Vertex-Disjoint $K_{1,t}$'s in Graphs

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Abstract

Let $\delta(G)$ denote the minimum degree of a graph G. We prove that for $t \geq 4$ and $k \geq 2$, a graph G of order at least $(t+1)k + 2t^2 - 4t + 2$ with $\delta(G) \geq k + t - 1$ contains k pairwise vertex-disjoint $K_{1,t}$'s.

1. Introduction

We consider only undirected graphs without loops or multiple edges. For a graph G, we denote by $V(G)$, $E(G)$ and $\delta(G)$ the vertex set, the edge set and the minimum degree of G, respectively. A graph F is called a t-claw if F is isomorphic to $K_{1,t}$.

Let $t, k \geq 2$ be integers. In [2], Ota made a conjecture that if G is a graph with $|V(G)| \geq (t+1)k + t^2 - t$ and $\delta(G) \geq k + t - 1$, then G contains k vertex-disjoint t-claws. As is shown in [2], in this conjecture, the condition of the minimum degree of G is sharp in the sense that for any fixed t and k, there exists a graph of arbitrarily large order which has minimum degree $k + t - 2$ but does not contain k vertex-disjoint t-claws and, if k is sufficiently large compared with t, then the condition on the order of G is also sharp in the sense that there exists a graph G with $|V(G)| = (t+1)k + t^2 - t - 1$ and $\delta(G) \geq k + t - 1$ such that G does not contain k vertex-disjoint t-claws. The conjecture is settled affirmatively for $t = 2$ in [2; Theorem 5] and for $t = 3$ in [1]. For $t \geq 4$, it is proved in [2; Theorem 1] that if G is a graph with $|V(G)| \geq (t+1)k + 2t^2 - 3t - 1$ and $\delta(G) \geq k + t - 1$, then G contains k vertex-disjoint t-claws. In this paper, we prove the following theorem.

Main Theorem. *Let $t \geq 4$, $k \geq 2$ be integers, and let G be a graph with $|V(G)| \geq (t+1)k + 2t^2 - 4t + 2$ and $\delta(G) \geq k + t - 1$. Then G contains k pairwise vertex-disjoint t-claws.*

We need the following notation and terminology. Let G be a graph. For a vertex $v \in V(G)$, we denote by $N(v) = N_G(v)$ and $d_G(v)$ the set of vertices adjacent to v and the degree of v, respectively. For a vertex set $S \subseteq V(G)$, we write $\langle S \rangle = \langle S \rangle_G$ for the subgraph of G induced by S. For disjoint subsets S and T of $V(G)$, we let $E(S,T) = E_G(S,T)$ denote the set of edges of G joining a vertex in S and a vertex in T. When S or T consists of a single vertex, say $S = \{x\}$ or $T = \{y\}$, we write $E(x,T)$ or $E(S,y)$ for $E(S,T)$.

2. Preparation for the proof of the theorem

By way of contradiction, suppose that there exists a graph G with $|V(G)| \geq (t+1)k + 2t^2 - 4t + 2$ and $\delta(G) \geq k + t - 1$ such that G does not contain k pairwise vertex-disjoint t-claws. We may assume that G is an edge-maximal counterexample. Then G contains $k-1$ vertex-disjoint t-claws, say $C^{(1)}, C^{(2)}, \ldots, C^{(k-1)}$. Let $H = G - (\bigcup_{i=1}^{k-1} V(C^{(i)}))$. Let $P^{(1)}, P^{(2)}, \ldots, P^{(s)}$ be the K_t components of H, i.e.,the components of H isomorphic to K_t. Define $U = \bigcup_{\alpha=1}^{s} V(P^{(\alpha)})$ and $W = V(H) - U$. We assume that $C^{(1)}, C^{(2)}, \ldots, C^{(k-1)}$ are chosen so that $|E(\langle W \rangle)| + \frac{2}{t}|E(\langle U \rangle)|$ is maximum.

By assumption, H contains no t-claw, or equivalently, every vertex of H has degree at most $t-1$. We define $n = |V(H)|$. Note that $n = |V(G)| - (t+1)(k-1) \geq 2t^2 - 3t + 3$. For each i, let $a^{(i)}$ be the center of $C^{(i)}$ and $B^{(i)} = \{b_1^{(i)}, b_2^{(i)}, \ldots, b_t^{(i)}\}$ be the set of leaves of $C^{(i)}$. In the following argument, we sometimes fix i and set $C = C^{(i)}$. In such case, we write $a, B, b_1, b_2, \ldots, b_t$ instead of $a^{(i)}, B^{(i)}, b_1^{(i)}, b_2^{(i)}, \ldots, b_t^{(i)}$, respectively.

We prove several basic lemmas concerning the number of edges between $V(C^{(i)})$ and $V(H)$, which will be used in the subsequent sections. Throughout the rest of this section, we fix i with $1 \leq i \leq k-1$. Thus as mentioned in the preceding paragraph, a denotes the center of $C = C^{(i)}$, and $B = \{b_1, b_2, \ldots, b_t\}$ denotes the set of leaves of C.

Lemma 2.1. *Let $v \in V(H)$, and suppose that $d_H(v) + |E(B,v)| \geq t$. Then $|E(a, V(H) - \{v\} - N_H(v))| \leq t - 1 - d_H(v)$.*

Proof. Suppose that $|E(a, V(H) - \{v\} - N_H(v))| \geq t - d_H(v)$. Then we can take $X \subset N(a) \cap (V(H) - \{v\} - N_H(v))$ such that $|X| = t - d_H(v)$. On the other hand, by the assumption that $|E(B,v)| \geq t - d_H(v)$, we can take $Y \subset N(v) \cap B$ such that $|Y| = t - d_H(v)$. Then $\langle \{a\} \cup X \cup (B - Y) \rangle$ contains a t-claw and $\langle Y \cup \{v\} \cup N_H(v) \rangle$ contains a t-claw. These are mutually vertex-disjoint t-claws in $\langle V(C) \cup V(H) \rangle$, a contradiction. □

Lemma 2.2. *If $E(a, V(H)) \neq \emptyset$, then $|E(b_p, V(H))| \leq t$ for every $b_p \in B$.*

Proof. If there exists $b_p \in B$ such that $|E(b_p, V(H))| \geq t+1$, then we can find in $\langle V(C) \cup V(H) \rangle$ a t-claw with center b_p and a t-claw with center a which are mutually vertex-disjoint, a contradiction. □

Lemma 2.3. *If $|(N(b_p) \cup N(b_q)) \cap V(H)| \geq 2t - 1$ for $b_p, b_q \in B$ with $p \neq q$, then $|E(b_p, V(H))| \leq t - 2$ or $|E(b_q, V(H))| \leq t - 2$.*

Proof. Otherwise, we can find two vertex-disjoint t-claws with centers b_p and b_q in $\langle V(C) \cup V(H) \rangle$, a contradiction. □

Lemma 2.4. *Let $v \in V(H)$, and suppose that $d_H(v) + |E(B,v)| \geq t + 1$. Then $|E(b_p, V(H) - \{v\} - N_H(v))| \leq t - 2$ for every $b_p \in B$.*

Proof. Suppose that there exists $b_p \in B$ such that $|E(b_p, V(H) - \{v\} - N_H(v))| \geq t - 1$. Then $\langle \{a, b_p\} \cup (V(H) - \{v\} - N_H(v)) \rangle$ contains a t-claw and $\langle (B - \{b_p\}) \cup \{v\} \cup N_H(v) \rangle$ contains a t-claw. These are mutually vertex-disjoint t-claws in $\langle V(C) \cup V(H) \rangle$, a contradiction. □

Lemma 2.5. *Let P be a K_t component of H, and suppose that there exists a vertex $v \in V(H) - V(P)$ such that $d_H(v) + |E(V(C), v)| \geq t + 1$. Then $E(V(C), V(P)) = \emptyset$.*

Proof. Suppose that $E(x, V(P)) \neq \emptyset$ for some $x \in V(C)$. Then $\langle \{x\} \cup V(P) \rangle$ contains a t-claw. Also, since $d_H(v) + |E(V(C) - \{x\}, v)| \geq t$, $\langle \{v\} \cup N_H(v) \cup (V(C) - \{x\}) \rangle$ contains a t-claw with center v. This is a contradiction. □

Lemma 2.6. *Let P be a K_t component of H, and suppose that there exists a vertex $v \in V(H) - V(P)$ such that $|E(V(C), v)| \geq 2$. Then $E(B, V(P)) = \emptyset$, and hence it follows that $|E(V(C), V(P))| \leq t$.*

Proof. If $b_p u \in E(G)$ for some $b_p \in B$ and $u \in V(P)$, then since $E(V(C) - \{b_p\}, v) \neq \emptyset$, by replacing C by a t-claw contained in $\langle b_p \cup V(P) \rangle$, we get a contradiction to the maximality of $|E(\langle W \rangle)| + \frac{2}{t}|E(\langle U \rangle)|$. □

Lemma 2.7. *Let P be a K_t component of H, and suppose that $E(V(C), V(H) - V(P)) \neq \emptyset$. Then $|E(V(C), V(P))| \leq t$.*

Proof. If $E(a, V(H) - V(P)) \neq \emptyset$, then by Lemma 2.1, $E(B, u) = \emptyset$ for every vertex $u \in V(P)$, and hence $|E(V(C), V(P))| = |E(a, V(P))| \leq t$. Thus we may assume that $E(a, V(H) - V(P)) = \emptyset$. Without loss of generality, we may assume that $E(b_1, V(H) - V(P)) \neq \emptyset$. If $E(b_p, V(P)) \neq \emptyset$ for some $p \neq 1$, then by replacing C by a t-claw contained in $\langle \{b_p\} \cup V(P) \rangle$, we get a contradiction to the maximality of $|E(\langle W \rangle)| + \frac{2}{t}|E(\langle U \rangle)|$. Thus $E(\{b_2, \ldots, b_t\}, V(P)) = \emptyset$. Suppose that $|E(V(C), V(P))| = |E(\{a, b_1\}, V(P))| \geq t + 1$. Then there exist two independent edges ax and $b_1 y$ with $x, y \in V(P)$. By replacing C by a t-claw contained in $\langle a, b_2, \ldots, b_t, x \rangle$, we get a contradiction to the maximality of $|E(\langle W \rangle)| + \frac{2}{t}|E(\langle U \rangle)|$. □

In the rest of this section, we consider the case where $s \geq t + 1$. For each α with $1 \leq \alpha \leq t + 1$, we take a vertex $u_\alpha \in V(P^{(\alpha)})$. Since

$$\sum_{i=1}^{k-1} \sum_{\alpha=1}^{t+1} |E(V(C^{(i)}), u_\alpha)| = \sum_{\alpha=1}^{t+1} (d_G(u_\alpha) - (t-1)) \geq (t+1)k,$$

there exists an index i with $1 \leq i \leq k - 1$ such that $\sum_{\alpha=1}^{t+1} |E(V(C^{(i)}), u_\alpha)| > t + 1$. Then there exist two edges xu_α and yu_β joining $V(C^{(i)})$ and $\{u_1, u_2, \ldots, u_{t+1}\}$ with $x, y \in V(C^{(i)})$, $x \neq y$ and $\alpha \neq \beta$. Replacing $C^{(i)}$ by t-claws contained in $\langle x \cup V(P^{(\alpha)}) \rangle$ and $\langle y \cup V(P^{(\beta)}) \rangle$, we obtain k vertex-disjoint t-claws in G. This is a contradiction. □

3. The case where $s = t$

We continue with the notation of the preceding section. In order to prove the main theorem, we shall choose some $C^{(i)}$'s, and show that they together with some vertices in H contain more t-claws, which contradicts the assumption that G is a counterexample. In this section, we consider the case where $s = t$.

Lemma 3.1. Suppose that $C = C^{(i)}$ satisfies $|E(V(C), \{u_1, u_2, \ldots, u_t, v\})| \geq t + 2$. Then the following hold.
(i)$2 \leq |E(V(C), v)| \leq t$.
(ii)$E(B, \{u_1, u_2, \ldots, u_t\}) = \emptyset$.

Proof. If $|E(V(C), v)| = t + 1$, then since $|E(V(C), \{u_1, u_2, \ldots, u_t, v\})| \geq t + 2$, there exists an edge xu_α with $x \in V(C)$ and $1 \leq \alpha \leq t$. Then both $\langle (V(C) - \{x\}) \cup \{v\} \rangle$ and $\langle \{x\} \cup V(P^{(\alpha)}) \rangle$ contain a t-claw, which is a contradiction. Hence $|E(V(C), v)| \leq t$.

If $E(V(C), v) = \emptyset$, then we have $|E(V(C), \{u_1, u_2, \ldots, u_t\})| \geq t+2$. This implies that there exist two independent edges joining $V(C)$ and $\{u_1, u_2, \ldots, u_t\}$, and hence $\langle V(C) \cup U \rangle$ contains two vertex-disjoint t-claws. This is a contradiction. Hence $|E(V(C), v)| \geq 1$.

To show (i) and (ii), we first assume that $av \in E(G)$. Then, any edge $b_p u_\alpha \in E(B, \{u_1, u_2, \ldots, u_t\})$ would make two vertex-disjoint t-claws in $\langle (V(C) - \{b_p\}) \cup \{v\} \rangle$ and $\langle \{b_p\} \cup V(P^{(\alpha)}) \rangle$, and hence (ii) follows. By (ii), $E(V(C), \{u_1, u_2, \ldots, u_t\}) = E(a, \{u_1, u_2, \ldots, u_t\})$, and hence $|E(V(C), \{u_1, u_2, \ldots, u_t\})| \leq t$, which implies that $|E(V(C), v)| \geq 2$. This shows (i).

Thus we may assume that $av \notin E(G)$. Since $|E(V(C), v)| \geq 1$, there exists an edge $b_p v$ with $1 \leq p \leq t$. We claim that $E(B - \{b_p\}, \{u_1, u_2, \ldots, u_t\}) = \emptyset$. Suppose that there exists an edge $b_q u_\alpha$ with $p \neq q$. If we replace C by the t-claw with center u_α contained in $\langle \{b_q\} \cup V(P^{(\alpha)}) \rangle$ and set $W' = W \cup (V(C) - \{b_q\})$ and $U' = U - V(P^{(\alpha)})$, then we have $|E(\langle W' \rangle)| + \frac{2}{t}|E(\langle U' \rangle)| > |E(\langle W \rangle)| + \frac{2}{t}|E(\langle U \rangle)|$, which contradicts the maximality of $|E(\langle W \rangle)| + \frac{2}{t}|E(\langle U \rangle)|$. Thus $E(B - \{b_p\}, \{u_1, u_2, \ldots, u_t\}) = \emptyset$, as claimed. Now since $E(V(C), \{u_1, u_2, \ldots, u_t\}) = E(\{a, b_p\}, \{u_1, u_2, \ldots, u_t\})$ cannot contain two independent edges, it contains at most t edges. This implies that $|E(V(C), v)| \geq 2$, and (i) follows. Now we have another edge $b_q v$ with $q \neq p$. Applying the previous claim to this edge, we have $E(B - \{b_q\}, \{u_1, u_2, \ldots, u_t\}) = \emptyset$, and hence $E(B, \{u_1, u_2, \ldots, u_t\}) = \emptyset$. This completes the proof. □

Define $J = \{i |\ 1 \leq i \leq k - 1,\ |E(V(C^{(i)}), \{u_1, u_2, \ldots, u_t, v\})| \geq t + 2\ \}$.

Lemma 3.2. $\sum_{i \in J} |E(V(C^{(i)}), v)| \geq |J| + t + 1$.

Proof. Let $|J| = m$. By the definition of J, if $i \notin J$ then $|E(V(C^{(i)}), \{u_1, u_2, \ldots, u_t, v\})| \leq t+1$. For $i \in J$, $|E(V(C^{(i)}), \{u_1, u_2, \ldots, u_t\})| \leq t$ by Lemma 3.1(ii). Since $\delta(G) \geq k+t-1$ and the maximum degree of H is $t - 1$,

$$(t+1)k \le \left|E\left(\bigcup_{i=1}^{k-1} V(C^{(i)}), \{u_1,u_2,\dots,u_t,v\}\right)\right| = \sum_{i\notin J} |E(V(C^{(i)}),\{u_1,u_2,\dots,u_t,v\})|$$
$$+\sum_{i\in J}|E(V(C^{(i)}),\{u_1,u_2,\dots,u_t\})| + \sum_{i\in J}|E(V(C^{(i)}),v)| \le (t+1)(k-1-m)+tm$$
$$+\sum_{i\in J}|E(V(C^{(i)}),v)|.$$

Thus $\sum_{i\in J}|E(V(C^{(i)}),v)| \ge |J|+t+1.$ □

We may assume that $J=\{1,2,\dots,m\}$ where $m=|J|$, and

$$|E(V(C^{(1)}),v)| \ge |E(V(C^{(2)}),v)| \ge \dots \ge |E(V(C^{(m)}),v)| \ge 2. \tag{3.1}$$

By Lemma 3.1(i) and Lemma 3.2, there exists $l\in J$ with $2\le l\le m$ such that

$$\sum_{i=1}^{l}(|E(V(C^{(i)}),v)|-1) \ge t \tag{3.2}$$

and such that $\sum_{i=1}^{l-1}(|E(V(C^{(i)}),v)|-1) \le t-1$. Then by Lemma 3.1(i),

$$\sum_{j=1}^{i}(|E(V(C^{(j)}),v)|-1) \le t-1 \text{ for each } 1\le i\le l-1. \tag{3.3}$$

Also by Lemma 3.1(i), $l-1 \le \sum_{i=1}^{l-1}(|E(V(C^{(i)}),v)|-1) \le t-1$. Thus

$$2\le l\le t. \tag{3.4}$$

Lemma 3.3. *$|E(a^{(i)},\{u_1,u_2,\dots,u_t\})| \ge i$ for each $1\le i\le l$*

Proof. We see from Lemma 3.1 that $|E(a^{(1)},\{u_1,u_2,\dots,u_t\})|\ge 1$. Thus we may assume $2\le i\le l$. By (3.1) and (3.3),

$$|E(V(C^{(i)}),v)|-1 \le \frac{\sum_{j=1}^{i-1}(|E(V(C^{(i)}),v)|-1)}{i-1} \le \frac{t-1}{i-1}.$$

Hence $|E(a^{(i)},\{u_1,u_2,\dots,u_t\})|-i \ge |E(V(C^{(i)}),\{u_1,\dots,u_t,v\})| - |E(V(C^{(i)}),v)|-i \ge (t+2)-(\frac{t-1}{i-1}+1)-i = \frac{(t-i)(i-2)}{i-1} \ge 0$ by (3.4). □

By Lemma 3.3, we can take l independent edges $a^{(i)}u_i$, $1\le i\le l$. On the other hand, (3.2) implies that $\sum_{i=1}^{l}|E(B^{(i)},v)|\ge t$. Hence we can take $X\subset\bigcup_{i=1}^{l}B^{(i)}$ such that $X\subset N(v)\cap\bigcup_{i=1}^{l}B^{(i)}$, $|X|=t$. Then each of $\langle X\cup\{v\}\rangle$ and $\langle a^{(i)}\cup V(P^{(i)})\rangle$ for $1\le i\le l$ contains a t-claw. These are $l+1$ vertex-disjoint t-claws in $\langle(\bigcup_{i=1}^{l}V(C^{(i)}))\cup V(H)\rangle$, which contradicts the assumption that G is a counterexample.

4. Counting argument

Throughout the rest of this paper, we assume that $s\le t-1$. In this section, we find a *good* vertex in H that can be used later to find an extra t-claw. Recall that U is the set of vertices contained in the K_t components of H, and $W = V(H) - U$. We define

$$I = \{i|\ 1 \le i \le k-1,\ E(V(C^{(i)}), W) = \emptyset\},$$
$$J = \{i|\ 1 \le i \le k-1,\ i \notin I,\ |E(V(C^{(i)}), V(H))| > n-s\}.$$

Note that since $n \ge 2t^2-3t+3$ and $s \le t-1$, it follows that $|E(V(C^{(i)}), V(H))| \ge 2t^2-4t+5$ if $i \in J$.

Lemma 4.1. *There exists a vertex $v \in W$ such that*

$$d_H(v) + \sum_{i\in J} |E(V(C^{(i)}), v)| \ge |J| + t.$$

Proof. We set $l = |I|$ and $m = |J|$, and assume that $d_H(v)+\sum_{i\in J} |E(V(C^{(i)}), v)| \le m+t-1$ for all $v \in W$. We first claim that $|E(V(C^{(i)}), U)| \le t(t+1)$ for each $i \in I$. If $V(C^{(i)})$ is joined by edges to at most one component of $\langle U\rangle$, then the claim is obvious. If $V(C^{(i)})$ is joined to at least two components of $\langle U\rangle$, then by Lemma 2.7, $|E(V(C^{(i)}), U)| \le ts < t(t+1)$. Thus the claim follows. Note that this claim implies that

$$\sum_{i\in I} |E(V(C^{(i)}), U)| \le t(t+1)l. \tag{4.1}$$

For $i \in J$, since $E(V(C^{(i)}), W) \ne \emptyset$, it follows from Lemma 2.7 that $|E(V(C^{(i)}), U)| \le ts$. Hence

$$\sum_{i\in J} |E(V(C^{(i)}), U)| \le tsm. \tag{4.2}$$

Also, by the definition of I and J, if $i \notin I \cup J$, then

$$|E(V(C^{(i)}), V(H))| \le n-s. \tag{4.3}$$

Now we estimate the following weighted sum of the degrees of vertices in H in two ways: $\frac{t-1}{t}\sum_{u\in U} d_G(u) + \sum_{v\in W} d_G(v)$. First, since $\delta(G) \ge k+t-1$,

$$\frac{t-1}{t}\sum_{u\in U} d_G(u) + \sum_{v\in W} d_G(v) \ge (k+t-1)\left(\frac{t-1}{t}|U| + |W|\right) = (k+t-1)(n-s). \tag{4.4}$$

On the other hand, by (4.1),(4.2) and (4.3),

$$\begin{aligned}
&\frac{t-1}{t}\sum_{u\in U} d_G(u) + \sum_{v\in W} d_G(v)\\
&= \frac{t-1}{t}\sum_{u\in U}\left(d_H(u) + \sum_{i=1}^{k-1} |E(V(C^{(i)}), u)|\right) + \sum_{v\in W}\left(d_H(v) + \sum_{i=1}^{k-1} |E(V(C^{(i)}), v)|\right)\\
&= \frac{t-1}{t}\left(\sum_{u\in U} d_H(u) + \left(\sum_{i\in I} + \sum_{i\in J} + \sum_{i\notin I\cup J}\right)|E(V(C^{(i)}), U)|\right)\\
&+ \sum_{v\in W}\left(d_H(v) + \sum_{i\in J} |E(V(C^{(i)}), v)|\right) + \sum_{i\notin I\cup J} |E(V(C^{(i)}), W)|\\
&\le \frac{t-1}{t}\left(\sum_{u\in U} d_H(u) + \sum_{i\in I} |E(V(C^{(i)}), U)| + \sum_{i\in J} |E(V(C^{(i)}), U)|\right)
\end{aligned}$$

$$+\sum_{v\in W}\left(d_H(v)+\sum_{i\in J}|E(V(C^{(i)}),v)|\right)+\sum_{i\notin I\cup J}|E(V(C^{(i)}),V(H))|$$
$$\le \frac{t-1}{t}(t(t-1)s+t(t+1)l+tsm)+(m+t-1)(n-ts)+(n-s)(k-1-l-m)$$
$$=(k+t-1)(n-s)+(t^2-1)l-(l+1)(n-s)$$
$$\le (k+t-1)(n-s)-(t^2-4t+5)l-(2t^2-4t+4).$$

Since $l\ge 0$, this contradicts (4.4). □

In the following argument, we consider the vertices in W satisfying the condition in Lemma 4.1. We define

$$W_0=\left\{v\in W\,\middle|\, d_H(v)+\sum_{i\in J}|E(V(C^{(i)}),v)|\ge |J|+t\right\},$$

which is not empty by Lemma 4.1. We also define

$$W_1=\{v\in W|\ \exists i\in J,\ d_H(v)+|E(V(C^{(i)}),v)|\ge t+1\},$$
$$W_2=\left\{v\in W-W_1\,\middle|\ \exists J_0\subset J,\ 2\le |J_0|\le t-d_H(v),\right.$$
$$\left. d_H(v)+\sum_{i\in J_0}|E(V(C^{(i)}),v)|\ge |J_0|+t\right\}.$$

Lemma 4.2. *The following statements hold:*
(i)$W_0\subset W_1\cup W_2$.
(ii)*If v is a vertex in W_0 with $d_H(v)=t-1$, then $v\in W_1$.*

Proof. Suppose that $v\in W_0$. By the definition of W_0,

$$\sum_{i\in J}(|E(V(C^{(i)}),v)|-1)\ge t-d_H(v).$$

Thus there exists $J_0\subset J$ with $1\le |J_0|\le t-d_H(v)\le t$ such that $|E(V(C^{(i)}),v)|-1\ge 1$ for each $i\in J_0$ and

$$\sum_{i\in J_0}(|E(V(C^{(i)}),v)|-1)\ge t-d_H(v).$$

This proves (i). Further if $d_H(v)=t-1$, then $|J_0|=1$. Thus (ii) holds. □

Lemma 4.3. Suppose that $W_1=\emptyset$. Fix $C=C^{(i)}$, and let $b_p\in B$, Suppose that $E(B,U)=\emptyset$ and $|E(b_p,V(H))|\ge t+1$, and let $x_1,x_2,\dots,x_{t-1}$ be $t-1$ vertices in $V(H)$ adjacent to b_p. Then the following inequality holds:

$$|E(a,V(H))|+|E(b_p,V(H))|+\sum_{i=1}^{t-1}(d_H(x_i)+|E(V(C),x_i)|)\ge |E(V(C),V(H))|+t-1+$$
$$|E(\langle a,x_1,x_2,\dots,x_{t-1}\rangle)|.$$

Proof. First we claim that $|E(B-\{b_p\},V(H)-\{x_1,x_2,\dots,x_{t-1}\})|\le \sum_{i=1}^{t-1}d_H(x_i)-e$, where $e=|E(\langle x_1,x_2,\dots,x_{t-1}\rangle)|$. We replace C by the t-claw contained in $\langle a,b_p,x_1,x_2,\dots,x_{t-1}\rangle$. Let $H'=\langle (V(H)-\{x_1,x_2,\dots,x_{t-1}\})\cup(V(C)-\{a,b_p\})\rangle$, and let U' be the union of the

vertex sets of the K_t components of H'. Also we set $S = (B - \{b_p\}) \cap U'$.
If $S = \emptyset$, then the claim immediately follows from the maximality of $|E(\langle W\rangle)|+\frac{2}{t}|E(\langle U\rangle)|$. Thus we may assume that $S \neq \emptyset$. Let $y \in N(b_p) \cap (V(H) - \{x_1, \ldots, x_{t-1}\})$. Then for any $b \in S$, b is adjacent to y since otherwise we can find two vertex-disjoint t-claws in $\langle a, b, N_{H'}(b)\rangle$ and in $\langle b_p, x_1, x_2, \ldots, x_{t-1}, y\rangle$, a contradiction. Hence there exists a K_t component $P' \subset \langle U'\rangle$ such that $\{y\} \cup S \subset V(P')$. Note that $|N(b_p) \cap (V(H) - \{x_1, \ldots, x_{t-1}\})| \geq 2$ by the assumption that $|E(b_p, V(H))| \geq t+1$. Since the above observation holds for any choice of $y \in N(b_p) \cap (V(H) - \{x_1, \ldots, x_{t-1}\})$, it follows that $1 \leq |S| \leq t-2$. This implies that $\langle V(C) - \{b_p\}\rangle \not\cong K_t$. On the other hand, since $W_1 = \emptyset$ and $d_H(y) + |E(y, V(C))| \geq d_H(y) + |E(B, y)| = |E(y, \{x_1, x_2, \ldots, x_{t-1}\})| + |E(y, V(H) - \{x_1, x_2, \ldots, x_{t-1}, y\})| + |S| + 1 = |E(y, \{x_1, x_2, \ldots, x_{t-1}\})| + d_{P'}(y) + 1 = |E(y, \{x_1, x_2, \ldots, x_{t-1}\})| + t$, we obtain $E(y, \{x_1, x_2, \ldots, x_{t-1}\}) = \emptyset$ and $d_H(y) = t - 1 - |S|$.

Now, replace C by the t-claw contained in $\langle b_p, x_1, x_2, \ldots, x_{t-1}, y\rangle$ and set $H'' = \langle (V(C) - \{b_p\}) \cup (V(H) - \{x_1, x_2, \ldots, x_{t-1}, y\})\rangle$. Then by the maximality of $|E(W)| + \frac{2}{t}|E(U)|$, it follows that

$$0 \leq (\sum_{i=1}^{t-1} d_H(x_i) + d_H(y) - e) - (|E(B - b_p, V(H) - \{x_1, x_2, \ldots, x_{t-1}\})| - |S| + t - 1) =$$

$$(\sum_{i=1}^{t-1} d_H(x_i) - e) - |E(B - b_p, V(H) - \{x_1, x_2, \ldots, x_{t-1}\})|,$$ as claimed.

Consequently,

$$\begin{aligned}
|E(a, V(H))| + |E(b_p, V(H))| + \sum_{i=1}^{t-1} |E(V(C), x_i)| &= |E(V(C), V(H))| + \\
&|E(\{a, b_p\}, \{x_1, x_2, \ldots, x_{t-1}\})| - |E(V(C) - \{a, b_p\}, V(H) - \{x_1, x_2, \ldots, x_{t-1}\})| \\
&\geq |E(V(C), V(H))| + (t - 1 + \sum_{i=1}^{t-1} |E(a, x_i)|) - (\sum_{i=1}^{t-1} d_H(x_i) - e) \\
&= |E(V(C), V(H))| + t - 1 + |E(\langle a, x_1, x_2, \ldots, x_{t-1}\rangle)| - (\sum_{i=1}^{t-1} d_H(x_i)).
\end{aligned}$$

This completes the proof of Lemma 4.3. □

5. Proof of the main theorem

In this section, we continue with the notation of the preceding sections, and complete the proof of the main theorem. We first consider the case where $W_1 = \emptyset$.

Case 1: $W_1 = \emptyset$

We take a vertex $v \in W_0$, and fix it. By Lemma 4.2(i), $v \in W_2$. Also, by Lemma 4.2(ii), $d_H(v) \leq t - 2$. We see from the proof of Lemma 4.2 that there exists $J_0 \subseteq J$ with $2 \leq |J_0| \leq t - d_H(v)$ such that $|E(V(C^{(i)}), v)| \geq 2$ for each $i \in J_0$ and such that $d_H(v) + \sum_{i \in J_0} |E(V(C^{(i)}), v)| \geq |J_0| + t$. Note that $d_H(v) + |J_0| \leq t$.

Lemma 5.1. For each $C = C^{(i)}$ with $i \in J_0$, one of the following statesments holds:

(i) $|E(a, V(H) - \{v\} - N_H(v))| \geq t - d_H(v)$.
(ii)$av \notin E(G)$ and $|E(b_p, V(H) - \{v\} - N_H(v))| \geq (t-1)|J_0|$ for some $b_p \in B$.

Proof. Since $i \in J_0 \subset J$, we have

$$|E(V(C), V(H))| \geq n - s + 1 \geq 2t^2 - 4t + 5. \tag{5.1}$$

Since $v \notin W_1$ and $i \in J_0$, $2 \leq |E(V(C), v)| \leq t$. Suppose that (i) does not hold. Then $|E(a, V(H) - \{v\})| \leq t - 1 - d_H(v) + |N_H(v)| = t - 1$. If $E(a, V(H)) \neq \emptyset$, then by Lemma 2.2, $|E(b, V(H))| \leq t$ for each $b \in B$, and hence $|E(V(C), V(H))| = |E(a, V(H))| + |E(B, V(H))| \leq t^2 + t$, which contradicts (5.1). Thus $E(a, V(H)) = \emptyset$. Since $|E(V(C), v)| \geq 2$, we see from Lemma 2.6 that

$$E(V(C^{(i)}), V(H)) = E(B, W). \tag{5.2}$$

Hence by (5.1), there exists $b_p \in B$ such that $|E(b_p, W)| \geq t + 1$. Let $x_1, x_2, \ldots, x_{t-1} \in N(b_p) \cap W$. Since $W_1 = \emptyset$, $d_H(x_i) + |E(x_i, V(C))| \leq t$ for each $1 \leq i \leq t - 1$. Consequently by Lemma 4.3, $|E(b_p, W)| \geq |E(V(C), W)| + t - 1 + |E(\langle x_1, x_2, \ldots, x_{t-1} \rangle)| - \left(\sum_{i=1}^{t-1} \{d_H(x_i) + |E(x_i, V(C))|\} \right) \geq |E(V(C), W)| - t^2 + 2t - 1 + |E(\langle x_1, x_2, \ldots, x_{t-1} \rangle)| \geq 2t^2 - 4t + 5 - t^2 + 2t - 1 \geq t^2 - 2t + 4 > 2t - 1$. By Lemma 2.3, for each $b \in B - \{b_p\}$, $|E(b, W)| \leq t - 2$. This shows that $|E(b_p, W)| \geq 2t^2 - 4t + 5 - (t-1)(t-2) = t^2 - t + 3$.

Hence $|E(b_p, V(H) - \{v\} - N_H(v))| = |E(b_p, W)| - |E(b_p, \{v\} \cup N_H(v))| \geq |E(b_p, W)| - 1 - |N_H(v)| \geq |E(b_p, W)| + |J_0| - t - 1 \geq t^2 - t + 3 + |J_0| - t - 1 \geq t(t-2) + |J_0| \geq |J_0|(t-2) + |J_0| \geq |J_0|(t-1)$.

This completes the proof of Lemma 5.1. □

Now, we shall find $|J_0| + 1$ vertex-disjoint t-claws in $\langle \bigcup_{i \in J_0} V(C^{(i)}) \cup V(H) \rangle$, which will contradict the assumption that G is a counterexample. We may assume that $J_0 = \{i | \ 1 \leq i \leq |J_0|\}$. We may also assume that for an integer h with $0 \leq h \leq |J_0|$, $C = C^{(i)}$ satisfies (i) in Lemma 5.1 for all $1 \leq i \leq h$, and $C = C^{(i)}$ satisfies (ii) in Lemma 5.1 for all $h + 1 \leq i \leq |J_0|$. Moreover, for $C = C^{(i)}$ with $h + 1 \leq i \leq |J_0|$, we may assume that b_1 is the vertex b_p satisfying the condition of Lemma 5.1(ii). By the choice of J_0,

$$\sum_{i \in J_0} (|E(V(C^{(i)}), v)| - 1) \geq t - d_H(v).$$

This inequality implies that for each $i (1 \leq i \leq |J_0|)$, we can choose a subset $X^{(i)} \subset N(v) \cap V(C^{(i)})$ such that

$$|X^{(i)}| \leq |E(V(C^{(i)}, v)| - 1, \ \sum_{i=1}^{|J_0|} |X^{(i)}| = t - d_H(v)$$

and

$$a^{(i)} \notin X^{(i)} \text{ for } 1 \leq i \leq h, \ b_1^{(i)} \notin X^{(i)} \text{ for } h + 1 \leq i \leq |J_0|.$$

Then we can find a t-claw with center v in $\langle \{v\} \cup N_H(v) \cup \bigcup_{i=1}^{|J_0|} X^{(i)} \rangle$. Note that by the condition in Lemma 5.1(ii), we have $a^{(i)} \notin X^{(i)}$ also for $h + 1 \leq i \leq |J_0|$.

We define $Y^{(i)} = V(C^{(i)}) - X^{(i)}$ for $1 \leq i \leq h$ and $Y^{(i)} = \{a^{(i)}, b^{(i)}\}$ for $h + 1 \leq i \leq |J_0|$. We take disjoint subsets $Z^{(i)}$ of $V(H) - \{v\} - N_H(v)$ for $1 \leq i \leq |J_0|$ such that

$$Z^{(i)} \subset N(a^{(i)}), \ |Z^{(i)}| = |X^{(i)}| \text{ for } 1 \leq i \leq h,$$

$Z^{(i)} \subset N(b_1^{(i)}),\ |Z^{(i)}| = t-1$ for $h+1 \le i \le |J_0|$.
This can be done by determining $Z^{(i)}$ from $i=1$ up to $|J_0|$, because for $1 \le i \le h$,

$$\sum_{j=1}^{i} |X^{(j)}| \le \sum_{j=1}^{|J_0|} |X^{(j)}| = t - d_H(v) \le |E(a^{(i)}, V(H) - \{v\} - N_H(v))|,$$

and for $h+1 \le i \le |J_0|$,

$$\sum_{j=1}^{h} |X^{(j)}| + (t-1)(i-h) \le (t-1)i \le (t-1)|J_0| \le |E(b_1^{(i)}, V(H) - \{v\} - N_H(v))|.$$

Then for each i with $1 \le i \le |J_0|$, $\langle Y^{(i)} \cup Z^{(i)} \rangle$ contains a t-claw with center $a^{(i)}$ or $b_1^{(i)}$ depending on whether $i \le h$ or $i \ge h+1$. Obviously, these t-claws and the t-claw in $\langle \{v\} \cup N_H(v) \cup \bigcup_{i=1}^{|J_0|} X^{(i)} \rangle$ are pairwise vertex-disjoint. This contradicts the assumption that G is a counterexample, and completes the proof for Case 1.
Case 2: $W_1 \neq \emptyset$.
Let $v \in W_1$. By the definition of W_1, we can take a t-claw $C = C^{(i)}$ with $i \in J$ such that $d_H(v) + |E(V(C), v)| \ge t+1$. Then by Lemma 2.5, we have $E(V(C), U) = \emptyset$, and hence

$$|E(V(C), W)| = |E(V(C), V(H))| \ge n - s + 1 \ge 2t^2 - 4t + 5. \quad (5.4)$$

Lemma 5.2 $E(a, W) = \emptyset$.

Proof. Suppose that $E(a, W) \neq \emptyset$. By Lemma 2.1, we have $|E(a, W - \{v\})| = |E(a, W - \{v\} - N_H(v))| + |E(a, N_H(v))| \le t - 1 - d_H(v) + |E(a, N_H(v))| \le t - 1 - d_H(v) + |N_H(v)| = t-1$, and hence $|E(a, W)| \le t$. On the other hand, we see from Lemma 2.2 that for each vertex $b_p \in B$, $|E(b_p, W)| \le t$. Hence $|E(V(C), W)| = |E(a, W)| + |E(B, W)| \le t + t^2$, which contradicts (5.4). □

Note that it follows from Lemma 5.2 that

$$d_H(v) + |E(B, v)| \ge t+1. \quad (5.5)$$

Hence by Lemma 2.4,

$$|E(b, W - \{v\} - N_H(v))| \le t-2 \text{ for every } b \in B, \quad (5.6)$$

which implies that $|E(b, W)| \le 2t-2$ for every $b \in B$. We see from (5.4) and Lemma 5.2 that there exist at least three vertices $b \in B$ such that $|E(b, W)| \ge 2t-3$. By (5.6), this implies that $|N_H(v)| \ge t-2$. Let $S = \{b \in B |\ |E(b, W)| \ge 2t-3\}$. Thus $|S| \ge 3$. We first consider the case where $|N_H(v)| = t-2$. For each $b \in S$, we have $|N(b) \cap (W - \{v\} - N_H(v))| = t-2$ and $N(b) \supset \{v\} \cup N_H(v)$ by (5.6), and hence $|E(b, W)| = 2t-3$. In particular, $S \subset N(v)$ and it follows from (5.4) and Lemma 5.2 that $|S| \ge 5$, and hence $|N_H(v)| + |N(v) \cap B| \ge |N_H(v)| + |S| \ge (t-2)+5 \ge t+2$. Consequently, taking $b_p \in S$ and $x \in N_H(v)$, we see that $\langle \{b_p, a, x\} \cup (N(b_p) \cap (W - \{v\} - N_H(v))) \rangle$ contains a t-claw and $\langle (N(v) \cap (B - \{b_p\})) \cup \{v\} \cup (N_H(v) - \{x\}) \rangle$ contains a t-claw, a contradiction.

We now consider the case where $|N_H(v)| = t-1$. If there exists $b_q \in S$ such that $b_q v \notin E(G)$, then $|N(b_q) \cap (W - \{v\} - N_H(v))| = t-2$ and $N(b_q) \supset N_H(v)$, and hence, taking $x \in N_H(v)$, we see from (5.5) that $\langle (N(v) \cap B) \cup \{v\} \cup (N_H(v) - \{x\}) \rangle$ contains a t-claw and $\langle \{b_q, a, x\} \cup (N(b_q) \cap (W - \{v\} - N_H(v))) \rangle$ contains a t-claw, a contradiction. Thus

$S \subset N(v)$. Suppose that there exists $b_p \in S$ such that $|N(b_p) \cap (W - \{v\} - N_H(v))| = t-2$. Note that $|N(b_p) \cap (\{v\} \cup N_H(v))| \geq t-1$; in particular, $N(b_p) \cap N_H(v) \neq \emptyset$. Also recall $|S| \geq 3$, and hence $|N_H(v)| + |N(v) \cap B| \geq (t-1)+3 = t-2$. Consequently, taking $x \in N(b_p) \cap N_H(v)$, we see that $\langle \{b_p, a, x\} \cup (N(b_p) \cap (W - \{v\} - N_H(v))) \rangle$ contains a t-claw and $\langle (N(v) \cap (B - \{b_p\})) \cup \{v\} \cup (N_H(v) - \{x\}) \rangle$ contains a t-claw, a contradiction. Thus $|N(b) \cap (W - \{v\} - N_H(v))| = t-3$ for all $b \in S$, which means that $|N(b) \cap W| = 2t-3$ for all $b \in S$. This together with (5.4) and Lemma 5.2 implies that $|S| \geq 5$, and hence $|N_H(v)| + |N(v) \cap B| \geq (t-1)+5 \geq t+3$. Consequently, taking $b_p \in S$ and $x, y \in N_H(v)$ with $x \neq y$, we see that $\langle \{b_p, a, x, y\} \cup ((N(b_p) \cap (W - \{v\} - N_H(v))) \rangle$ contains a t-claw and $\langle (N(v) \cap (B - \{b_p\})) \cup \{v\} \cup (N_H(v) - \{x, y\}) \rangle$ contains a t-claw, a contradiction.

This completes the proof of the main theorem. □

References

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