W_2| \leq k$. Since $Q_2 \cup Q_3$ is not contained in any k-cutset, this forces $W_3 = \emptyset$. However, this contradicts Lemma 6.2(i). Thus $H_1 = \emptyset$, and we can similarly obtain $H_3 = \emptyset$. Then by Lemma 5.3, we have $|H_2| = |H| \geq k-1$. Since either $|Q_1| \leq (k+1)/2$ or $|Q_3| \leq (k+1)/2$ holds, By symmetry, we may assume that $|Q_1| \leq (k+1)/2$. This forces $W_1 = \emptyset$ because $|Q_1 \cup Q_2 \cup W_2| \leq 2 + (k+1)/2 < k$. Then by the minimality of $|H|$, we have $|Q_1| \geq k-1$. This is a contradiction. Thus we have $H' \cap C = \emptyset$ or $W' \cap C = \emptyset$.

∎

Now we are in a position to obtain the following lemma which plays an important role in the rest of the proof.

Lemma 6.4. *Let $e \in E(H)$. Then there exists a good element $C \in \mathcal{F}$ such that C contains e (and hence there exists a good cutset $Q' \in J_3$ which contains C).*

Proof. The assertion immediately follows from Lemmas 5.2 and 6.3.

∎

In the rest of the proof, by Lemmas 6.3 and 6.4, we may assume that $Q \in J_3$ and for any edge $e \in E(H)$, e is contained in a good element $C_e \in \mathcal{F}$ and there exists a cutset $Q'_e \in J_3$ which contains C_e. In the rest of the proof, when the choice of e is not important in the argument and there is no fear of confusion, Q'_e is written as Q' briefly. (This means that in the following lemmas, the assumption that "Suppose that there exists a good cutset $Q' \in K$." is often omitted.) Also, let $C \in \mathcal{F}$ be a good element in Q. Then by Lemma 6.3, C is contained in either $Q_1 \cup Q_2$ or $Q_2 \cup Q_3$. Without loss of generality, we may assume that C is contained in $Q_1 \cup Q_2$. We prove the following lemmas.

Lemma 6.5. *$H_3 = \emptyset$ and $H_1 \neq \emptyset$.*

Proof. First assume $H_3 \neq \emptyset$. By the minimality of H, we have $|H_2 \cup Q_2 \cup Q_3| \geq k + 2$. Then $|Q_1 \cup Q_2 \cup W_2| \leq k$. Since now $Q_1 \cup Q_2 \cup W_2$ contains a k-contractible edge, this implies $W_1 = \emptyset$. However, this contradicts Lemma 6.2(ii). Thus $H_3 = \emptyset$ holds. Next assume that $H_1 = \emptyset$. Since now $H_3 = \emptyset$, $|H_2| \geq k-1$. Then either $|Q_1 \cup Q_2 \cup W_2| < k$ or $|Q_2 \cup Q_3 \cup W_2| < k$ holds, which means either $W_1 = \emptyset$ or $W_3 = \emptyset$ holds. If $W_1 = \emptyset$, then by the minimality of $|H|$, $|Q_1| \geq k - 1$, which implies $W_3 = \emptyset$. Similarly, if $W_3 = \emptyset$, then we can obtain $W_1 = \emptyset$. Consequently, $W_1 = W_3 = \emptyset$ holds. Then $|W| = |W_2| < k - 1$. This contradicts the minimality of $|H|$.

∎

Lemma 6.6. *$|Q_1 \cup Q_2 \cup H_2| = k + 2$ and $|W_2 \cup Q_2 \cup Q_3| = k$.*

Proof. By Lemma 6.5, we have $H_1 \neq \emptyset$ and $H_3 = \emptyset$. Then by Lemma 6.2(i), $W_3 \neq \emptyset$. Hence by the minimality of $|H|$, $|Q_1 \cup Q_2 \cup H_2| \geq k+2$. By the connectivity of G, $|W_2 \cup Q_2 \cup Q_3| \geq k$. But $2k + 2 = \sum_{i=1}^{3} |Q_i| + |W_2| + |Q_2| + |H_2| = |Q_1 \cup Q_2 \cup H_2| + |W_2 \cup Q_2 \cup Q_3|$, hence the equalities hold.

∎

Since $|Q_1 \cup Q_2 \cup Q_3| = k + 1$, by Lemma 6.6, we have $|H_2| = |Q_3| + 1$. Also, since $|H_2| \geq 2$, we have $|Q_3| \geq 1$. Next, we prove the following lemmas.

Lemma 6.7. *$|N(U) \cap H| \geq |U| + 1$ for all nonempty subsets U of $Q - C$.*

Proof. Suppose there exists a nonempty subset U of $Q - C$ with $|N(U) \cap H| \leq |U|$. In this case, $|U| \leq |Q - C| \leq k - 2 < k - 1 \leq |H|$. So, $H - N(U) \neq \emptyset$. Then, $(Q - U) \cup (N(U) \cap H)$ is $(k + 1)$-cutset containing C and separating $H - N(U)$ from $W \cup U$. Since $C \in \mathcal{F}$, note that $(Q - U) \cup (N(U) \cap H) \in J_3$. Then, since $|H - N(U)| < |H|$, this contradicts the minimality of $|H|$.

■

By Lemma 6.7, since $N(Q_3) \cap H \subset H_2$ and $|H_2| = |Q_3| + 1$, we have $N(Q_3) \cap H = H_2$. By Lemmas 6.3- 6.7, we can obtain the following fact:

For each edge $e_i \in E(H)$ with $1 \le i \le |E(H)|$, e_i is contained in a good element C_{e_i} such that $C_{e_i} \in \mathcal{F}$, and moreover, there exists $(k+1)$-cutset Q^i in J_3 such that Q^i contains C_{e_i}. Let H^i, W^i be components in $G - Q^i$ such that $H^i \neq \emptyset, W^i \neq \emptyset$ and $G - Q^i = \langle V(H^i) \cap V(W^i) \rangle$. Let H_1^i, H_2^i and H_3^i denote $H \cap H^i, H \cap Q^i$ and $H \cap W^i$, respectively. Also, let W_1^i, W_2^i and W_3^i denote $W \cap H^i, W \cap Q^i$ and $W \cap W^i$, respectively. Let Q_1^i, Q_2^i and Q_3^i denote $Q \cap H^i, Q \cap Q^i$ and $Q \cap W^i$, respectively. We may assume $C \in Q_1^i \cup Q_2^i$. Then $H_3^i = \emptyset, |H_2^i| = |Q_3^i| + 1$ and $N(Q_3^i) \cap H = H_2^i$.

We prove the following lemma.

Lemma 6.8. *For each j with $2 \le j \le |E(H)|$, if $(N(Q_3^j) \cap H) \cap (\bigcup_{i=1}^{j-1} N(Q_3^i) \cap H) \neq \emptyset$, then $|\bigcup_{i=1}^{j} N(Q_3^i) \cap H| \le |\bigcup_{i=1}^{j} Q_3^i| + 1$.*

Proof. We prove by induction on j. Suppose $j = 2$. Since $N(Q_3^1) \cap H = H_2^1$ and $N(Q_3^2) \cap H = H_2^2$, we may assume that $H_2^1 \cap H_2^2 \neq \emptyset$. If $Q_3^1 \cap Q_3^2 = \emptyset$, then $|(N(Q_3^1) \cup N(Q_3^2)) \cap H| \le |H_2^1 \cup H_2^2| \le |H_2^1| + |H_2^2| - |H_2^1 \cap H_2^2| \le |Q_3^1| + 1 + |Q_3^2| + 1 - 1 = |Q_3^1 \cup Q_3^2| + 1$. If $Q_3^1 \cap Q_3^2 \neq \emptyset$, then by Lemma 6.7, we have $|N(Q_3^1 \cap Q_3^2) \cap H| \ge |Q_3^1 \cap Q_3^2| + 1$, and hence $|(N(Q_3^1) \cup N(Q_3^2)) \cap H| \le |H_2^1| + |H_2^2| - (|Q_3^1 \cap Q_3^2| + 1) = |Q_3^1| + 1 + |Q_3^2| + 1 - (|Q_3^1 \cap Q_3^2| + 1) = |Q_3^1 \cup Q_3^2| + 1$. Thus the result follows.

Assume $j \ge 3$. If $Q_3^j \subset \bigcup_{i=1}^{j-1} Q_3^i$, the result follows by the induction hypothesis. Assume $Q_3^j \not\subset \bigcup_{i=1}^{j-1} Q_3^i$, and let $R = Q_3^j \cap \bigcup_{i=1}^{j-1} Q_3^i$ and $S = (N(Q_3^j) \cap H) \cap (\bigcup_{i=1}^{j-1} N(Q_3^i) \cap H)$. Then $|S| \ge |R| + 1$ by the assumption of the lemma or by Lemma 6.7 according as $R = \emptyset$ or $R \neq \emptyset$. Hence we have $|\bigcup_{i=1}^{j} N(Q_3^i) \cap H| \le |\bigcup_{i=1}^{j-1} Q_3^i| + 1 + |Q_3^j| + 1 - |R| - 1 = |\bigcup_{i=1}^{j} Q_3^i| + 1$.

■

Since H is connected, there exists an edge e_{j+1} joining from $\bigcup_{i=1}^{j} N(Q_3^i) \cap H$ to $H - \bigcup_{i=1}^{j} N(Q_3^i)$ when $H - \bigcup_{i=1}^{j} N(Q_3^i) \neq \emptyset$. Also, e_{j+1} is contained in some good element $C_{e_{j+1}}$ and $C_{e_{j+1}}$ is contained in some $(k+1)$-cutset $Q^{j+1} \in J_3$. Hence we have $\bigcup_{i=1}^{|E(H)|} N(Q_3^i) \cap H = H$ and by repeated applications of Lemma 6.8, we have $|H| \le |\bigcup_{i=1}^{|E(H)|} Q_3^i| + 1 \le |Q - C| + 1 \le k - 2$. Since $|H| \ge k - 1$, this is a contradiction. This completes the proof of Theorem 1.

■

References

[1] M. Kriesell, Contractible subgraphs in 3-connected graphs, *J. Combin. Theory Ser. B* **80** (2000), 32–48.

[2] W. Mader, Endlichkeitssätze für k-kritische graphen, *Math. Ann.* **229** (1977), 143–153.

[3] W. Mader, On k-critically n-connected graphs, in Progress in graph theory, J. A. Bondy and U. S. R. Murty, Editors, (Academic Press, 1984), 389–398.

[4] W. Mader, Disjunkte fragmente in kritisch n-fach zusammenhängende Graphen, *Europ. J. Combinatorics* **6** (1985), 353–359.

[5] W. Mader, Generalizations of critical connectivity of graphs, *Discrete Math.* **72** (1988), 267–283.

[6] W. Mader, On k-con-critically n-connected graphs, *J. Combin. Theory Ser. B* **86** (2002), 296–314.

[7] W. Mader, High connectivity keeping sets in n-connected graphs, *Combinatorica* **24** (2004), 441–458.

[8] W. Mader, High connectivity keeping sets in graphs and digraphs, *Discrete Math.* **302** (2005), 173–187.

[9] W. McCuaig and K. Ota, Contractible triples in 3-connected graphs, *J. Combin. Theory Ser. B* **60** (1994), 308–314.

[10] C. Thomassen, Non-separating cycles in k-connected graphs, *J. Graph Theory* **5** (1981), 351–354.

[11] W. Tutte, How to draw a graph, *Proc. London Math. Soc.* **13** (1963), 743–767.