THE BALANCED DECOMPOSITION NUMBER AND VERTEX CONNECTIVITY

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Abstract. The *balanced decomposition number* $f(G)$ of a graph G was introduced by Fujita and Nakamigawa [*Discr. Appl. Math.*, 156 (2008), pp. 3339-3344]. A *balanced colouring* of a graph G is a colouring of some of the vertices of G with two colours, such that there is the same number of vertices in each colour. Then, $f(G)$ is the minimum integer s with the following property: For any balanced colouring of G, there is a partition $V(G) = V_1 \dot{\cup} \cdots \dot{\cup} V_r$ such that, for every i, V_i induces a connected subgraph, $|V_i| \leq s$, and V_i contains the same number of coloured vertices in each colour. Fujita and Nakamigawa studied the function $f(G)$ for many basic families of graphs, and demonstrated some applications.

In this paper, we shall continue the study of the function $f(G)$. We give a characterisation for non-complete graphs G of order n which are $\lfloor \frac{n}{2} \rfloor$-connected, in view of the balanced decomposition number. We shall prove that a necessary and sufficient condition for such $\lfloor \frac{n}{2} \rfloor$-connected graphs G is $f(G) = 3$. We shall also determine $f(G)$ when G is a complete multipartite graph, and when G is a generalised Θ-graph (i.e., a graph which is a subdivision of a multiple edge). Some applications will also be discussed. Further results about the balanced decomposition number also appear in two subsequent papers of Fujita and Liu.

Key words. Graph decomposition, vertex colouring, k-connected

AMS subject classifications. 05C15, 05C40, 05C70

1. Introduction

In this paper, we only consider finite undirected graphs. For such a graph G, its *vertex set* and *edge set* are denoted by $V(G)$ and $E(G)$ respectively.

Our definitions concerning graphs throughout the paper are fairly standard. For a graph G and $U \subset V(G)$, the subgraph of G induced by U is denoted by $G[U]$. The graph $G-U$ is the subgraph of G induced by $V(G) \setminus U$. We write $G-u$ for $G-\{u\}$. The *open neighbourhood* of U is $N(U) = \{v \in V(G-U) : vu \in E(G)$ for some $u \in U\}$. The set U is a *cut-set* of G if $G-U$ is a disconnected graph. For $k \in \mathbb{N}$, G is a *k-connected* graph if $|V(G)| \geq k+1$, and G has no cut-set of size at most $k-1$. Every non-empty graph is 0-connected. The maximum k for which G is k-connected is the *connectivity* of G, and is denoted by $\kappa(G)$. For $U, W \subset V(G)$ and $U \cap W = \emptyset$, we write $E(U, W)$ for the edges of G which intersect with both U and W. We write $E(u, W)$ for $E(\{u\}, W)$. A *$U-W$ path* of G is a sub-path where one end-vertex is in U, the other is in W, and no other vertex (if any) is in $U \cup W$. Finally, for $u, v \in V(G)$, the *distance* from u to v in G is denoted by $d_G(u, v)$.

We refer the reader to [1] or [4] for any undefined terms.

In [9], Fujita and Nakamigawa introduced the *balanced decomposition number* of a graph. For a graph G with $|V(G)| = n \in \mathbb{N}$, a *balanced colouring* of G is a pair (R, B), where $R, B \subset V(G)$, $R \cap B = \emptyset$, and $0 \leq |R| = |B| \leq \lfloor \frac{n}{2} \rfloor$. We shall refer the vertices of R (B) as the *red (blue) vertices*, those of $V(G) \setminus (R \cup B)$

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the *uncoloured vertices*, and those of $R \cup B$ the *coloured vertices*. A set $U \subset V(G)$ is a *balanced set* if $|U \cap R| = |U \cap B|$, and $G[U]$ is connected. A *balanced decomposition* of G, or of (R, B), is a partition $V(G) = V_1 \dot\cup \cdots \dot\cup V_r$ (for some $r \geq 1$), such that each V_i is a balanced set. We may also write the balanced decomposition as $\mathcal{P} = \{V_1, \ldots, V_r\}$. The *size* of $\mathcal{P}$ is then defined as the maximum of $|V_1|, \ldots, |V_r|$.

Now, observe that, if G is a disconnected graph, then we can take a balanced colouring so that G does not have a balanced decomposition at all, simply by colouring one vertex in one component red, and another vertex in another component blue. So, from now on, we shall only consider balanced decompositions for connected graphs.

Let G be a connected graph on n vertices, and $k \in \mathbb{Z}$, $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$. We define

$$\begin{aligned} f(k, G) \ = \ \min\{s \in \mathbb{N} : & \text{ every balanced colouring } (R, B) \text{ of } G \text{ with } |R| = |B| = k \\ & \text{has a balanced decomposition of size at most } s\}. \end{aligned}$$

Note that $f(k, G) \leq n$, so $f(k, G)$ is well-defined. The *balanced decomposition number* of G is then defined as

$$f(G) = \max\left\{ f(k, G) : 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \right\}.$$

Fujita and Nakamigawa [9] studied the function $f(G)$ in many directions. Among these were the following.

- A graph G with $f(G) = 2$ if and only if G is a complete graph with at least two vertices.
- $f(K_{m,n}) = \lfloor \frac{n-2}{m} \rfloor + 3$, if $K_{m,n}$ is the complete bipartite graph with $2 \leq m \leq n$.
- The result that $f(C_n) = \lfloor \frac{n}{2} \rfloor + 1$, if C_n is the cycle graph on n vertices, and the conjecture that $f(G) \leq \lfloor \frac{n}{2} \rfloor + 1$, if G is a 2-connected graph on n vertices.

In this paper, we shall prove further results in the direction of each of the above.

- In Section 2, we characterise connected graphs G with balanced decomposition number 3.
- In Section 3, we determine the balanced decomposition number of complete multipartite graphs, which extends the complete bipartite graph result.
- In Section 4, we prove an asymptotically tight bound for $f(G)$ when G is a generalised Θ-graph, which is a graph obtained by subdividing a multiple edge.

As we will observe from our results, the balanced decomposition number seems to have a deep relationship with vertex connectivity of graphs. In Section 2, we show that a graph G of order n is $\lfloor \frac{n}{2} \rfloor$-connected if and only if $f(G) = 2$ or 3. This result will point us to a new direction for the study of vertex connectivity in graphs. The problem of finding non-trivial characterisations for a graph to be k-connected has been well studied, with the cases $k = 2$ (Whitney [17]) and $k = 3$ (Tutte [16]) the most well known (see also, e.g., Sections 3.1 and 3.2 of [4]). On the other hand, if a graph G does not have high connectivity, say, G is 2-connected, then $f(G)$ is likely to be large (see the above conjecture for 2-connected graphs). In view of this, we believe that the balanced decomposition number can be a new criterion to measure the connectivity of a graph. In Section 5, we propose a problem about the relationship between the balanced decomposition number and connectivity. Also, some applications are discussed. We shall see that, we can decide whether a graph satisfies $f(G) \in \{2, 3\}$ or not with an algorithm in polynomial time. We shall also have a discussion about the relation of the balanced decomposition number with "non-separating subgraphs".

In two subsequent papers [7, 8], more results about the balanced decomposition number are proved. These include further applications [7], as well as partial results of the above conjecture, in the cases when the graph is a subdivided K_4, and when it is a 2-connected series-parallel graph [8].

2. Graphs G with $f(G) = 3$

We first recall a trivial remark from [9].

Proposition 1 (Remark 2 of [9]). *Let G be a connected graph with at least 2 vertices. Then $f(G) = 2$ if and only if G is a complete graph.*

So it is natural to ask: "Which connected graphs G have $f(G) = 3$?". With $f(G)$ small, we would expect such graphs G to have many edges. So, these graphs may conceivably be highly connected as well.

Before we proceed, we need Theorem 2 below, which is a well known consequence of Menger's theorem (see, for example, [1], Ch. III, Corollary 6, which implies Theorem 2).

Theorem 2. *Let $k \in \mathbb{N}$, and let G be a k-connected graph with $|V(G)| \geq 2k$. Then, for any $U, W \subset V(G)$ with $U \cap W = \emptyset$ and $|U| = |W| = k$, there exist k vertex disjoint $U - W$ paths.*

Now, here is the characterisation result.

Theorem 3. *Let G be a connected graph with $n \geq 3$ vertices and $G \neq K_n$. Then $f(G) = 3$ if and only if G is $\lfloor \frac{n}{2} \rfloor$-connected.*

Proof. Suppose that $f(G) = 3$, and assume for contradiction that G is not $\lfloor \frac{n}{2} \rfloor$-connected. Then, let $C \subset V(G)$ be a cut-set with $|C| = c \leq \lfloor \frac{n}{2} \rfloor - 1$, and let $V(G-C) = P \,\dot\cup\, Q$, where $|P| = p > 0$, $|Q| = q > 0$, and $E(P, Q) = \emptyset$. Assume without loss of generality that $p \geq q$. Then, $p \geq \lceil \frac{1}{2}(n-c) \rceil \geq \lceil \frac{1}{2}(2c+2-c) \rceil = \lceil \frac{c}{2} \rceil + 1$.

- If $q \geq \lfloor \frac{c}{2} \rfloor + 1$, then we take the balanced colouring (R, B) with $|P \cap R| = \lceil \frac{c}{2} \rceil + 1$, $|C \cap R| = \lfloor \frac{c}{2} \rfloor$, $|Q \cap B| = \lfloor \frac{c}{2} \rfloor + 1$, and $|C \cap B| = \lceil \frac{c}{2} \rceil$, so that $|R| = |B| = c + 1 \leq \lfloor \frac{n}{2} \rfloor$.
- If $q \leq \lfloor \frac{c}{2} \rfloor$, then $p = n - q - c \geq 2c + 2 - q - c = c - q + 2$. We take the balanced colouring (R, B) where $|P \cap R| = c - q + 2$, $|C \cap R| = q - 1$, $|Q \cap B| = q$, and $|C \cap B| = c - q + 1$. So again, $|R| = |B| = c + 1 \leq \lfloor \frac{n}{2} \rfloor$. Note that $c - q + 2, c - q + 1 \geq q + 1 > 0$.

In either case, it is easy to check that $f(G) \geq 4$. Indeed, for any such balanced colouring, if we can find a balanced decomposition $\mathcal{P}$ of size at most 3, then we cannot have two coloured vertices of the same colour in any member of $\mathcal{P}$. So, the vertices of $P \cap R$ are in distinct members of $\mathcal{P}$, say, $A_1, \ldots, A_{|P \cap R|}$. Also, for every i, A_i cannot contain a vertex of $Q \cap B$. Otherwise, since $E(P, Q) = \emptyset$, we would have A_i also containing a vertex of C, which is always coloured. So, A_i must contain a vertex of $C \cap B$. Now, distinct A_i and A_j must contain distinct vertices of $C \cap B$, and this is impossible since $|P \cap R| = |C \cap B| + 1$. Hence, G is $\lfloor \frac{n}{2} \rfloor$-connected.

Conversely, suppose that G is $\lfloor \frac{n}{2} \rfloor$-connected. Since $G \neq K_n$, by Proposition 1, we have $f(G) \geq 3$. We shall show, by downward induction on k, that $f(k, G) \leq 3$ for every $k \leq \lfloor \frac{n}{2} \rfloor$, which will suffice.

Firstly, for $k = \lfloor \frac{n}{2} \rfloor$, suppose that we have a balanced colouring (R, B) with $|R| = |B| = \lfloor \frac{n}{2} \rfloor$. Since G is $\lfloor \frac{n}{2} \rfloor$-connected, applying Theorem 2 with $k = \lfloor \frac{n}{2} \rfloor$, $U = R$ and $W = B$ gives $f(\lfloor \frac{n}{2} \rfloor, G) \leq 3$.

Now, suppose that the implication holds for $\ell + 1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, but not for $k = \ell$. That is, ℓ is the maximal integer such that $f(\ell + 1, G) \leq 3$ and $f(\ell, G) > 3$. Let (R, B) be a balanced colouring of G with $|R| = |B| = \ell$, and let $U = V(G) \setminus (R \cup B)$ be the uncoloured vertices of (R, B). Let (R', B') be a balanced colouring of G, where $R' = R \cup \{y'\}$, $B' = B \cup \{x'\}$, and $y', x' \in U$. In other words, to get the sets R' and B', we take vertices $y', x' \in U$, "colour y' red" and append it to R, and, "colour x' blue" and append it to B. By the induction hypothesis, there exists a balanced decomposition $\mathcal{P}$ for (R', B') of size at most 3. We may assume without loss of generality that the structure of $\mathcal{P}$ is as follows. A set of size 2 induces an edge of G with one end-vertex in R', the other in B'; call these sets $\mathcal{P}'_2$. We may assume that $|\mathcal{P}'_2|$ is maximal, so that a set of size 3 induces a path of length 2 in G with one end-vertex in R', the other in B' (so there is no edge of G joining these end-vertices); call these sets $\mathcal{P}'_3$. A set of size 1 is a vertex of $U \setminus \{x', y'\}$.

If y' and x' are in the same balanced set of $\mathcal{P}$, then $\mathcal{P}$ will also be a balanced decomposition of size at most 3 for (R, B). So, we may assume that $x' \in A_r$ and $y' \in A_b$, where $A_r, A_b \in \mathcal{P}$ are distinct, and $A_r \cap R = \{x\}$, $A_b \cap B = \{y\}$, for some $x \in R$ and $y \in B$.

Define $\mathcal{P}_i = \mathcal{P}'_i \setminus \{A_r, A_b\}$ for $i = 2, 3$, and $U_1 = U \setminus \bigcup\{A : A \in \mathcal{P}_3\}$. We have the following observation.

Observation 1. *$V(G - \{x, y\})$ is a disjoint union of members of $\mathcal{P}_2, \mathcal{P}_3$ and U_1.* □

For a set $C \in \mathcal{P}_3$, let $\{c_r\} = C \cap R$, $\{c_b\} = C \cap B$, and $\{c_u\} = C \cap U$.

Now, we construct a subgraph $H \subset G$ containing x and y. Moreover, we shall derive a partition $V(H) = X' \,\dot\cup\, Y' \,\dot\cup\, Z$ which we will need later in the proof. The subgraph H will contain many trees in a specific form which we shall describe first.

Suppose that we have subsets $\mathcal{Q}_2 \subset \mathcal{P}_2$, $\mathcal{Q}_3 \subset \mathcal{P}_3$ and $U' \subset U_1$. Suppose that we also have a red vertex $w \in R \setminus \bigcup\{A : A \in \mathcal{Q}_2 \cup \mathcal{Q}_3\}$. We shall "grow" a tree by successively attaching elements of $\mathcal{Q}_2, \mathcal{Q}_3$ and U' to what we already have. This will be based on two operations. Start with w. If at some stage we have the tree T, then we can form a new tree $T' \supset T$ by doing one of the following.

- If $u \in V(T) \cap R$, then join $v \in U' \setminus V(T)$ to u if $vu \in E(G)$. That is, $T' = T \cup vu$.
- If $u \in V(T) \cap (R \cup U)$, $A \in \mathcal{Q}_2 \cup \mathcal{Q}_3$, $A \not\subset V(T)$, and $\{v\} = A \cap B$, then unite $G[A]$ with T by joining vu, if $vu \in E(G)$. That is, $T' = T \cup G[A] \cup vu$.

We shall call a tree that can be constructed by successively performing these two operations in some order a *red tree*, and such a red vertex w is the *seed* of the red tree. Denote this red tree by R_w. Similarly, switching the roles of red and blue, we call a tree $B_{w'}$ that can be constructed from a blue vertex $w' \in B \setminus \bigcup\{A : A \in \mathcal{Q}_2 \cup \mathcal{Q}_3\}$ as a result a *blue tree*. Examples of these trees are shown in Figure 1. Groups of vertices in the boxes are elements of $\mathcal{P}_2$ and $\mathcal{P}_3$. The uncoloured vertices not inside the boxes are elements of U_1. The vertices w and w' are the seeds. In subsequent diagrams, red, blue and uncoloured vertices will be depicted as in Figure 1.

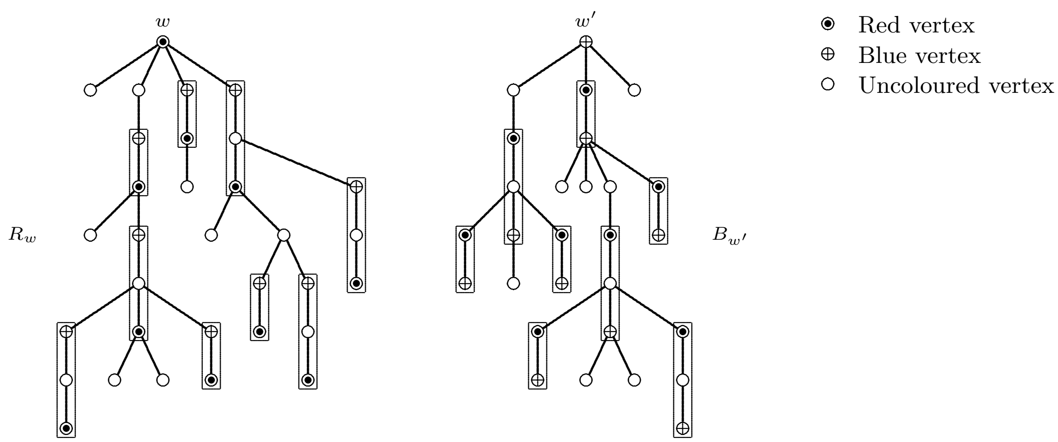

Figure 1. A red tree and a blue tree.

So with this, using $\mathcal{Q}_2 = \mathcal{P}_2$, $\mathcal{Q}_3 = \mathcal{P}_3$ and $U' = U_1$, we can construct a red tree R_x with seed x, and a blue tree B_y with seed y. Choose R_x and B_y such that $V(R_x) \cap V(B_y) = \emptyset$, and $|V(R_x) \cup V(B_y)|$ is maximal. Let $G_0 = R_x \cup B_y$ (so, G_0 is a disconnected graph with $V(G_0) = V(R_x) \cup V(B_y)$).

Now, suppose that we have constructed a subgraph $\tilde{G}$ with $G_0 \subset \tilde{G} \subset G$ satisfying the following properties.

(i) There are families of red trees and blue trees, $\mathcal{R}$ and $\mathcal{B}$, and some $\mathcal{C}' \subset \mathcal{P}_3$ such that the following hold.
- $R_x \in \mathcal{R}$, $B_y \in \mathcal{B}$, and the members of $\mathcal{R} \cup \mathcal{B}$ are vertex disjoint.
- For every $T \in \mathcal{R} \cup \mathcal{B}$ and $C' \in \mathcal{C}'$, we have $C' \notin \{C \in \mathcal{P}_3 : C \subset V(T)\}$.
- For all $C \in \mathcal{C}'$, we have $V(R_x) \cap C = V(B_y) \cap C = \emptyset$.
- If $T \in (\mathcal{R} \cup \mathcal{B}) \setminus \{R_x, B_y\}$ and w is the seed of T, we have $V(T - w) \cap C = \emptyset$ for every $C \in \mathcal{C}'$, and there is a unique $C' \in \mathcal{C}'$ such that $V(T) \cap C' = \{w\}$.
- $V(\tilde{G}) = \bigcup\{V(T) : T \in \mathcal{R} \cup \mathcal{B}\} \cup \bigcup\{C : C \in \mathcal{C}'\}$.

Hence, $V(\tilde{G} - \{x, y\})$ is a union of members of $\mathcal{P}_2, \mathcal{P}_3$ and U_1. Note that, if we have $C \in \mathcal{C}'$ and c_r (c_b) has no red (blue) tree with order at least 2 constructed onto it, then c_r (c_b) may or may not be a red (blue) tree itself, and we must declare such a status for c_r (c_b).

(ii) (Adjacency conditions). If $C \in \mathcal{C}'$, then in $\tilde{G}$, one of the following holds.

(a) c_u sends at least one edge of G to a red vertex in some red tree in $\mathcal{R} \setminus \{R_{c_r}\}$. Moreover, no edge of G joins c_u to any blue vertex of any blue tree in $\mathcal{B}$, and c_b is not the seed of a blue tree.

(b) c_u sends at least one edge of G to a blue vertex in some blue tree in $\mathcal{B} \setminus \{B_{c_b}\}$. Moreover, no edge of G joins c_u to any red vertex of any red tree in $\mathcal{R}$, and c_r is not the seed of a red tree.

(c) c_u sends at least one edge of G to a red vertex in some red tree in $\mathcal{R} \setminus \{R_{c_r}\}$, and to a blue vertex in some blue tree in $\mathcal{B} \setminus \{B_{c_b}\}$, and c_r and c_b are both seed vertices, of a red tree in $\mathcal{R}$ and a blue tree in $\mathcal{B}$, respectively.

We partition $\mathcal{C}' = \tilde{\mathcal{C}}_r \dot\cup \tilde{\mathcal{C}}_b \dot\cup \tilde{\mathcal{C}}$ as follows.

$$\begin{aligned} \tilde{\mathcal{C}}_r &= \{C \in \mathcal{C}' : c_u \text{ satisfies (a) above}\}, \\ \tilde{\mathcal{C}}_b &= \{C \in \mathcal{C}' : c_u \text{ satisfies (b) above}\}, \\ \tilde{\mathcal{C}} &= \{C \in \mathcal{C}' : c_u \text{ satisfies (c) above}\}. \end{aligned}$$

In addition, the following hold.

(d) In $\tilde{G}$, every $C^1 \in \tilde{\mathcal{C}}_r \cup \tilde{\mathcal{C}}$ has the following property. For some $t \geq 1$, there is a sequence of distinct sets $C^1, \ldots, C^t \in \tilde{\mathcal{C}}_r \cup \tilde{\mathcal{C}}$ such that for every i, we have a red tree $R_{c_r^i} \in \mathcal{R}$, and c_u^i sends an edge of G to $V(R_{c_r^{i+1}}) \cap R$, where by convention, $R_{c_r^{t+1}} = R_x$.

(e) A similar statement to (d) holds when we switch the roles of red and blue, x and y, $\mathcal{R}$ and $\mathcal{B}$, and $\tilde{\mathcal{C}}_r$ and $\tilde{\mathcal{C}}_b$.

(iii) (Maximality condition). Every red tree and blue tree of $\tilde{G}$ cannot be "extended" in the following sense: There is no element of $\{C \in \mathcal{P}_2 \cup \mathcal{P}_3 \cup U_1 : C \not\subset V(\tilde{G})\}$ that can be appended to any red tree or blue tree, in accordance to the rules of the construction of the red trees and blue trees.

Note that $\tilde{G} = G_0$ satisfies (i) and (ii) vacuously, and also (iii). Figure 2 shows a possible structure of $\tilde{G}$. Each large box represents a red tree or a blue tree, with the seed vertex at the top of each tree. For the boxed 3-sets, we have $C_1, C_4 \in \tilde{\mathcal{C}}_r$, $C_3, C_7 \in \tilde{\mathcal{C}}_b$, and $C_2, C_5, C_6 \in \tilde{\mathcal{C}}$.

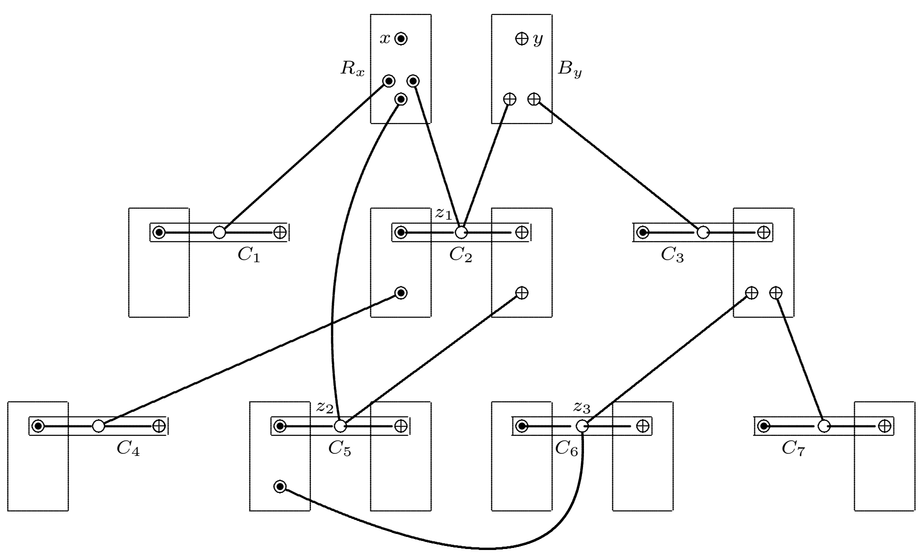

Figure 2. The structure of $\tilde{G}$.

Now, we want to extend $\tilde{G}$ to some subgraph $\hat{G}$ with $\tilde{G} \subset \hat{G} \subset G$. We shall do this with the following algorithm.

Step 1. Let $\tilde{G} \supset G_0$ satisfy properties (i) to (iii), with $\mathcal{R}, \mathcal{B}, \tilde{\mathcal{C}}_r, \tilde{\mathcal{C}}_b$ and $\tilde{\mathcal{C}}$ defined as before. Now, suppose that we have $C \in \mathcal{P}_3$, $C \not\subset V(\tilde{G})$ with c_u sending an edge of G to a red vertex in some red tree in $\mathcal{R}$, or to a blue vertex in some blue tree in $\mathcal{B}$, or both.

- For the first case, append the edges $\{c_u w \in E(G) : w \in V(T) \cap R \text{ for some } T \in \mathcal{R}\}$ to $\tilde{G}$. Using the members of $\mathcal{P}_2 \cup \mathcal{P}_3 \cup U_1$ "available" to us; that is, the family $\{A \in \mathcal{P}_2 \cup \mathcal{P}_3 \cup U_1 : A \not\subset V(\tilde{G}) \cup C\}$, construct a red tree R_{c_r}, while c_b is left alone and is not considered as a blue tree. Choose R_{c_r} so that $|V(R_{c_r})|$ is maximal. Set $\mathcal{R}^0 = \mathcal{R} \cup \{R_{c_r}\}$, $\mathcal{B}^0 = \mathcal{B}$, $\tilde{\mathcal{C}}_r^0 = \tilde{\mathcal{C}}_r \cup \{C\}$, $\tilde{\mathcal{C}}_b^0 = \tilde{\mathcal{C}}_b$ and $\tilde{\mathcal{C}}^0 = \tilde{\mathcal{C}}$.
- Similarly, for the second case, append the edges $\{c_u w \in E(G) : w \in V(T) \cap B \text{ for some } T \in \mathcal{B}\}$ to $\tilde{G}$. Using the members of $\{A \in \mathcal{P}_2 \cup \mathcal{P}_3 \cup U_1 : A \not\subset V(\tilde{G}) \cup C\}$, construct a blue tree B_{c_b}, while c_r is left alone and is not considered as a red tree. Choose B_{c_b} so that $|V(B_{c_b})|$ is maximal. Set $\mathcal{R}^0 = \mathcal{R}$, $\mathcal{B}^0 = \mathcal{B} \cup \{B_{c_b}\}$, $\tilde{\mathcal{C}}_r^0 = \tilde{\mathcal{C}}_r$, $\tilde{\mathcal{C}}_b^0 = \tilde{\mathcal{C}}_b \cup \{C\}$ and $\tilde{\mathcal{C}}^0 = \tilde{\mathcal{C}}$.
- For the last case, append the edges $\{c_u w \in E(G) : w \in V(T) \cap R \text{ for some } T \in \mathcal{R}\}$ and $\{c_u w \in E(G) : w \in V(T) \cap B \text{ for some } T \in \mathcal{B}\}$ to $\tilde{G}$, and note that there is at least one edge of each type. We then construct a red tree R_{c_r} and a blue tree B_{c_b}, again using members of $\{A \in \mathcal{P}_2 \cup \mathcal{P}_3 \cup U_1 : A \not\subset V(\tilde{G}) \cup C\}$, so that $V(R_{c_r}) \cap V(B_{c_b}) = \emptyset$ and $|V(R_{c_r}) \cup V(B_{c_b})|$ is maximal. Set $\mathcal{R}^0 = \mathcal{R} \cup \{R_{c_r}\}$, $\mathcal{B}^0 = \mathcal{B} \cup \{B_{c_b}\}$, $\tilde{\mathcal{C}}_r^0 = \tilde{\mathcal{C}}_r$, $\tilde{\mathcal{C}}_b^0 = \tilde{\mathcal{C}}_b$ and $\tilde{\mathcal{C}}^0 = \tilde{\mathcal{C}} \cup \{C\}$.

In all three cases, let $\tilde{G}^0$ be the new graph obtained. If we cannot perform Step 1, then we set $\hat{G} = \tilde{G}$.

Step 2. Now, suppose that, for some $t \geq 0$, we have found the families $\mathcal{R}^i$, $\mathcal{B}^i$, $\tilde{\mathcal{C}}_r^i$, $\tilde{\mathcal{C}}_b^i$, $\tilde{\mathcal{C}}^i$, with $\tilde{\mathcal{C}}_r^i \cup \tilde{\mathcal{C}}_b^i \cup \tilde{\mathcal{C}}^i = \tilde{\mathcal{C}}_r \cup \tilde{\mathcal{C}}_b \cup \tilde{\mathcal{C}} \cup \{C\}$, and the graphs $\tilde{G}^i$, for $0 \leq i \leq t$, but not for $i = t+1$, with $G_0 \subset \tilde{G} \subset \tilde{G}^0 \subset \cdots \subset \tilde{G}^t$. Suppose that $\tilde{G}^t$ "partially satisfies" properties (i) to (iii) as follows. Instead of properties (ii)(a-c), assume that the following is satisfied instead.

(ii′) In $\tilde{G}^t$, one of the following holds.

(a′) If $C' \in \tilde{\mathcal{C}}_r^t$, then c'_u sends at least one edge of G to some red vertex of some red tree in $\mathcal{R}^t \setminus \{R_{c'_r}\}$, and c_b is not the seed of a blue tree in $\mathcal{B}^t$.

(b′) If $C' \in \tilde{\mathcal{C}}_b^t$, then c'_u sends at least one edge of G to some blue vertex of some blue tree in $\mathcal{B}^t \setminus \{B_{c'_b}\}$, and c_r is not the seed of a red tree in $\mathcal{R}^t$.

(c′) If $C' \in \tilde{\mathcal{C}}^t$, then c'_u sends at least one edge of G to a red vertex in some red tree in $\mathcal{R}^t \setminus \{R_{c'_r}\}$, and to a blue vertex in some blue tree in $\mathcal{B}^t \setminus \{B_{c'_b}\}$, and c'_r and c'_b are both seed vertices, of a red tree in $\mathcal{R}^t$ and a blue tree in $\mathcal{B}^t$, respectively.

As for properties (i) and (iii), they are all satisfied with $\tilde{G}^t$, $\mathcal{R}^t$, $\mathcal{B}^t$, $\tilde{\mathcal{C}}_r^t$, $\tilde{\mathcal{C}}_b^t$ and $\tilde{\mathcal{C}}^t$ in place of $\tilde{G}$, $\mathcal{R}$, $\mathcal{B}$, $\tilde{\mathcal{C}}_r$, $\tilde{\mathcal{C}}_b$ and $\tilde{\mathcal{C}}$. Let (i′) and (iii′) be the modified properties. Note that of course $\tilde{G}^0$ satisfies the properties (i′) to (iii′).

Now, suppose that we have $E(V(R_{d_r}) \cap R, d'_u) \neq \emptyset$ for some $D \in \tilde{\mathcal{C}}_r^t \cup \tilde{\mathcal{C}}^t$, some red tree $R_{d_r} \in \mathcal{R}^t$, and some $D' \in \tilde{\mathcal{C}}_b^t$ (If not, then set $t' = t$ and go to Step 3). Add all edges of $E(V(R_{d_r}) \cap R, d'_u)$ to $\tilde{G}^t$. Now, set $\tilde{\mathcal{C}}_r^{t+1} = \tilde{\mathcal{C}}_r^t$, $\tilde{\mathcal{C}}_b^{t+1} = \tilde{\mathcal{C}}_b^t \setminus \{D'\}$ and $\tilde{\mathcal{C}}^{t+1} = \tilde{\mathcal{C}}^t \cup \{D'\}$. Next, construct a new red tree $R_{d'_r}$ with elements from $\{A \in \mathcal{P}_2 \cup \mathcal{P}_3 \cup U_1 : A \not\subset V(\tilde{G}^t)\}$, and with $|V(R_{d'_r})|$ maximal. Set $\mathcal{R}^{t+1} = \mathcal{R}^t \cup \{R_{d'_r}\}$ and $\mathcal{B}^{t+1} = \mathcal{B}^t$. Let $\tilde{G}^{t+1} \supset \tilde{G}^t$ be the new graph obtained. Note that indeed, we have $\tilde{\mathcal{C}}_r^0 = \cdots = \tilde{\mathcal{C}}_r^{t+1}$. It is easy to check that $\tilde{G}^{t+1}$ also satisfies the properties (i′) to (iii′).

Repeat this procedure successively, starting with $\tilde{G}^0$. Note that the construction of new red trees enhances this possibility. For some $t' \geq 0$, we must have $E(V(S) \cap R, d_u) = \emptyset$ for all red trees $S \in \mathcal{R}^{t'}$ (including $S = R_x$) and $D \in \tilde{\mathcal{C}}_b^{t'}$. This is because in each step, we are moving a set from some $\tilde{\mathcal{C}}_b^t$ to some $\tilde{\mathcal{C}}^t$. These sets are not moved again, so this procedure cannot last forever.

Step 3. Next, we do a similar thing as Step 2, switching the roles of red and blue. We now have a graph $\tilde{G}^{t'}$. Suppose that for some $t \geq t' \geq 0$, we have found $\mathcal{R}^i$, $\mathcal{B}^i$, $\tilde{\mathcal{C}}_r^i$, $\tilde{\mathcal{C}}_b^i$, $\tilde{\mathcal{C}}^i$ and $\tilde{G}^i$ for $0 \leq i \leq t$, but not for $i = t+1$, with $G_0 \subset \tilde{G} \subset \tilde{G}^0 \subset \cdots \subset \tilde{G}^{t'} \subset \cdots \subset \tilde{G}^t$. Moreover, suppose that $\tilde{G}^t$ satisfies the properties (i′) to (iii′). Now, suppose that $E(V(B_{d_b}) \cap B, d'_u) \neq \emptyset$ for some blue tree $B_{d_b} \in \mathcal{B}^t$, $D \in \tilde{\mathcal{C}}_b^t \cup \tilde{\mathcal{C}}^t$, and $D' \in \tilde{\mathcal{C}}_r^t$. Carry out Step 2 in an analogous manner, switching the roles of red and blue, and $\tilde{\mathcal{C}}_r^t$ and $\tilde{\mathcal{C}}_b^t$. We obtain $\mathcal{R}^{t+1}$, $\mathcal{B}^{t+1}$, $\tilde{\mathcal{C}}_r^{t+1}$, $\tilde{\mathcal{C}}_b^{t+1}$, $\tilde{\mathcal{C}}^{t+1}$ and $\tilde{G}^{t+1} \supset \tilde{G}^t$, again satisfying the properties (i′) to (iii′). Note that at this stage, it is not possible to apply Step 2, since we do not construct any new red trees in Step 3, and indeed, we have $\tilde{\mathcal{C}}_b^{t'} = \cdots = \tilde{\mathcal{C}}_b^{t+1}$. So, starting with $\tilde{G}^{t'}$, repeat Step 3 successively until it stops. For some $t'' \geq t' \geq 0$, we obtain $\mathcal{R}^{t''}$, $\mathcal{B}^{t''}$, $\tilde{\mathcal{C}}_r^{t''}$, $\tilde{\mathcal{C}}_b^{t''}$, $\tilde{\mathcal{C}}^{t''}$ and $\tilde{G}^{t''}$ such that, $E(V(S) \cap R, d_u) = \emptyset$ for all red trees $S \in \mathcal{R}^{t''}$ and $D \in \tilde{\mathcal{C}}_b^{t''}$; $E(V(T) \cap B, d'_u) = \emptyset$ for all blue trees $T \in \mathcal{B}^{t''}$ and $D' \in \tilde{\mathcal{C}}_r^{t''}$. So now, the graph $\tilde{G}^{t''}$ satisfies the properties (i) to (iii) that the graph $\tilde{G}$ at the beginning satisfied. Now, we set $\hat{G} = \tilde{G}^{t''}$.

Roughly speaking, in Step 1 we attempt to "extend" a graph $\tilde{G}$ that we already have to a new graph $\tilde{G}^0$, by adding a permissible element of $\mathcal{P}_3$ and constructing new red and blue trees. In Steps 2 and 3, we "tidy up" the new graph $\tilde{G}^0$ by moving vertices and constructing more red and blue trees, so that we end up with $\hat{G}$ having a similar structure to the previous graph $\tilde{G}$.

Now, starting with G_0, with $\tilde{\mathcal{C}}_r = \tilde{\mathcal{C}}_b = \tilde{\mathcal{C}} = \emptyset$, run the above algorithm successively, replacing $\hat{G}$ by $\tilde{G}$ each time we move from Step 3 back to Step 1. This process must terminate at some stage, because we are using more and more elements of $\mathcal{P}_3$ every time we apply Step 1. When we cannot apply Step 1, let H be the final state of the graph $\tilde{G}$, and let $\mathcal{C}_r, \mathcal{C}_b$ and $\mathcal{C}$ be the final states of $\tilde{\mathcal{C}}_r, \tilde{\mathcal{C}}_b$ and $\tilde{\mathcal{C}}$. We have now constructed a graph H with $G_0 \subset H \subset G$. Moreover, H satisfies properties (i) to (iii) (with H in place of $\tilde{G}$).

Now, let X' (Y') be the union of the vertex sets of the red (blue) trees in H with $\bigcup\{C : C \in \mathcal{C}_r\}$ ($\bigcup\{C : C \in \mathcal{C}_b\}$), and $Z = \bigcup\{\{c_u\} : C \in \mathcal{C}\}$. Note that we have $V(H) = X' \mathbin{\dot\cup} Y' \mathbin{\dot\cup} Z$. Let $W = U \setminus V(H)$, and $W' = (R \cup B) \setminus V(H)$. If $|(X' \cup W) \cap U| \leq |Y' \cap U|$, set $X = X' \cup W$ and $Y = Y' \cup W'$. Otherwise, set $X = X' \cup W \cup W'$ and $Y = Y'$. Note that $V(G) = X \mathbin{\dot\cup} Y \mathbin{\dot\cup} Z$.

Our aim now will be to delete a cut-set of G of size at most $\lfloor \frac{n}{2} \rfloor - 1$, which will be a final contradiction. This cut-set will be $(X \cap (B \cup U)) \mathbin{\dot\cup} (Y \cap R) \mathbin{\dot\cup} Z$ if $|X \cap U| \leq |Y \cap U|$, and $(X \cap B) \mathbin{\dot\cup} (Y \cap (R \cup U)) \mathbin{\dot\cup} Z$ if $|X \cap U| > |Y \cap U|$. We must therefore prove certain non-adjacencies in G.

In order to tackle this, we shall digress and describe a special type of tree which will be crucial to our discussion. For this, we shall forget about G for the moment, as well as R, B and U.

Let $\mathcal{F}_2$ be a family of edges and $\mathcal{F}_3$ be a family of paths of length 2, where each member has one end-vertex coloured red, the other coloured blue, and in the case of a member of $\mathcal{F}_3$, the middle vertex uncoloured. Let F_1 be a set of uncoloured vertices. Also, the members of $\mathcal{F}_2, \mathcal{F}_3$ and F_1 are mutually vertex disjoint, and we may think of each of $\mathcal{F}_2, \mathcal{F}_3$ and F_1 having infinitely many members. Let w (w') be another red (blue) vertex. Let $\tilde{R}, \tilde{B}$ and $\tilde{U}$ denote the red, blue and uncoloured vertices.

We say that a tree T with at least two vertices is *alternating* if T can be constructed as follows.

- Start with the vertex $w \in \tilde{R}$. We will now successively append members of $\mathcal{F}_2 \cup \mathcal{F}_3 \cup F_1$.
- Suppose that at some stage, we have constructed a tree S, and $A \in \mathcal{F}_2 \cup \mathcal{F}_3 \cup F_1 \cup \{w\}$ is the last subgraph appended to S. Then, we can obtain a new tree $S' \supset S$ by doing either one of the following.
  - If we have $u \in V(A) \cap \tilde{R}$, we may join $v \in F_1 \setminus V(S)$ to u. That is, $S' = S \cup vu$.
  - If we have $u \in V(A) \cap (\tilde{R} \cup \tilde{U})$, $C \in \mathcal{F}_2 \cup \mathcal{F}_3$, $C \not\subset S$, and $\{v\} = V(C) \cap \tilde{B}$, we may unite C with S by joining vu. That is, $S' = S \cup C \cup vu$.

  We do this successively, and stop at any point we wish. Let T_r be the final tree obtained, and C_r be the final subgraph appended, where $C_r \in \mathcal{F}_2 \cup \mathcal{F}_3 \cup F_1 \cup \{w\}$.
- Now, repeat the above two steps, starting with the vertex $w' \in \tilde{B}$, switching the roles of red and blue, and that any subgraph that we append is not involved in the construction of T_r. We obtain a similar tree T_b, vertex disjoint from T_r. Let C_b be the final subgraph appended, where $C_b \in \mathcal{F}_2 \cup \mathcal{F}_3 \cup F_1 \cup \{w'\}$. Furthermore, assume that we do not have both $C_r \in F_1$ and $C_b \in F_1$.

- If we have $z \in V(C_r) \cap (\tilde{R} \cup \tilde{U})$ and $z' \in V(C_b) \cap \tilde{B}$, or $z \in V(C_r) \cap \tilde{R}$ and $z' \in V(C_b) \cap \tilde{U}$ (note that such z and z' always exist), then we unite T_r and T_b by joining zz'. Let T be the tree obtained. That is, $T = T_r \cup T_b \cup zz'$.

Also, we call T_r a *red alternating tree*, and the red vertex w is the *seed* of T_r. Similarly, T_b is a *blue alternating tree*, with seed w'. See Figure 3. The subgraphs of order 2 and 3 in boxes are elements of $\mathcal{F}_2$ and $\mathcal{F}_3$. The uncoloured vertices not in the boxes are elements of F_1.

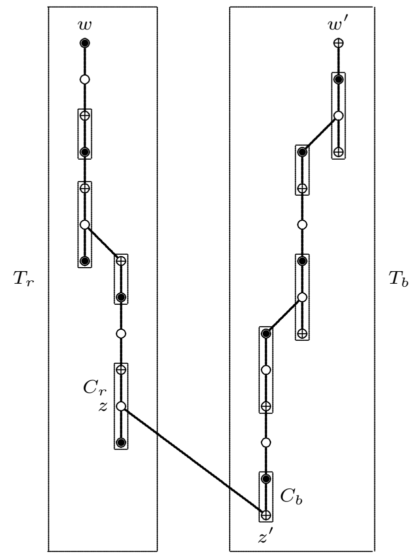

Figure 3. An alternating tree.

We have the following claim.

<u>Claim 2.</u> *$(\tilde{R}, \tilde{B})$ is a balanced colouring for T, and $(\tilde{R}, \tilde{B})$ has a balanced decomposition of size at most* 3.

Proof. Let $\mathcal{F}_2, \mathcal{F}_3, F_1, w, w', T_r, T_b, C_r$ and C_b be defined as before. Obviously, $|\tilde{R}| = |\tilde{B}|$, since $|V(T_r) \cap \tilde{R}| = |V(T_r) \cap \tilde{B}| + 1$ and $|V(T_b) \cap \tilde{B}| = |V(T_b) \cap \tilde{R}| + 1$. We define a linear ordering $\prec_r$ on $V(T_r)$ as follows. We have $a \prec_r a'$ if either $d_{T_r}(w, a) < d_{T_r}(w, a')$, or $d_{T_r}(w, a) = d_{T_r}(w, a')$ and $a \in \tilde{R}$, $a' \in \tilde{B}$. Note that the only way to have $d_{T_r}(w, a) = d_{T_r}(w, a')$ is when we join the uncoloured vertex in some $C \in \mathcal{F}_3$ to a blue vertex in some $D \in \mathcal{F}_2 \cup \mathcal{F}_3$, whence $\{a\} = V(C) \cap \tilde{R}$ and $\{a'\} = V(D) \cap \tilde{B}$.

Now, let $s = |V(T_r) \cap \tilde{R}| - 1 = |V(T_r) \cap \tilde{B}| \geq 0$. Suppose that, starting with w, as we move along $V(T_r)$, with respect to $\prec_r$, the coloured vertices that we come across are, in order, $c_1, \ldots, c_{2s+1}$. We have $c_1 = w$. It is easy to see, for example, by induction on the number of elements of $\mathcal{F}_2 \cup \mathcal{F}_3 \cup F_1 \cup \{w\}$ used in T_r, that $c_1, c_3, \ldots, c_{2s+1} \in \tilde{R}$ and $c_2, c_4, \ldots, c_{2s} \in \tilde{B}$. If $s \geq 1$, then let $K \subset V(T_r)$ be the vertices coming after c_{2s}. We obtain a balanced decomposition for $T_r - K$ as follows. For $1 \leq i \leq s$, let

$$[c_{2i-1}, c_{2i}] = \begin{cases} V(c_{2i-1} \cdots c_{2i}), & \text{if } d_{T_r}(w, c_{2i-1}) < d_{T_r}(w, c_{2i}), \\ \{u, c_{2i-1}, c_{2i}\}, & \text{if } d_{T_r}(w, c_{2i-1}) = d_{T_r}(w, c_{2i}), \end{cases}$$

where $c_{2i-1} \cdots c_{2i}$ is the sub-path of T_r with end-vertices c_{2i-1} and c_{2i} (which has order at most 3), and u is the uncoloured vertex preceding c_{2i-1}. Note that we have $uc_{2i-1}, uc_{2i} \in E(T_r)$. So, we have the following balanced decomposition with size at most 3 for $T_r - K$.

$$V(T_r - K) = \bigcup_{i=1}^{s}[c_{2i-1}, c_{2i}] \cup \bigcup \left\{ \{u'\} : u' \in V(T_r - K) \setminus \bigcup_{i=1}^{s}[c_{2i-1}, c_{2i}] \right\}.$$

Of course, if $s = 0$, then we just ignore this balanced decomposition.

We can carry out a similar procedure on T_b, switching the roles of red and blue. Defining a similar linear ordering $\prec_b$ on $V(T_b)$, we have a similar set of coloured vertices $d_1, \dots, d_{2t+1}$, for some $t \geq 0$, which alternate in colour (starting with blue), and a similar set $L \subset V(T_b)$ containing the vertices coming after $d_{2t} \in \tilde{R}$, with respect to $\prec_b$. Again, we can find a balanced decomposition of $T_b - L$ with size at most 3.

Finally, it is easy to check, by a simple case by case analysis, that $T[K \cup L]$ has a balanced decomposition of size at most 3, using the fact that the edge zz' exists (In fact, there are four possible cases for the structure of K (L), each one containing one red (blue) vertex and at most two uncoloured vertices). We shall not go into details here. □

Now we return to the graph G. Before we prove the claim regarding non-adjacencies in G, we first consider "extracting" a red (blue) alternating tree from a red (blue) tree in $H \subset G$, and introduce a notation. Suppose that R_u is a red tree in H with seed u, and $v \in V(R_u)$. Let $v \in C_1 \in \{u\} \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup U_1$ for some $C_1 \subset R_u$. If $C_1 \neq \{u\}$, then, when C_1 was constructed in R_u, it was appended by being joined to some $C_2 \in \{u\} \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup U_1$, $C_2 \subset R_u$. Repeat this procedure with C_2, and successively. We obtain distinct sets $C_1, \dots, C_t \in \{u\} \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup U_1$ in R_u, for some $t \geq 1$, with $v \in C_1$ and $C_t = \{u\}$. This is indeed the case. To see this, consider the linear ordering $\prec$ on the sets of $\{u\} \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup U_1$ used in R_u, in the order that they appeared in the construction of R_u, and observe that $C_t \prec \cdots \prec C_1$. Now, define

$$\langle R_u, v \rangle = R_u[C_1 \cup \cdots \cup C_t].$$

Similarly, if $B_{u'}$ is a blue tree in H with seed u', and $v' \in V(B_{u'})$, we define $\langle B_{u'}, v' \rangle$ analogously. With this definition, we have the following observation.

Observation 3. *$\langle R_u, v \rangle$ ($\langle B_{u'}, v' \rangle$) is a red (blue) alternating tree with seed u (u'). Moreover, $V(R_u) \setminus \langle R_u, v \rangle$ ($V(B_{u'}) \setminus \langle B_{u'}, v' \rangle$) is a disjoint union of members of $\mathcal{P}_2, \mathcal{P}_3$ and U_1.* □

Now, we are ready to prove the following non-adjacencies claim.

Claim 4.

(a) *If $|(X' \cup W) \cap U| \leq |Y' \cap U|$, so that $X = X' \cup W$ and $Y = Y' \cup W'$, then*

$$E(X \cap R, Y \cap (B \cup U)) = \emptyset.$$

(b) *If $|(X' \cup W) \cap U| > |Y' \cap U|$, so that $X = X' \cup W \cup W'$ and $Y = Y'$, then*

$$E(X \cap (R \cup U), Y \cap B) = \emptyset.$$

Proof. (a) It suffices to prove that

$$E(X' \cap R, (Y' \cap (B \cup U)) \cup (W' \cap B)) = \emptyset. \tag{1}$$

Let $u \in X' \cap R$ and $v \in (Y' \cap (B \cup U)) \cup (W' \cap B)$, and assume that $uv \in E(G)$. We shall prove that the existence of the edge uv either contradicts the maximality of some red tree or blue tree in H (property (iii)), or it implies that (R, B) has a balanced decomposition of size at most 3; hence, $f(\ell, G) \leq 3$.

Now, u is a red vertex in some red tree $R_{c_r^1}$ of H where, with a slight abuse of notation, either $\{c_r^1\} = C^1 \cap R$ for some $C^1 \in \mathcal{C}_r \cup \mathcal{C}$, or $c_r^1 = x$. If $v \in W' \cap B$, then this contradicts the maximality of $R_{c_r^1}$. So let $v \in Y' \cap (B \cup U)$. Then v is a blue or an uncoloured vertex in some blue tree $B_{d_b^1}$ in H, where $\{d_b^1\} = D^1 \cap B$ for some $D^1 \in \mathcal{C}_b \cup \mathcal{C}$, or $d_b^1 = y$. Our aim now will be to find a family $\mathcal{T}$ of vertex disjoint alternating trees such that $V(G) \setminus \bigcup\{V(S) : S \in \mathcal{T}\}$ is a disjoint union of members of $\mathcal{P}_2, \mathcal{P}_3$ and U_1. Then $f(\ell, G) \leq 3$ follows from Claim 2.

If $c_r^1 \neq x$, then in H, by property (ii)(d), for some $s \geq 1$, we have distinct sets $C^1, \dots, C^{s-1} \in \mathcal{C}_r \cup \mathcal{C}$, and vertices $c^2 \in V(R_{c_r^2}) \cap R, \dots, c^s \in V(R_{c_r^s}) \cap R$ such that $c^2 c_u^1, \dots, c^s c_u^{s-1} \in E(G)$, where $c_r^s = x$ and $C^i = \{c_r^i, c_b^i, c_u^i\}$ as usual for $1 \le i \le s-1$. Note that this is all vacuous if $s = 1$, except that $c_r^1 = x$. Similarly, by property (ii)(e), for some $t \ge 1$, we have distinct $D^1, \dots, D^{t-1} \in \mathcal{C}_b \cup \mathcal{C}$, and vertices $d^2 \in V(B_{d_b^2}) \cap B, \dots, d^t \in V(B_{d_b^t}) \cap B$ such that $d^2 d_u^1, \dots, d^t d_u^{t-1} \in E(G)$, where $d_b^t = y$ and $D^j = \{d_r^j, d_b^j, d_u^j\}$ for $1 \le j \le t-1$. This is all vacuous if $t = 1$, except that $d_b^1 = y$.

Now, it may be the case that we have some of $C^1, \dots, C^{s-1}$ coinciding with some of $D^1, \dots, D^{t-1}$. Let $\mathcal{D} = \{C^i : C^i \in \{D^1, \dots, D^{t-1}\}\}$. Note that we have $\mathcal{D} \subset \mathcal{C}$. Now, we claim that we can take $\mathcal{T} = \{T\} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, where

$$\begin{aligned} T &= \langle R_{c_r^1}, u \rangle \cup \langle B_{d_b^1}, v \rangle \cup uv, \\ \mathcal{T}_1 &= \big\{ \langle R_{c_r^i}, c^i \rangle \cup c^i c_u^{i-1} c_b^{i-1} : 2 \le i \le s, C^{i-1} \notin \mathcal{D} \big\}, \\ \mathcal{T}_2 &= \big\{ \langle B_{d_b^j}, d^j \rangle \cup d^j d_u^{j-1} d_r^{j-1} : 2 \le j \le t, D^{j-1} \notin \mathcal{D} \big\}, \\ \mathcal{T}_3 &= \big\{ \langle R_{c_r^p}, c^p \rangle \cup \langle B_{d_b^q}, d^q \rangle \cup c^p c_u^{p-1} d^q : 2 \le p \le s, 2 \le q \le t, C^{p-1} = D^{q-1} \in \mathcal{D} \big\}. \end{aligned}$$

To prove that this choice of $\mathcal{T}$ works, we prove the following claim.

<u>Subclaim 5.</u>

(A) *$\mathcal{T}$ is a family of vertex disjoint alternating trees.*

(B) *We have*

$$\begin{aligned} \bigcup \{V(S) : S \in \mathcal{T}\} &= \{x, y\} \,\dot\cup\, V\big(\langle R_{c_r^1}, u \rangle - c_r^1\big) \,\dot\cup\, V\big(\langle B_{d_b^1}, v \rangle - d_b^1\big) \\ &\quad \dot\cup \bigcup_{i=2}^{s} V\big(\langle R_{c_r^i}, c^i \rangle - c_r^i\big) \,\dot\cup \bigcup_{j=2}^{t} V\big(\langle B_{d_b^j}, d^j \rangle - d_b^j\big) \\ &\quad \dot\cup\, (C^1 \cup \dots \cup C^{s-1} \cup D^1 \cup \dots \cup D^{t-1}). \end{aligned} \tag{2}$$

Hence, $V(G) \setminus \big(\bigcup \{V(S) : S \in \mathcal{T}\} \big)$ is a disjoint union of members of $\mathcal{P}_2, \mathcal{P}_3$ and U_1.

Proof. (A) Clearly, by Observation 3, T and members of $\mathcal{T}_1$ and $\mathcal{T}_2$ are alternating trees. Also, if $S \in \mathcal{T}_3$, then $S = \langle R_{c_r^p}, c^p \rangle \cup \langle B_{d_b^q}, d^q \rangle \cup c^p c_u^{p-1} d^q$ for some $2 \le p \le s, 2 \le q \le t$ with $C^{p-1} = D^{q-1}$. We have $c_u^{p-1} = d_u^{q-1}$, so that $c^p c_u^{p-1}, c_u^{p-1} d^q \in E(H)$. It is clear from this and Observation 3 that S is an alternating tree.

Now, we prove that these alternating trees of $\mathcal{T}$ are vertex disjoint.

- Obviously, T is vertex disjoint from members of $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$.
- Let $S_1 = \langle R_{c_r^i}, c^i \rangle \cup c^i c_u^{i-1} c_b^{i-1} \in \mathcal{T}_1$ and $S_2 = \langle B_{d_b^j}, d^j \rangle \cup d^j d_u^{j-1} d_r^{j-1} \in \mathcal{T}_2$, for some $2 \le i \le s$, $2 \le j \le t$ and $C^{i-1}, D^{j-1} \notin \mathcal{D}$. Obviously, we have $V\big(\langle R_{c_r^i}, c^i \rangle\big) \cap V\big(\langle B_{d_b^j}, d^j \rangle\big) = \emptyset$. Now, $c_u^{i-1} \ne d_u^{j-1}$ since $C^{i-1} \ne D^{j-1}$, and $c_b^{i-1} \ne d_b^j$, since $C^{i-1} \ne D^j$ if $j \le t-1$, or $c_b^{i-1} \ne y$ if $j = t$. It follows that $c_u^{i-1}, c_b^{i-1} \notin V(S_2)$. Similarly, $d_u^{j-1}, d_r^{j-1} \notin V(S_1)$. Hence, $V(S_1) \cap V(S_2) = \emptyset$.
- Let $S_1 = \langle R_{c_r^i}, c^i \rangle \cup c^i c_u^{i-1} c_b^{i-1} \in \mathcal{T}_1$ and $S_3 = \langle R_{c_r^p}, c^p \rangle \cup \langle B_{d_b^q}, d^q \rangle \cup c^p c_u^{p-1} d^q \in \mathcal{T}_3$, for some $2 \le i, p \le s$, $2 \le q \le t$, $C^{i-1} \notin \mathcal{D}$, and $C^{p-1} = D^{q-1} \in \mathcal{D}$. Since $i \ne p$, clearly we have $V\big(\langle R_{c_r^i}, c^i \rangle\big) \cap V\big(\langle R_{c_r^p}, c^p \rangle \cup \langle B_{d_b^q}, d^q \rangle\big) = \emptyset$. Also, $c_u^{i-1} \ne c_u^{p-1}$, and $c_b^{i-1} \ne d_b^q$ as before: either $C^{i-1} \ne D^q$ if $q \le t-1$, or $c_b^{i-1} \ne y$ if $q = t$. It follows that $V(S_1) \cap V(S_3) = \emptyset$.
- Similarly, $V(S_2) \cap V(S_3) = \emptyset$ for every $S_2 \in \mathcal{T}_2$ and $S_3 \in \mathcal{T}_3$.

(B) Note that the last part follows from Observation 1 and (2), since each set on the right hand side of (2), apart from $\{x, y\}$, is a union of members of $\mathcal{P}_2$, $\mathcal{P}_3$ and U_1, and that we have the disjoint union as indicated in (2). So, it remains to prove (2). As usual, any meaningless terms, such as C^s, will be considered non-existent.

We prove that the left hand side of (2) is contained in the right hand side.

- Obviously, $V(T) \subset \{x, y\} \cup V\big(\langle R_{c_r^1}, u \rangle - c_r^1\big) \cup V\big(\langle B_{d_b^1}, v \rangle - d_b^1\big) \cup C^1 \cup D^1$.

- Let $S_1 = \langle R_{c_r^i}, c^i\rangle \cup c^i c_u^{i-1} c_b^{i-1} \in \mathcal{T}_1$, where $2 \le i \le s$ and $C^{i-1} \notin \mathcal{D}$. Then, $V(S_1) \subset \{x\} \cup V\big(\langle R_{c_r^i}, c^i\rangle - c_r^i\big) \cup C^i \cup C^{i-1}$. A similar argument holds if $S_2 \in \mathcal{T}_2$.
- Let $S_3 = \langle R_{c_r^p}, c^p\rangle \cup \langle B_{d_b^q}, d^q\rangle \cup c^p c_u^{p-1} d^q \in \mathcal{T}_3$, where $2 \le p \le s$, $2 \le q \le t$, and $C^{p-1} = D^{q-1}$. Then, $V(S_3) \subset \{x, y\} \cup V\big(\langle R_{c_r^p}, c^p\rangle - c_r^p\big) \cup V\big(\langle B_{d_b^q}, d^q\rangle - d_b^q\big) \cup C^i \cup C^{i-1}$.

Now, we prove the opposite containment.

- $\{c_r^1, d_b^1\} \cup V\big(\langle R_{c_r^1}, u\rangle - c_r^1\big) \cup V\big(\langle B_{d_b^1}, v\rangle - d_b^1\big) \subset V(T)$. In particular, $x \in V(T)$ if $s = 1$, and $y \in V(T)$ if $t = 1$.
- If $s \ge 2$ and $2 \le i \le s$, then $\{c_r^i\} \cup V\big(\langle R_{c_r^i}, c^i\rangle - c_r^i\big) \subset V(S_1)$ for some $S_1 \in \mathcal{T}_1$ if $C^{i-1} \notin \mathcal{D}$; or $\{c_r^i\} \cup V\big(\langle R_{c_r^i}, c^i\rangle - c_r^i\big) \subset V(S_3)$ for some $S_3 \in \mathcal{T}_1$ if $C^{i-1} \in \mathcal{D}$. A similar argument holds for $V\big(\langle B_{d_b^j}, d^j\rangle - d_b^j\big)$ if $t \ge 2$ and $2 \le j \le t$. Note that we have considered the set $\{x, y\}$ here.
- Let $s \ge 2$ and $1 \le i \le s-1$. If $C^i \notin \mathcal{D}$, then $c_u^i, c_b^i \in V(S_1)$ for some $S_1 \in \mathcal{T}_1$. If $C^i \in \mathcal{D}$, then $t \ge 2$, and $C^i = D^j$ for some $1 \le j \le t-1$. We have $c_u^i \in V(S_3)$ for some $S_3 \in \mathcal{T}_3$, and $c_b^i \in V(T)$ if $C^i = D^1$; $c_b^i \in V(S_2)$ for some $S_2 \in \mathcal{T}_2$ if $D^{j-1} \notin \mathcal{D}$, $j \ge 2$ and $t \ge 3$; and $c_b^i \in V(S_3)$ for some $S_3 \in \mathcal{T}_3$ if $D^{j-1} \in \mathcal{D}$, $j \ge 2$ and $t \ge 3$. A similar argument holds for d_u^j and d_r^j if $1 \le j \le t-1$ and $t \ge 2$.

We must also check that the right hand side of (2) is a disjoint union as stated. Indeed,

$$\mathcal{S} = \big\{\langle R_{c_r^1}, u\rangle - c_r^1, \langle B_{d_b^1}, v\rangle - d_b^1\big\} \cup \big\{\langle R_{c_r^i}, c^i\rangle - c_r^i : 2 \le i \le s\big\} \cup \big\{\langle B_{d_b^j}, d^j\rangle - d_b^j : 2 \le j \le t\big\}$$

is a family of graphs, each of which is a red or blue alternating tree in a red or blue tree, with the seed deleted. So for any $F \in \mathcal{S}$, none of $\{x, y\}, C_1, \ldots, C_{s-1}, D_1, \ldots, D_{t-1}$ intersects with F. The members of $\mathcal{S}$ are also vertex disjoint themselves. Finally, x and y do not belong to any of $C_1, \ldots, C_{s-1}, D_1, \ldots, D_{t-1}$.

So, (2) holds. This proves part (B), and the proof of Subclaim 5 is complete. □

Finally, by Claim 2 and Subclaim 5(A), $G\big[\bigcup\{V(S) : S \in \mathcal{T}\}\big]$ has a balanced decomposition of size at most 3. With the last part of Subclaim 5(B), it follows that $f(\ell, G) \le 3$. This proves part (a) of Claim 4.

(b) It suffices to prove that

$$E((X' \cap (R \cup U)) \cup W \cup (W' \cap R), Y' \cap B) = \emptyset.$$

Switching the roles of X' and Y', and of R and B, and with the fact that now $W' \subset X$ (instead of $W' \subset Y$ in (a)), (1) implies that

$$E((X' \cap (R \cup U)) \cup (W' \cap R), Y' \cap B) = \emptyset.$$

So, it remains to prove that $E(W, Y' \cap B) = \emptyset$. Let $u \in W$ and $v \in Y' \cap B$, and assume $uv \in E(G)$. Then u is an uncoloured vertex not used in H, and v is a blue vertex in some blue tree S in H. If $u \in U_1$, then this contradicts the maximality of S. If $u \in A \in \mathcal{P}_3$ for some $A \not\subset V(H)$, then this contradicts the termination of the algorithm when H was constructed.

This completes the proof of Claim 4. □

Now we can easily finish the proof of Theorem 3. If $|(X' \cup W) \cap U| \le |Y' \cap U|$, it follows that $|X \cap U| \le |Y \cap U|$. We delete from G the set $(X \cap (B \cup U)) \,\dot\cup\, (Y \cap R) \,\dot\cup\, Z$. This leaves the sets $X \cap R$ and $Y \cap (B \cup U)$, which, by Claim 4(a), are disconnected. Now, in X, with the exception of x and $|Z|$ other red vertices, the rest of the set has the same number of red and blue vertices, and so, $|X \cap B| = \frac{1}{2}(|X| - |Z| - 1 - |X \cap U|)$.

Similarly, $|Y \cap R| = \frac{1}{2}(|Y| - |Z| - 1 - |Y \cap U|)$. So,

$$\begin{aligned} |(X \cap (B \cup U)) \dot\cup (Y \cap R) \dot\cup Z| &= |X \cap B| + |X \cap U| + |Y \cap R| + |Z| \\ &\le \frac{1}{2}(|X| - |Z| - 1 - |X \cap U|) + |X \cap U| + \\ &\qquad \frac{1}{2}(|Y| - |Z| - 1 - |X \cap U|) + |Z| \\ &= \frac{1}{2}(|X| + |Y|) - 1 \le \frac{n}{2} - 1. \end{aligned}$$

Similarly, if $|(X' \cup W) \cap U| > |Y' \cap U|$, then $|X \cap U| > |Y \cap U|$. We delete from G the set $(X \cap B) \dot\cup (Y \cap (R \cup U)) \dot\cup Z$. This leaves the sets $X \cap (R \cup U)$ and $Y \cap B$, which, by Claim 4(b), are disconnected. Again, we have $|X \cap B| = \frac{1}{2}(|X| - |Z| - 1 - |X \cap U|)$, and $|Y \cap R| = \frac{1}{2}(|Y| - |Z| - 1 - |Y \cap U|)$. So a similar calculation gives

$$|(X \cap B) \dot\cup (Y \cap (R \cup U)) \dot\cup Z| < \frac{n}{2} - 1.$$

In either case, we have deleted from G a cut-set of size at most $\lfloor \frac{n}{2} \rfloor - 1$, which is a final contradiction. This completes the proof of Theorem 3. □

3. Complete Multipartite Graphs

Our next aim is to determine $f(G)$, where G is a complete multipartite graph with at least 3 classes. The case for complete bipartite graphs was solved in [9]. We may easily combine Theorems 1 and 2 of [9] to get the following.

Theorem 4 (Theorems 1 and 2 of [9])**.** *Let $1 \le m \le n$. Then, $f(K_{m,n}) = \lfloor \frac{n-2}{m} \rfloor + 3$.*

It turns out that the complete multipartite graphs version is a simple consequence of Proposition 1 and Theorems 3 and 4. In this section, we denote the complete multipartite graph with class sizes $k_1 \ge \cdots \ge k_t$ by $K_{k_1,\dots,k_t}$, where $t \ge 3$. Also, let V_i be the vertex class with order k_i, for $1 \le i \le t$. Then, we have the following extension to Theorem 4.

Theorem 5. *Let $k_1 \ge \cdots \ge k_t \ge 1$, where $t \ge 3$. Then,*

$$f(K_{k_1,\dots,k_t}) = \left\lfloor \frac{k_1 - 2}{\sum_{i=2}^t k_i} \right\rfloor + 3. \tag{3}$$

Proof. Let $p = |V(K_{k_1,\dots,k_t})| = k_1 + \cdots + k_t$. If $k_1 = 1$, then $K_{k_1,\dots,k_t} \cong K_t$, so Proposition 1 implies that $f(K_{k_1,\dots,k_t}) = 2$, and (3) holds. Now, assume that $k_1 \ge 2$, so that $K_{k_1,\dots,k_t}$ is not a complete graph. We shall consider two cases.

Case 1. $k_1 \le \lceil \frac{p}{2} \rceil$.

In this case, we shall apply Theorem 3. For every $1 \le i \le t$, we have $|V(K_{k_1,\dots,k_t}) \setminus V_i| \ge \lfloor \frac{p}{2} \rfloor$. This implies that $K_{k_1,\dots,k_t}$ is $\lfloor \frac{p}{2} \rfloor$-connected. Indeed, if $C \subset V(K_{k_1,\dots,k_t})$ and $|C| \le \lfloor \frac{p}{2} \rfloor - 1$, then $K_{k_1,\dots,k_t} - C$ must contain vertices from at least two different classes. But, two vertices from different classes are neighbours, and they form a dominating set for $K_{k_1,\dots,k_t}$, so that $K_{k_1,\dots,k_t} - C$ is connected. Hence by Theorem 3, we have $f(K_{k_1,\dots,k_t}) = 3$, and this is easily seen to be consistent with (3), since we have $\sum_{i=2}^t k_i \ge \lfloor \frac{p}{2} \rfloor$, so that $\lfloor (k_1 - 2)/\left(\sum_{i=2}^t k_i\right) \rfloor = 0$.

Case 2. $k_1 \ge \lceil \frac{p}{2} \rceil + 1$.

For this case, we shall use the following simple observation.

Observation 6. *If $H \supset G$ are connected graphs and $V(H) = V(G)$, then $f(H) \le f(G)$* □

We shall apply Theorem 4 and Observation 6 with $n = k_1$ and $m = \sum_{i=2}^t k_i$. Since $K_{k_1,\dots,k_t} \supset K_{m,n}$ and $V(K_{k_1,\dots,k_t}) = V(K_{m,n})$, we have $f(K_{k_1,\dots,k_t}) \le \lfloor \frac{n-2}{m} \rfloor + 3$.

Also, since V_1 is an independent set, it turns out that we can take balanced colourings for $K_{k_1,\ldots,k_t}$ similar to those for $K_{m,n}$ as described in [9] (leading to the lower bound). That is, if $V' = V_2 \cup \cdots \cup V_t$, we take the balanced colouring (R, B) for $K_{k_1,\ldots,k_t}$ where,

- if $(2\ell-2)m + 2 \le n \le (2\ell-1)m + 1$ for some $\ell \in \mathbb{N}$, then $|V_1 \cap R| = |V_1 \cap B| = \lfloor \frac{n}{2} \rfloor$;
- if $(2\ell-1)m + 2 \le n \le 2\ell m + 1$ for some $\ell \in \mathbb{N}$, then $|V' \cap R| = m$, $|V_1 \cap R| = \lfloor \frac{n-m}{2} \rfloor$, and $|V_1 \cap B| = \lfloor \frac{n+m}{2} \rfloor$.

The exact same arguments as in [9] again lead to $f(K_{k_1,\ldots,k_t}) \ge \lfloor \frac{n-2}{m} \rfloor + 3$. Indeed, for the first balanced colouring, any balanced decomposition has size at least

$$2\left\lceil \frac{\lfloor \frac{n}{2} \rfloor}{m} \right\rceil + 1 \ge 2\left\lceil \frac{(\ell-1)m+1}{m} \right\rceil + 1 = 2\ell + 1 = \left\lfloor \frac{n-2}{m} \right\rfloor + 3.$$

For the second balanced colouring, any balanced decomposition has size at least

$$2\left(\left\lceil \frac{\lfloor \frac{n-m}{2} \rfloor}{m} \right\rceil + 1 \right) \ge 2\left\lceil \frac{(\ell-1)m+1}{m} \right\rceil + 2 = 2\ell + 2 = \left\lfloor \frac{n-2}{m} \right\rfloor + 3.$$

So, (3) holds in this case, and we are done. □

4. Generalised Θ-graphs

In this section, we shall study the function $f(G)$, where G is a generalised Θ-graph. That is, G is a graph which is a subdivision of a multiple edge. More precisely, G is the graph union of $t \ge 2$ paths, $Q_1, \ldots, Q_t$ say, with each having the same two end-vertices, x and y say, so that $V(Q_i) \cap V(Q_j) = \{x, y\}$ for any $i \ne j$. In other words, the Q_i are pairwise internally vertex disjoint paths. In addition, all but at most one of the Q_i have order at least 3.

We begin by recalling a result and a conjecture from [9].

Theorem 6 (Theorem 4 of [9]). *Let $n \ge 3$. Then, $f(C_n) = \lfloor \frac{n}{2} \rfloor + 1$, where C_n is the cycle of order n.*

Conjecture 7 (Conjecture 1 of [9]). *Let G be a 2-connected graph of order n. Then $f(G) \le \lfloor \frac{n}{2} \rfloor + 1$.*

So, studying $f(G)$ for generalised Θ-graphs G is related to both Theorem 6 and Conjecture 7, since cycles are generalised Θ-graphs, and generalised Θ-graphs are 2-connected. In Theorem 8, we shall prove that the same upper bound as in Theorem 6 holds for generalised Θ-graphs, and that we have a matching lower bound which is asymptotically tight. This result can be considered as a partial solution to Conjecture 7.

Theorem 8. *Let G be a generalised Θ-graph, formed by uniting the pairwise internally vertex disjoint paths $Q_1, \ldots, Q_t$, where $t \ge 2$, $|V(Q_1)| \ge \cdots \ge |V(Q_t)| \ge 2$, and $|V(Q_{t-1})| \ge 3$. Let $|V(G)| = n \ge t+1$. Then, we have the following.*

(a) *$f(G) \ge \lceil \frac{n}{2} \rceil$, if $|V(Q_1)| \ge \lfloor \frac{n}{2} \rfloor + 1$. $f(G) \ge \lceil \frac{n-t+1}{2} \rceil$ if $|V(Q_i)| \le \lfloor \frac{n}{2} \rfloor$ for every i.*

(b) *$f(G) \le \lfloor \frac{n}{2} \rfloor + 1$.*

In particular, if t is fixed, then $f(G) = \frac{n}{2} + O(1)$.

Proof. Let the common end-vertices of the paths $Q_1, \ldots, Q_t$ be x and y. That is, $\{x, y\} = V(Q_i) \cap V(Q_j)$ for any $i \ne j$.

(a) Let $|V(Q_1)| \ge \lfloor \frac{n}{2} \rfloor + 1$. We take a balanced colouring (R, B) for G where $R = \{v \in V(Q_1) : d_{Q_1}(v, x) \le \lfloor \frac{n}{2} \rfloor - 1\}$, and $B = V(G) \setminus R$ if n is even, $B = V(G - z) \setminus R$ if n is odd, where $z \in V(Q_1)$ is the vertex with $d_{Q_1}(z, x) = \lfloor \frac{n}{2} \rfloor$ (so, z is the only uncoloured vertex). We have $|R| = |B| = \lfloor \frac{n}{2} \rfloor$. Now, any balanced decomposition $\mathcal{P}$ of G can only have at most two sets containing vertices of R. Otherwise, if $\mathcal{P}$ has at least three such sets, then $\mathcal{P}$ must contain a set A such that $A \cap R \subset R \setminus \{x, x'\}$, where $x' \in V(Q_1)$ is the vertex with $d_{Q_1}(x', x) = \lfloor \frac{n}{2} \rfloor - 1$. But then, $G[A \cap R]$ is disconnected from $G[B]$, a contradiction. Now, since there is at most one uncoloured vertex, we have $|\mathcal{P}| \le 3$. If $|\mathcal{P}| = 3$, then n is odd, and $\{z\}$ is a member of $\mathcal{P}$. But then, it is not possible to have a balanced decomposition of $G - z$ into two sets, since a balanced set containing x' must contain all other red vertices. Hence, we have $|\mathcal{P}| \leq 2$, which implies that $f(G) \geq \lceil \frac{n}{2} \rceil$.

Now, let $|V(Q_i)| \leq \lfloor \frac{n}{2} \rfloor$ for every i. Note that we have $t \geq 3$ in this case. We take a balanced colouring (R, B) for G as follows. For every i, let $q_i = |V(Q_i)| - 2$ be the number of internal vertices of Q_i. Let $m = n - 2 = \sum_{i=1}^{t} q_i$ be the total number of internal vertices of all the paths Q_i. We claim the following

Claim 7. *There exists a partition* $[t] = U \dot{\cup} W$, where $U, W \neq \emptyset$, such that $\sum_{i \in U} q_i \geq \frac{m}{3}$ and $\sum_{j \in W} q_j \geq \frac{m}{3}$.

Proof. We have $q_1 \geq \cdots \geq q_t \geq 0$ and $q_{t-1} \geq 1$. We are done if $q_1 \geq \frac{m}{3}$, since then we have $\frac{m}{3} \leq q_1 \leq \lfloor \frac{m}{2} \rfloor - 1$, and we can take $U = \{1\}$ and $W = \{2, \ldots, t\}$. So, assume $q_1 < \frac{m}{3}$. Let $p \geq 2$ be the integer such that $\sum_{i=1}^{p-1} q_i < \frac{m}{3}$ and $\sum_{i=1}^{p} q_i \geq \frac{m}{3}$. We are done if $\sum_{i=1}^{p} q_i \leq \frac{2m}{3}$ (in which case $p < t$), since then we can take $U = \{1, \ldots, p\}$ and $W = \{p+1, \ldots, t\}$. So, assume $\sum_{i=1}^{p} q_i > \frac{2m}{3}$. But then, we have $q_p > \frac{m}{3} > q_1$, a contradiction. □

Now, take a partition $[t] = U \dot{\cup} W$ as given by Claim 7, and assume that $\sum_{i \in U} q_i \leq \sum_{j \in W} q_j$. Take $R = \bigcup_{i \in U} V(Q_i - \{x, y\})$, so that $\frac{n-2}{3} \leq |R| \leq \lfloor \frac{n-2}{2} \rfloor$. Now, we want to take $B \subset \bigcup_{j \in W} V(Q_j - \{x, y\})$ with $|B| = |R|$, spreading the blue vertices "as evenly as possible". More precisely, note that

$$\frac{|B|}{\sum_{j \in W} q_j} \geq \frac{\left(\frac{n-2}{3}\right)}{\left(\frac{2(n-2)}{3}\right)} = \frac{1}{2},$$

so that we can take $B \subset \bigcup_{j \in W} V(Q_j - \{x, y\})$ where no two adjacent vertices in $\bigcup_{j \in W} V(Q_j - \{x, y\})$ are uncoloured. Note that x and y are uncoloured vertices.

Let $\mathcal{P}$ be a balanced decomposition for (R, B). Firstly, suppose that x and y are in the same member C of $\mathcal{P}$. Then, this implies that $C \supset R \cup B$. Otherwise, if we have a vertex $u \in R \setminus C$ ($u \in B \setminus C$), then in $G - C$, u is disconnected from all other blue (red) vertices. Now, note that $|C| \geq n - |W|$, for otherwise $|C| < n - |W|$, then C misses at least two uncoloured vertices in some $Q_j - \{x, y\}$ with $j \in W$, which is not possible by the choice of B, and $G - (R \cup B) \subset \bigcup_{j \in W} V(Q_j - \{x, y\})$. Hence, we have $f(G) \geq |C| \geq n - |W| \geq n - t + 1 \geq \lceil \frac{n-t+1}{2} \rceil$.

Now, suppose that x and y are in different members of $\mathcal{P}$, say, C_x and C_y. We can apply a similar argument to the previous case with $C_x \cup C_y$ in place of C, to get $C_x \cup C_y \supset R \cup B$, and $|C_x \cup C_y| \geq n - t + 1$. Hence, $f(G) \geq \max\{|C_x|, |C_y|\} \geq \lceil \frac{n-t+1}{2} \rceil$.

(b) Let (R, B) be a balanced colouring for G. We shall consider two cases: when $|V(Q_i)| \leq \lfloor \frac{n}{2} \rfloor + 1$ for every i, and when $|V(Q_1)| \geq \lfloor \frac{n}{2} \rfloor + 2$.

Case 1. $|V(Q_i)| \leq \lfloor \frac{n}{2} \rfloor + 1$ *for every* i.

Suppose that we cannot find a suitable balanced decomposition for G. We begin by proving two claims. These will then help us to describe an algorithm which finds a balanced decomposition of G with size at most $\lfloor \frac{n}{2} \rfloor + 1$, and with at most three parts, which will prove Case 1.

Claim 8. *There exists a balanced set* $A \subset V(G)$ *with* $|A| \leq \lfloor \frac{n}{2} \rfloor$ *such that exactly one of* x, y *is in* A.

Proof. If either x or y is uncoloured, then we can simply take $A = \{x\}$ or $A = \{y\}$, whichever is uncoloured. So, assume that both x and y are coloured, and without loss of generality that $x \in R$. Suppose that such a balanced set A cannot be found. Then, for any $1 \leq i \leq t$ and any $x' \in V(Q_i - \{x, y\})$, since $|V(Q_i - y)| \leq \lfloor \frac{n}{2} \rfloor$, the path $xQ_ix' \subset Q_i$ satisfies $|V(xQ_ix') \cap R| > |V(xQ_ix') \cap B|$. So in particular, we have $|V(Q_i - y) \cap R| \geq |V(Q_i - y) \cap B| + 1$ for each i. So,

$$\begin{aligned} |R| &\geq \sum_{i=1}^{t} |V(Q_i - y) \cap R| - (t-1) \geq \sum_{i=1}^{t} \big(|V(Q_i - y) \cap B| + 1\big) - (t-1) \\ &= \sum_{i=1}^{t} |V(Q_i - y) \cap B| + 1 \geq |B|. \end{aligned}$$

Equality holds if and only if $y \in B$ and $|V(Q_i - y) \cap R| = |V(Q_i - y) \cap B| + 1$ for every i. But then $\{V(Q_1), V(Q_2 - \{x, y\}), \ldots, V(Q_t - \{x, y\})\}$ is a balanced decomposition for G, with size at most $\lfloor \frac{n}{2} \rfloor + 1$, which is a contradiction. This proves Claim 8. □

Claim 9. *Let $A \subset V(G-y)$ be a balanced set with $x \in A$, $|A| \leq \lfloor \frac{n}{2} \rfloor - 1$, and $N(A) \setminus \{y\} \subset R$. Then, for some $1 \leq j \leq t$, we can find a non-empty balanced set $C \subset V(Q_j - y) \setminus A$, with $N(C) \cap A \neq \emptyset$, and $|C| \leq \lfloor \frac{n}{2} \rfloor - 1$.*

Proof. Define $I \subset [t]$ by $I = \{j \in [t] : V(Q_j - y) \setminus A \neq \emptyset\}$. Note that $|I| \geq 2$, otherwise, $|A| \geq n - \lfloor \frac{n}{2} \rfloor > \lfloor \frac{n}{2} \rfloor - 1$, a contradiction. If no such balanced set C exists, then for each $j \in I$, since the vertex of $N(A) \cap V(Q_j)$ is red, and $|V(Q_j - \{x, y\})| \leq \lfloor \frac{n}{2} \rfloor - 1$, we have $|(V(Q_j - y) \setminus A) \cap R| \geq |(V(Q_j - y) \setminus A) \cap B| + 1$, by a similar argument as in Claim 8. But then, we have

$$\begin{aligned}|(V(G) \setminus A) \cap R| &\geq \sum_{j \in I} |(V(Q_j - y) \setminus A) \cap R| \geq \sum_{j \in I} |(V(Q_j - y) \setminus A) \cap B| + 2 \\ &> |(V(G) \setminus A) \cap B|.\end{aligned}$$

We have a contradiction, since $V(G) \setminus A$ is a balanced set. Claim 9 follows. □

We remark that with A as in Claim 8 or Claim 9, we have $G - A$ is connected. We can now describe the algorithm.

Step 1. By Claim 8, without loss of generality, we can find a balanced set $A_1 \subset V(G - y)$ with $x \in A_1$, and $|A_1| \leq \lfloor \frac{n}{2} \rfloor$. If $|A_1| = \lfloor \frac{n}{2} \rfloor$, stop; we have a suitable balanced decomposition $\{A_1, V(G) \setminus A_1\}$ for G. Otherwise, $|A_1| \leq \lfloor \frac{n}{2} \rfloor - 1$; go to Step 2.

Step 2. If $N(A_1) \setminus \{y\} \subset R \cup B$, go to Step 3. Otherwise, there exists an uncoloured vertex in $N(A_1) \setminus \{y\}$; append it to A_1. We have another balanced set A_2 with $A_1 \subset A_2 \subset V(G - y)$ and $|A_2| = |A_1| + 1$. If $|A_2| = \lfloor \frac{n}{2} \rfloor$, stop; we have a suitable balanced decomposition $\{A_2, V(G) \setminus A_2\}$ for G. Otherwise, $|A_2| \leq \lfloor \frac{n}{2} \rfloor - 1$; repeat Step 2, using A_2 for A_1.

Step 3. If $N(A_1) \setminus \{y\} \subset R$ or $N(A_1) \setminus \{y\} \subset B$, go to Step 4. Otherwise, there exist a red vertex and a blue vertex in $N(A_1) \setminus \{y\}$; append them to A_1. We have another balanced set A_3 with $A_1 \subset A_3 \subset V(G-y)$ and $|A_3| = |A_1| + 2$. If $|A_3| = \lfloor \frac{n}{2} \rfloor$ or $\lfloor \frac{n}{2} \rfloor + 1$, stop; we have a suitable balanced decomposition $\{A_3, V(G) \setminus A_3\}$ for G. Otherwise, $|A_3| \leq \lfloor \frac{n}{2} \rfloor - 1$; go back to Step 2, using A_3 for A_1.

Step 4. By Claim 9, for some $1 \leq j \leq t$, we can find a non-empty balanced set $C \subset V(Q_j - y) \setminus A_1$, with $N(C) \cap A_1 \neq \emptyset$, and $|C| \leq \lfloor \frac{n}{2} \rfloor - 1$. We have a balanced set $A_1 \cup C \subset V(G - y)$ with $|A_1 \cup C| > |A_1|$. If $|A_1 \cup C| \geq \lfloor \frac{n}{2} \rfloor$, stop; we have a suitable balanced decomposition $\{A_1, C, V(G) \setminus (A_1 \cup C)\}$ for G. Otherwise, $|A_1 \cup C| \leq \lfloor \frac{n}{2} \rfloor - 1$; go back to Step 2, using $A_1 \cup C$ for A_1.

This algorithm must terminate, since as we go through each step, we are creating new balanced sets with strictly increasing orders. When the algorithm terminates, we end up with a balanced decomposition of size at most $\lfloor \frac{n}{2} \rfloor + 1$ for G (and with at most three parts), a final contradiction. This completes the proof of Case 1.

Case 2. $|V(Q_1)| \geq \lfloor \frac{n}{2} \rfloor + 2$.

We shall use a similar idea to the proof in [9] of the upper bound of Theorem 6, that $f(C_n) \leq \lfloor \frac{n}{2} \rfloor + 1$, where C_n is the cycle with n vertices.

We number the vertices of G with $1, \ldots, n$ as follows. The vertices of Q_1 are numbered with $1, \ldots, |V(Q_1)|$, with vertex $v \in V(Q_1)$ receiving the number $d_{Q_1}(v, x) + 1$. Next, the vertices of $Q_2 - \{x, y\}$ are numbered with the next $|V(Q_2)| - 2$ integers, ordered by distance from y. We then repeat this with $Q_3 - \{x, y\}, \ldots, Q_t - \{x, y\}$ successively.

Now, for $i \in [n]$, define $A(i) \subset V(G)$ as follows.

$$A(i) = \begin{cases} \left\{i, i+1, \ldots, i + \left\lfloor \dfrac{n}{2} \right\rfloor - 1\right\} & \text{if } 1 \leq i \leq \left\lceil \dfrac{n}{2} \right\rceil + 1, \\ \{i, i+1, \ldots, n\} \cup \left\{1, \ldots, i - \left\lceil \dfrac{n}{2} \right\rceil - 1\right\} & \text{if } \left\lceil \dfrac{n}{2} \right\rceil + 2 \leq i \leq n. \end{cases}$$

In other words, with the numbering, $A(i)$ is the $\lfloor \frac{n}{2} \rfloor$ consecutive vertices of $V(G)$, starting at i, modulo n. We have the following claim.

Claim 10. *Let $A \subset V(G)$ be a set of consecutive vertices in the numbering, modulo n, with $|A|, |V(G) \setminus A| \geq \lceil \frac{n}{2} \rceil - 1$. Then, both $G[A]$ and $G[V(G) \setminus A]$ are connected.*

Proof. Let y be numbered with j. Since $|V(Q_1)| \geq \lfloor \frac{n}{2} \rfloor + 2$, we have $|\{j+1, \ldots, n\}| \leq n - (\lfloor \frac{n}{2} \rfloor + 2) = \lceil \frac{n}{2} \rceil - 2$. So, neither A nor $V(G) \setminus A$ can be a subset of $\{j+1, \ldots, n\}$, since $|A|, |V(G) \setminus A| \geq \lceil \frac{n}{2} \rceil - 1$. It is then easy to check the following.

- If $|A \cap \{x, y\}| = 0$ or 2, then one of A and $V(G) \setminus A$ is a subset of $V(Q_1 - \{x, y\})$ which induces a sub-path, and the other induces a subgraph of G which is the union of $Q_2, \ldots, Q_t$ with two disjoint sub-paths of Q_1, one containing x, the other containing y.
- If $|A \cap \{x, y\}| = 1$, then A and $V(G) \setminus A$ are both subdivided stars, one with centre x, the other with centre y.

In either case, both $G[A]$ and $G[V(G) \setminus A]$ are connected. So Claim 10 follows. □

We may now complete Case 2 in a similar way to the proof of Theorem 6 in [9]. For $i \in [n]$, define $g(i) = |A(i) \cap R| - |A(i) \cap B|$. For every i, we have $|g(i+1) - g(i)| \leq 2$ (modulo n), and $\sum_{i=1}^{n} g(i) = \lfloor \frac{n}{2} \rfloor (|R| - |B|) = 0$. We have two possibilities.

- Either, there exists i with $g(i) = 0$, whence $\{A(i), V(G) \setminus A(i)\}$ is a suitable balanced decomposition for G, since $|A(i)| = \lfloor \frac{n}{2} \rfloor \geq \lceil \frac{n}{2} \rceil - 1$, and $|V(G) \setminus A(i)| = \lceil \frac{n}{2} \rceil \geq \lceil \frac{n}{2} \rceil - 1$, so that Claim 10 applies.
- Or, there exists i with $\{g(i), g(i+1)\} = \{-1, 1\}$, modulo n. If $g(i) = -1$ and $g(i+1) = 1$, then, with respect to the numbering, modulo n, the first vertex of $A(i)$ is blue, $A(i)$ has one more blue vertex than red, and the first vertex outside of $A(i)$ is red. If w is this red vertex, then $\{A(i) \cup \{w\}, V(G) \setminus (A(i) \cup \{w\})\}$ is a suitable balanced decomposition for G, since $A(i) \cup \{w\}$ and $V(G) \setminus (A(i) \cup \{w\})$ are still consecutive vertices with respect to the numbering, modulo n, and $|A(i) \cup \{w\}| = \lfloor \frac{n}{2} \rfloor + 1 \geq \lceil \frac{n}{2} \rceil - 1$ and $|V(G) \setminus (A(i) \cup \{w\})| = \lceil \frac{n}{2} \rceil - 1$, so that Claim 10 applies. A similar argument holds if $g(i) = 1$ and $g(i+1) = -1$ for some i; we just switch the roles of red and blue.

The proof of Case 2 is now complete, and this proves Theorem 8. □

5. Concluding Remarks

Theorem 3 and Conjecture 7 suggest a more general problem.

Problem 9. *Let $n, t \in \mathbb{N}$ with $n - 1 \geq t \geq 2$. Let G be a graph of order n, with n sufficiently large. Determine a function $g(n, t)$ such that the following holds: If G is $g(n, t)$-connected, then $f(G) \leq t + 1$.*

Looking at Theorem 3, we find that $g(n, 2) = \lfloor \frac{n}{2} \rfloor$ is suitable, and moreover, the converse also holds. Along with Conjecture 7, we guess that $g(n, t) = \lfloor \frac{n}{t} \rfloor$ is a good candidate. But we remark if $g(n, t) = \lfloor \frac{n}{t} \rfloor$, then the converse is not true for $t \geq 3$. There is a counter-example: Take the graph formed by a K_{n-1} with another vertex joining to a vertex of the K_{n-1}.

We suspect that Problem 9 is not easy, given the difficulty of Conjecture 7. But a partial result for Problem 9 may be of interest. For example, the case $t = 3$.

Proposition 1 and Theorem 3 also imply that we can determine the time complexity for deciding whether a graph G satisfies $f(G) \in \{2, 3\}$ or not. The problem of determining a fast algorithm for finding the vertex connectivity $\kappa = \kappa(G)$ of a graph G has been considered by many people (Kleitman [13], Hopcroft and Tarjan [12], Even [5], Even and Tarjan [6], Galil [10], Cheriyan and Thurimella [2, 3], Nagamochi and Ibaraki [14, 15], and Henzinger et al [11], among others). It is known that such an algorithm can be carried out in polynomial time. A result of Henzinger et al [11] states that, for a graph G of order n and with connectivity κ, there is an algorithm which determines the connectivity of G in time $O(\min(\kappa^3 + n, \kappa n)\kappa n)$.

Corollary 10. *Let G be a connected graph of order n. Then, we can decide whether $f(G) = 2$, or $f(G) = 3$, or $f(G) \geq 4$, using an algorithm with running time $O(n^4)$.* □

The balanced decomposition number will assure us the existence of a good structure, especially when it is small. For example, from Theorem 3, if a graph G of order n is $\lfloor \frac{n}{2} \rfloor$-connected, then we can always find a "nearly balanced matching" for any arbitrary number and position of red and blue vertices (Here, "nearly balanced matching" means vertex-disjoint paths of order at most 3 whose end-vertices are coloured by red and blue, respectively).

Moreover, these are related to an existence of so called "non-separating subgraphs", that is, the subgraphs whose deletion preserve high connectivity. For any disjoint subsets $R, B \subset V(G)$ with $|R| = |B| = k$ in an m-connected graph G with $m \geq k$, by using Menger's theorem, there are k vertex disjoint paths $Q_1, \ldots, Q_k$ from R to B. However, in general, we can never hope for high connectivity of $G - \bigcup_{i=1}^{k} V(Q_i)$.

As for this problem, our results give the following.

Corollary 11. *Let $m, n \in \mathbb{N}$ with $m \geq \lfloor \frac{n}{2} \rfloor$, and let G be an m-connected graph of order $n \geq 2$. Then, for any disjoint subsets $R, B \subset V(G)$ with $k = |R| = |B| < \frac{m}{3}$, there are k vertex disjoint paths $Q_1, \ldots, Q_k$ from R to B, such that $G - \bigcup_{i=1}^{k} V(Q_i)$ is $(m - 3k)$-connected.*

Proof. Since G is $\lfloor \frac{n}{2} \rfloor$-connected, by Proposition 1 and Theorem 3, we have $f(G) \in \{2, 3\}$. So the balanced colouring (R, B) for G has a balanced decomposition $\mathcal{P}$ of size at most 3. This means that there are exactly k vertex disjoint sets of $\mathcal{P}$ containing coloured vertices, with each one having exactly one red vertex and one blue vertex. Let $A_1, \ldots, A_k \in \mathcal{P}$ be these k sets. For each A_i, since $|A_i| \leq 3$, one can easily check that there exists $A'_i \subset A_i$ such that $G[A'_i]$ is an $R - B$ sub-path of G. The corollary follows by letting $Q_i = G[A'_i]$ for every i, since clearly the Q_i are vertex disjoint, and we have $|\bigcup_{i=1}^{k} V(Q_i)| \leq 3k$, so that $G - \bigcup_{i=1}^{k} V(Q_i)$ is $(m - 3k)$-connected. □

Thus, a graph with a small balanced decomposition number is likely to have a good structure in view of non-separating subgraphs. In this sense, further study in this area will be interesting and significant.

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