Further Results on the Balanced Decomposition Number

Shinya Fujita[*]
Department of Mathematics
Gunma National College of Technology
580 Toriba, Maebashi 371-8530, Japan
fujita@nat.gunma-ct.ac.jp

Henry Liu[†]
Departamento de Matemática
Faculdade de Ciências e Tecnologia
Universidade Nova de Lisboa
Quinta da Torre, 2829-516 Caparica, Portugal
h.liu@fct.unl.pt

Abstract

A *balanced colouring* of a graph G is a colouring of some of the vertices of G with two colours, say red and blue, such that there is the same number of vertices in each colour. The *balanced decomposition number* $f(G)$ of G is the minimum integer s with the following property: For any balanced colouring of G, there is a partition $V(G) = V_1 \dot\cup \cdots \dot\cup V_r$ such that, for every $1 \le i \le r$, V_i induces a connected subgraph of order at most s, and contains the same number of red and blue vertices. The function $f(G)$ was introduced by Fujita and Nakamigawa [12], and was further studied by Fujita and Liu [10, 11]. In [12], Fujita and Nakamigawa conjectured that $f(G) \le \lfloor \frac{n}{2} \rfloor + 1$ if G is a 2-connected graph on n vertices. Partial results of this conjecture have been proved in [10, 11, 12]. In this paper, we shall prove another partial result, in the case when the number of coloured vertices is six. We shall also derive some consequences from known results about the balanced decomposition number and other results.

1 Introduction

In this paper, unless otherwise stated, all graphs will be simple and finite (i.e., undirected, and without multiple edges or loops). For such a graph G, its *vertex set* and *edge set* are denoted by $V(G)$ and $E(G)$ respectively. We write $e(G) = |E(G)|$.

Our definitions concerning graphs throughout the paper are fairly standard. Let G be a graph. For $U \subset V(G)$, the graph $G - U$ is the subgraph of G induced by $V(G) \setminus U$. For $k \in \mathbb{N}$, G is a *k-connected* graph if $|V(G)| \geq k + 1$, and for any $U \subset V(G)$ with $|U| \leq k - 1$, the graph $G - U$ is connected. If Q is a path, then an *internal vertex* of Q is a vertex which is not an end-vertex. We write $\operatorname{int} Q$ for the sub-path of Q induced by its internal vertices (This is vacuous if $|V(Q)| \leq 2$). If $U, W \subset V(G)$ and $U \cap W = \emptyset$, a $U - W$ *path* of G is a sub-path where one end-vertex is in U, the other is in W, and no internal vertex (if any) is in $U \cup W$. In particular, we call a $\{u\} - W$ path a $u - W$ *path*. If $H \subset G$ is a subgraph, then an *H-path* is a sub-path $P \subset G$ with distinct end-vertices u and v, such that $V(P) \cap V(H) = \{u, v\}$ and $E(P) \cap E(H) = \emptyset$. Note that $e(P) \geq 1$.

We refer the reader to [3] for any undefined terms.

In [12], Fujita and Nakamigawa introduced the *balanced decomposition number* of a graph. For a graph G with $|V(G)| = n \in \mathbb{N}$, a *balanced colouring* of G is a pair (R, B), where $R, B \subset V(G)$, $R \cap B = \emptyset$, and $0 \leq |R| = |B| \leq \lfloor \frac{n}{2} \rfloor$. We shall refer the vertices of R (B) as the *red (blue) vertices*, and those of $V(G) \setminus (R \cup B)$ as the *uncoloured vertices*. A *balanced decomposition* of G, or of (R, B), is a partition $V(G) = V_1 \dot{\cup} \cdots \dot{\cup} V_r$ (for some $r \geq 1$), such that for each $1 \leq i \leq r$, we have $|V_i \cap R| = |V_i \cap B|$, and V_i induces a connected subgraph of G. The *size* of the balanced decomposition is defined as the maximum of $|V_1|, \ldots, |V_r|$.

Now, observe that, if G is a disconnected graph, then we can take a balanced colouring so that G does not have a balanced decomposition at all. Simply, colour one vertex in one component red, and another vertex in another component blue. So, from now on, we shall only consider balanced decompositions for connected graphs.

So, if G is a connected graph on n vertices, and $k \in \mathbb{Z}$, $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$, we define

$$
\begin{aligned}
f(k, G) \ = \ \min\{s \in \mathbb{N} : \ & \text{Every balanced colouring } (R, B) \text{ of } G \\
& \text{with } |R| = |B| = k \text{ has a balanced} \\
& \text{decomposition of size at most } s\}.
\end{aligned}
$$

Note that $f(k, G) \leq n$, so $f(k, G)$ is well-defined. The *balanced decomposition number* of G is then defined as

$$
f(G) = \max\left\{ f(k, G) : 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \right\}.
$$

The function $f(G)$ was first studied by Fujita and Nakamigawa [12], and subsequently by Fujita and Liu [10, 11]. This paper is organised as follows. In Section 2, we shall review the known results about $f(G)$ which appeared in [10, 11, 12]. In Section 3, we shall prove a partial result of a conjecture of Fujita and Nakamigawa which concerns the balanced decomposition number of 2-connected graphs. In Section 4, we shall derive some results concerning the balanced decomposition number of certain graphs, using known results from [11, 12], as well as other results.

2 Known results

In [10, 11, 12], the balanced decomposition number has been determined exactly for many classes of graphs, and bounds were also obtained for some other classes. We begin by reviewing these results.

The balanced decomposition number has been determined exactly for trees, cycles, and complete multipartite graphs.

Theorem 1 (Theorem 1 of [12]) *Let T be a tree on n vertices. Then, $f(T) = n$.*

Theorem 2 (Theorem 4 of [12]) *Let C_n be the cycle on $n \geq 3$ vertices. Then, $f(C_n) = \lfloor \frac{n}{2} \rfloor + 1$.*

Theorem 3 (Theorems 1 and 2 of [12], Theorem 5 of [10]) *Let $K_{k_1, \ldots, k_t}$ be the complete multipartite graph with class sizes $k_1 \geq \cdots \geq k_t \geq 1$, where $t \geq 2$. Then,*

$$f(K_{k_1, \ldots, k_t}) = \left\lfloor \frac{k_1 - 2}{\sum_{i=2}^{t} k_i} \right\rfloor + 3.$$

In Theorem 3, the cases $t = 1, 2$ were solved in [12], and the cases $t \geq 3$ were solved in [10].

For some other classes of graphs, bounds have been obtained for their balanced decomposition numbers. These include the cases for generalised Θ-graphs, subdivisions of K_4, and 2-connected series-parallel graphs.

A *generalised Θ-graph (with t paths)* is the union of $t \geq 2$ paths, $Q_1, \ldots, Q_t$ say, with each having the same two end-vertices, x and y say, so that $V(Q_i) \cap V(Q_j) = \{x, y\}$ for any $i \neq j$. In other words, the Q_i are pairwise internally vertex disjoint paths. In addition, all but at most one of the Q_i have order at least 3. We have the following result.

Theorem 4 (Theorem 8 of [10]) *Let G be a generalised Θ-graph with t paths on n vertices. Then, $\lceil \frac{n-t+1}{2} \rceil \leq f(G) \leq \lfloor \frac{n}{2} \rfloor + 1$. In particular, if t is fixed, then $f(G) = \frac{n}{2} + O(1)$.*

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We write TK_4 for any graph which is a subdivided K_4. We have the following result.

Theorem 5 (Theorem 5 of [11]) *Let G be a TK_4 on n vertices. Then, $f(G) \leq \lfloor \frac{n}{2} \rfloor + 1$.*

A *series-parallel graph* is a simple graph which can be obtained as follows. Start with a single edge. We then perform a sequence of operations successively, each of which is one of the following.

(i) Subdivide an edge.

(ii) Replace an edge by a multiple edge.

It is well-known (Duffin [7]) that a series-parallel graph is 2-connected if and only if it can be obtained as described above, but with "single edge" replaced by "K_3". In other words, starting with a single edge, the first operation must be of type (ii) above. It is also well-known that if a 2-connected graph is not series-parallel, then it contains a subgraph which is a TK_4.

We have the following result.

Theorem 6 (Theorem 6 of [11]) *Let G be a 2-connected series-parallel graph on n vertices. Then, $f(G) \leq \lfloor \frac{n}{2} \rfloor + 1$.*

Now, another direction to the study of the balanced decomposition number is to ask the following question: For an integer $c \geq 2$, characterise all connected graphs G with $f(G) = c$. For $c = 2$, the answer is simple and was first remarked in [12]: $f(G) = 2$ precisely when G is a complete graph with at least two vertices. However, the cases $c \geq 3$ are much more complicated. For $c = 3$, such a characterisation was proved in [10].

Theorem 7 (Theorem 3 of [10]) *Let G be a connected graph on n vertices. Then, $f(G) = 3$ if and only if $n \geq 3$, G is $\lfloor \frac{n}{2} \rfloor$-connected, and is not a complete graph.*

Theorem 7 is pleasantly surprising. It gives us a new chacterisation for the family of graphs on n vertices which are $\lfloor \frac{n}{2} \rfloor$-connected. Of course, there are well-known characterisations for the structure of 2-connected graphs (Whitney [20]) and 3-connected graphs (Tutte [19]). See also, e.g., Proposition 3.1.2 and Theorem 3.2.2 of [5]. So, studying the function $f(G)$ can open up a new direction to the study of the vertex connectivity of a graph.

3 A conjecture about 2-connected Graphs

The following conjecture concerning the balanced decomposition number for 2-connected graphs was posed in [12].

Conjecture 8 (Conjecture 1 of [12]) *Let G be a 2-connected graph on n vertices. Then, $f(G) \leq \lfloor \frac{n}{2} \rfloor + 1$.*

Conjecture 8 is still open. Note that the graphs involved in Theorems 2, 4, 5 and 6 are all 2-connected, and each has the same upper bound of $f(G) \leq \lfloor \frac{n}{2} \rfloor + 1$ (where G is the corresponding graph and has order n). Hence, each of these four theorems is a partial result to Conjecture 8. The following partial result was also proved in [12].

Theorem 9 (Theorem 5 of [12]) *Let G be a 2-connected graph on $n \geq 4$ vertices. Then $f(2, G) \leq \lfloor \frac{n}{2} \rfloor + 1$.*

In this section, we shall extend this to the following result.

Theorem 10 *Let G be a 2-connected graph on $n \geq 6$ vertices. Then $f(3, G) \leq \lfloor \frac{n}{2} \rfloor + 1$.*

Before we prove Theorem 10, we state some consequences of Menger's theorem [15] which will be useful. These can be easily derived from, e.g., Ch. III, Corollary 6 of [3].

Theorem 11 *Let G be a 2-connected graph.*

    (a) *For any two vertices $x, y \in V(G)$, there exists a cycle of G containing both x and y.*

    (b) *Let $A \subset V(G)$ with $|A| \geq 2$. Let $v \in V(G - A)$. Then, there are $v - A$ paths $P, Q \subset G$ such that $V(P) \cap V(Q) = \{v\}$.*

We are now ready to give a proof of Theorem 10, which will also require some of the results in Section 2. Indeed, in the proof of Theorem 9 as given in [12], Theorems 2 and 11 were used. Of course, the simpler version of Theorems 9 and 10, that $f(1, G) \leq \lfloor \frac{n}{2} \rfloor + 1$ if G is a 2-connected graph on $n \geq 3$ vertices, is fairly trivial. For this, we take a balanced colouring (R, B) of G with $|R| = |B| = 1$. Apply Theorem 11(a) to get a cycle $C \subset G$ containing the red vertex and the blue vertex. Then by Theorem 2, C has a balanced decomposition $V_1 \dot{\cup} V_2$ with size at most $\lfloor \frac{n}{2} \rfloor + 1$. Hence, $V_1 \cup V_2 \cup \bigcup \{\{v\} : v \notin V(C)\}$ is a balanced decomposition for G with size at most $\lfloor \frac{n}{2} \rfloor + 1$.

Proof of Theorem 10. Let (R, B) be a balanced colouring of G with $|R| = |B| = 3$. Choose a subgraph $H \subset G$ such that H is 2-connected, $V(H) \supset R \cup B$, and $e(H)$ is minimal (Note that H exists since G is 2-connected and $V(G) \supset R \cup B$).

If H is a series-parallel graph, then Theorem 6 gives us a balanced decomposition for H, and hence for G, of size at most $\lfloor \frac{n}{2} \rfloor + 1$, and we are done (Indeed, to obtain the balanced decomposition for G, take a balanced

decomposition for H with size at most $\lfloor \frac{n}{2} \rfloor + 1$, along with the isolated uncoloured vertices of $V(G) \setminus V(H)$). So, H is not a series-parallel graph, and hence it contains a TK_4. Choose $H_0 \subset H$, where H_0 is a TK_4, such that $|V(H_0) \cap (R \cup B)|$ is maximal.

If $|V(H_0) \cap (R \cup B)| = 6$, then by Theorem 5, H_0, and hence G, has a balanced decomposition of size at most $\lfloor \frac{n}{2} \rfloor + 1$ (with a similar argument as before).

Hence, $|V(H_0) \cap (R \cup B)| \leq 5$. There exists a sub-path $Q_0 \subset H_0$ whose end-vertices are branch vertices of H_0 (they have degree 3 in H_0), whose internal vertices (if any) are subdividing vertices of H_0 (they have degree 2 in H_0), and $V(\operatorname{int} Q_0) \cap (R \cup B) = \emptyset$. Let $H_0' \subset H_0$ be the subgraph obtained by deleting all internal vertices and all edges of Q_0 from H_0. Note that H_0' is a generalised Θ-graph and so it is 2-connected, and that Q_0 is an H_0'-path. We now construct a sequence $H_0 \subset H_1 \subset \cdots \subset H_m$ of subgraphs of H (for some $m \geq 1$) which has the following properties.

(i) For each $0 \leq i \leq m$, $H_i = H_i' \cup Q_i$, where H_i' is 2-connected, and Q_i is an H_i'-path.

(ii) $Q_0 \supset Q_1 \supset \cdots \supset Q_m$.

(iii) $V(H_m') \supset R \cup B$.

Suppose that for some $i \geq 0$, we have found the subgraphs $H_0 \subset H_1 \subset \cdots \subset H_i$ of H, with $H_j = H_j' \cup Q_j$, where H_j' is 2-connected and Q_j is an H_j'-path, for every $0 \leq j \leq i$. Also, assume that $Q_0 \supset Q_1 \supset \cdots \supset Q_i$ holds. If possible, we shall find the subgraph H_{i+1} and the corresponding H_{i+1}' and Q_{i+1}.

Let q_1 and q_2 be the end-vertices of Q_i, i.e., $V(Q_i) \cap V(H_i') = \{q_1, q_2\}$. If $V(H_i) \supset R \cup B$ (and hence $V(H_i') \supset R \cup B$, since $\operatorname{int} Q_i \subset \operatorname{int} Q_0$ and $V(\operatorname{int} Q_0) \cap (R \cup B) = \emptyset$), set $m = i$. Otherwise, there exists a vertex $v \in R \cup B$ with $v \in V(H) \setminus V(H_i)$. By Theorem 11(b), there are $v - V(H_i)$ paths $R_1, R_2 \subset H$ with $V(R_1) \cap V(R_2) = \{v\}$. Let $V(R_\ell) \cap V(H_i) = \{u_\ell\}$ for $\ell = 1, 2$. We cannot have $u_1, u_2 \in V(Q_i)$, because then $u_1, u_2 \in V(Q_0)$, so that the graph $H_0 \cup R_1 \cup R_2$ contains a TK_4 which contradicts the choice of H_0 that $|V(H_0) \cap (R \cup B)|$ is maximal.

Observe that if we have any 2-connected graph $F \subset H$ and an F-path $P \subset H$, then the graph $F \cup P$ is also 2-connected.

- Without loss of generality, if $u_1 \in V(\operatorname{int} Q_i)$ and $u_2 \notin V(Q_i)$ then let $R_1' = u_1 \cdots q_1 \subset Q_i$. Note that $R_1 \cup R_1' \cup R_2$ is an H_i'-path, so the graph $H_{i+1}' = H_i' \cup R_1 \cup R_1' \cup R_2$ is 2-connected. Let $Q_{i+1} = u_1 \cdots q_2 \subset Q_i$.

- If $u_1, u_2 \notin V(\operatorname{int} Q_i)$, then $R_1 \cup R_2$ is an H_i'-path, and so $H_{i+1}' = H_i' \cup R_1 \cup R_2$ is 2-connected. Let $Q_{i+1} = Q_i$.

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In each case, Q_{i+1} is an H'_{i+1}-path. So, the graph $H_{i+1} = H'_{i+1} \cup Q_{i+1}$ is a required subgraph. Repeat this process. We must stop at some $H_m = H'_m \cup Q_m \subset H$, and it is clear that the sequence $H_0 \subset H_1 \subset \cdots \subset H_m$ satisfies (i) to (iii).

But, the graph $H'_m \subset H$ is 2-connected, with $V(H'_m) \supset R \cup B$, and we have $e(Q_m) \geq 1$. So, H'_m contradicts the minimality of $e(H)$. $\square$

4   Some consequences from known results

In this section, we shall derive some consequences of Theorems 2 and 6. There are many well-known results which say that a graph G must contain a certain type of cycle (e.g., a Hamilton cycle) if G satisfies certain conditions. Hence, we may apply such results along with Theorem 2 to deduce results about the balanced decomposition number for such graphs. There is also a similar result with "2-connected series-parallel graph" in place of "cycle", and so we can do something similar, using Theorem 6.

Perhaps the most well-known and famous of these results is Dirac's theorem [6], which says that if G is a graph of order $n \geq 3$, and G has minimum degree at least $\frac{n}{2}$, then G is Hamiltonian. So by Theorem 2, we have the following result.

Corollary 12 *Let G be a graph of order $n \geq 3$ with minimum degree at least $\frac{n}{2}$. Then $f(G) \leq \lfloor \frac{n}{2} \rfloor + 1$.*

There are many other results. A theorem of Elmallah and Colbourn [9] says that if G is a 3-connected planar graph, then G has a spanning 2-connected series-parallel graph. So by Theorem 6, we have the following result.

Corollary 13 *Let G be a 3-connected planar graph of order n. Then $f(G) \leq \lfloor \frac{n}{2} \rfloor + 1$.*

A more well-known theorem of Tutte [18] says something similar: If G is a 4-connected planar graph, then G is Hamiltonian. So this result, along with Theorem 2, imply a weaker result than Corollary 13 with "4-connected" in place of "3-connected".

A theorem of Ellingham et al. [8] says that if G is a 3-connected cubic graph with at least 10 vertices, then there exists a cycle containing 10 prescribed vertices of G, as long as G cannot be contracted to the Petersen graph in such a way that the 10 vertices each map to a distinct vertex of the Petersen graph. So by Theorem 2, we have the following result.

Corollary 14 *Let G be a 3-connected cubic graph of order $n \geq 10$ such that, for any 10 vertices of G, there is no contraction of G to the Petersen graph so that the 10 vertices each map to a distinct vertex of the Petersen graph. Then $f(5, G) \leq \lfloor \frac{n}{2} \rfloor + 1$.*

In the same direction, a theorem of Häggkvist and Mader [13] says that if G is a k-connected k-regular graph, then for every set of $k + \lfloor \frac{1}{3}\sqrt{k} \rfloor$ vertices of G, there exists a cycle containing them. So by Theorem 2, we have the following result.

Corollary 15 *Let G be a k-connected k-regular graph of order $n \geq k + \lfloor \frac{1}{3}\sqrt{k} \rfloor$. Then $f\left(\lfloor \frac{1}{2}k + \frac{1}{2}\lfloor \frac{1}{3}\sqrt{k} \rfloor\rfloor, G\right) \leq \lfloor \frac{n}{2} \rfloor + 1$.*

Finally, we can consider the balanced decomposition number for *random graphs*. There are many well-known models of random graphs. These models, as well as some other basic terms in the theory of random graphs, are defined in the appendix.

We consider the property of a random graph being Hamiltonian. There are many well-known results which state that a random graph is Hamiltonian w.h.p. if a certain condition is satisfied.

Komlós and Szemerédi [14] were the first to prove that, for $\mathcal{G}(n, p)$, $G_{n,p}$ is Hamiltonian w.h.p. if $p \geq (1 + o(1))\frac{\log n}{n}$, and for $\mathcal{G}(n, M)$, $G_{n,M}$ is Hamiltonian w.h.p. if $M \geq (\frac{1}{2} + o(1))n \log n$ (See also Bollobás [2]). More precisely, with Theorem 2, we have the following results.

Corollary 16 *Let $\omega(n)$ be any function such that $\omega(n) \to \infty$ as $n \to \infty$, and let $p \geq (\log n + \log\log n + \omega(n))/n$. Then w.h.p., $f(G_{n,p}) \leq \lfloor \frac{n}{2} \rfloor + 1$.*

Corollary 17 *Let $\omega(n)$ be any function such that $\omega(n) \to \infty$ as $n \to \infty$, and let $M \geq \lfloor n(\log n + \log\log n + \omega(n))/2 \rfloor$. Then w.h.p., $f(G_{n,M}) \leq \lfloor \frac{n}{2} \rfloor + 1$.*

For $\mathcal{G}(n, r\text{-reg})$, a result of Robinson and Wormald [16, 17] says that $G_{n,r\text{-reg}}$ is Hamiltonian w.h.p. if $r \geq 3$. By Theorem 2, we have the following result.

Corollary 18 *Let $r \geq 3$. Then w.h.p., $f(G_{n,r\text{-reg}}) \leq \lfloor \frac{n}{2} \rfloor + 1$.*

Finally, for $\mathcal{G}(n, k\text{-out})$, a very recent result of Bohman and Frieze [1] says that $G_{n,k\text{-out}}$ is Hamiltonian w.h.p. if $k \geq 3$. By Theorem 2, we have the following result.

Corollary 19 *Let $k \geq 3$. Then w.h.p., $f(G_{n,k\text{-out}}) \leq \lfloor \frac{n}{2} \rfloor + 1$.*

Further results of the balanced decomposition number in relation to random graphs will be of great interest for future research.

5 Appendix: Models of Random Graphs

We shall define a few basic terms from the theory of random graphs. For any other undefined terms, see [4].

Let $N = \binom{n}{2}$, and $0 < p < 1$. The $\mathcal{G}(n,p)$ *model* consists of all 2^N graphs on n vertices, where for each graph, we select an edge with probability p, independently from one another. An element of $\mathcal{G}(n,p)$ is denoted by $G_{n,p}$. The space $\mathcal{G}(n,p)$ becomes a probability space where for a $G_{n,p} \in \mathcal{G}(n,p)$ with m edges, we have $\mathbb{P}(G_{n,p}) = p^m(1-p)^{N-m}$.

Let $N = \binom{n}{2}$, and $1 \leq M \leq N$. The $\mathcal{G}(n,M)$ *model* consists of all $\binom{N}{M}$ graphs on n vertices with M edges. An element of $\mathcal{G}(n,M)$ is denoted by $G_{n,M}$. The space $\mathcal{G}(n,M)$ becomes a probability space with the uniform distribution: For every $G_{n,M} \in \mathcal{G}(n,M)$, we have $\mathbb{P}(G_{n,M}) = \binom{N}{M}^{-1}$.

Let $3 \leq r \leq n-1$ and let rn be even. The $\mathcal{G}(n,r\text{-reg})$ *model* consists of all r-regular graphs on n vertices (The set $\mathcal{G}(n,r\text{-reg})$ is not empty by our assumption on r). An element of $\mathcal{G}(n,r\text{-reg})$ is denoted by $G_{n,r\text{-reg}}$. The space $\mathcal{G}(n,r\text{-reg})$ becomes a probability space with the uniform distribution.

Let $1 \leq k \leq n-1$. The $\mathcal{G}(n,k\text{-out})$ *model* is defined as follows. We consider the space $\vec{\mathcal{G}}(n,k\text{-out})$ which consists of all directed graphs on n vertices with outdegree k. The space $\vec{\mathcal{G}}(n,k\text{-out})$ becomes a probability space with the uniform distribution. Then, the space $\mathcal{G}(n,k\text{-out})$ consists of the set of graphs on n vertices, each of which is the underlying simple graph of some member of $\vec{\mathcal{G}}(n,k\text{-out})$. An element of $\mathcal{G}(n,k\text{-out})$ is denoted by $G_{n,k\text{-out}}$. Note that each $G_{n,k\text{-out}}$ has minimum degree at least k, and has at most kn edges. The space $\mathcal{G}(n,k\text{-out})$ becomes a probability space where for $G_{n,k\text{-out}} \in \mathcal{G}(n,k\text{-out})$, we have

$$\mathbb{P}(G_{n,k\text{-out}}) = \mathbb{P}(\{\vec{G} \in \vec{\mathcal{G}}(n,k\text{-out}) : G_{n,k\text{-out}} \text{ is the underlying graph of } \vec{G}\}).$$

An element of any one of these spaces is a *random graph*, and in particular, an element of $\mathcal{G}(n,r\text{-reg})$ is a *random r-regular graph*. One is usually interested in the situation when n is large.

Let $\mathcal{G}_n$ be any one of these four models. A *property* of random graphs is a subset $\mathcal{P} \subset \mathcal{G}_n$ where each member of $\mathcal{P}$ has a certain property. We say that the property $\mathcal{P}$ holds *with high probability (w.h.p.)* in $\mathcal{G}_n$ if $\mathbb{P}(\mathcal{P}) \to 1$ as $n \to \infty$.

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