# Vertex-Disjoint Copies of $K_1 + (K_1 \cup K_2)$ in Claw-Free Graphs

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**Abstract**

A graph $G$ is said to be claw-free if $G$ does not contain an induced subgraph isomorphic to $K_{1,3}$. Let $k$ be an integer with $k \geq 2$. We prove that if $G$ is a claw-free graph of order at least $7k-6$ and with minimum degree at least 3, then $G$ contains $k$ vertex-disjoint copies of $K_1 + (K_1 \cup K_2)$.

**Keywords:** claw-free, minimum degree, vertex-disjoint.

## 1 Introduction

In this paper, we consider only finite, simple, undirected graphs with no loops and no multiple edges. For a graph $G$, we denote by $V(G)$, $E(G)$ and $\delta(G)$ the vertex set, the edge set and the minimum degree of $G$, respectively. For a vertex $x$ of a graph $G$, the neighborhood of $x$ in $G$ is denoted by $N_G(x)$, and let $d_G(x) := |N_G(x)|$. For a noncomplete graph $G$, let $\sigma_2(G)$ :=min$\{d_G(x) + d_G(y)|\ xy \notin E(G)\}$; if $G$ is a complete graph, let $\sigma_2(G) := \infty$.

For a subset $S$ of $V(G)$, the subgraph in $G$ induced by $S$ is denoted by $\langle S\rangle$. For a subgraph $H$ of $G$, $G-H = \langle V(G)-V(H)\rangle$ and for a vertex $x$ of $G$, $G-x = \langle V(G)-\{x\}\rangle$. For disjoint subsets $S$ and $T$ of $V(G)$, we let $E(S,T)$ denote the set of edges of $G$ joining a vertex in $S$ and a vertex in $T$. When $S$ or $T$ consists of a single vertex, say $S = \{x\}$ or $T = \{y\}$, we write $E(x,T)$ or $E(S,y)$ for $E(S,T)$. A graph $G$ is said to be *$K_{1,r}$-free* if $G$ does not contain an induced subgraph isomorphic to $K_{1,r}$. In particular, a graph $G$ is said to be *claw-free* when $G$ is $K_{1,3}$-free. Let $K_n$ be a complete graph of order $n$. Also, let $F$ be the graph obtained from $K_4$ by removing two edges which have an endvertex in common; thus $F = K_1 + (K_1 \cup K_2)$.

In this paper, we are concerned with degree conditions for the existence of vertex-disjoint subgraphs and forbidden subgraphs. It is proved in Sumner[4] that a connected claw-free graph of order $2k$ contains a perfect matching, i.e., $k$ vertex-disjoint copies of $K_2$. As for the existence of $k$ vertex-disjoint copies of $K_3$ in claw-free graphs, Wang[5] proved the following theorem.

**Theorem A.**([5]) *Let $k$ be an integer with $k \geq 2$. If $G$ is a claw-free graph of order at least $6k-5$ with $\delta(G) \geq 3$, then $G$ contains $k$ vertex-disjoint copies of $K_3$.*

It would be an interesting thesis for us to consider the relationship between degree conditions for the existence of other vertex-disjoint subgraphs and other forbidden subgraphs. Now we propose the following conjecture:

**Conjecture.** *Let $k, r, t$ be integers with $k \geq 2$ , $r \geq 3$ and $t \geq 2$.If $G$ is a $K_{1,r}$-free graph of order at least $(k-1)\{t(r-1)+1\}+1$ with $\delta(G) \geq t$, then $G$ contains $k$ vertex-disjoint copies of $K_{1,t}$.*

If the conjecture is true, the bound on $|V(G)|$ is best possible. To see this, let $B_i = K_t$ for each $1 \leq i \leq r-1$, and consider $G = \cup_{i=1}^{k-1} A_i$ where $A_i = K_1 + \cup_{j=1}^{r-1} B_j$ for each $1 \leq i \leq k-1$. Then $G$ is a $K_{1,r}$-free graph of order $(k-1)\{t(r-1)+1\}$ with $\delta(G) \geq t$. It is easy to check that $G$ does not contain $k$ vertex-disjoint copies of $K_{1,t}$.

Recently, in [2], the author proved that the conjecture is true for $t = 2$. Our purpose of this paper is to prove the following theorem which gives a partial result for this conjecture.

**Main Theorem.** *Let $k$ be an integer with $k \geq 2$. If $G$ is a claw-free graph of order at least $7k-6$ with $\delta(G) \geq 3$, then $G$ contains $k$ vertex-disjoint copies of $F$.*

As an immediate corollary of the main theorem, we see that the conjecture is true for $t = r = 3$ because $F$ contains $K_{1,3}$.

There are two known results concerning the existence of $k$ vertex-disjoint copies of $F$ in general graphs. In the rest of this section, we list these results as a reference for readers. Kawarabayashi[3] proved the following theorem.

**Theorem B.**([3])*Let $k$ be an integer with $k \geq 2$, and let $G$ be a graph with $|V(G)| \geq 4k$ and $\sigma_2(G) \geq |V(G)| + k$. Then $G$ contains $k$ vertex-disjoint copies of $F$.*

As a related result concerning Theorem B, the author[1] proved the following theorem.

**Theorem C.**([1])*Let $k$ be an integer with $k \geq 2$. Let $G$ be a graph with $|V(G)| \geq 4k$ and $\sigma_2(G) \geq |V(G)|/2 + 2k - 1$, and suppose that $G$ contains $k$ vertex-disjoint triangles. In the case where $|V(G)| = 4k+2$, suppose further that $G \not\cong K_{4t+3} \cup K_{4k-4t-1}$ for any $t$ with $0 \leq t \leq k-1$. Then $G$ contains $k$ vertex-disjoint copies of $F$.*

# 2 Preparation for the proof of the Main Theorem

Let $k, G$ be as in the Main Theorem. By Theorem A, $G$ has $k$ vertex-disjoint triangles. Let $k_1, k_2, k_3$ be integers with $k_1+k_2+k_3 = k$. Let $S_1, \ldots, S_{k_1}, T_1, \ldots, T_{k_2}, U_1, \ldots, U_{k_3}$ be $k$ vertex-disjoint subgraphs of $G$ such that for each $i$ with $1 \leq i \leq k_1$, either $S_i$ contains $K_1+(K_2 \cup K_2)$ as a spanning subgraph (i.e., $|V(S_i)| = 5$), or $S_i \cong K_1+(K_1 \cup K_3)$, for each $i$ with $1 \leq i \leq k_2$, $T_i$ contains $F$ as a spanning subgraph (i.e., $|V(T_i)| = 4$), and for each $i$ with $1 \leq i \leq k_3$, $U_i \cong K_3$. Let $S := \langle \cup_{i=1}^{k_1} V(S_i) \rangle$, $T := \langle \cup_{i=1}^{k_2} V(T_i) \rangle$, $U := \langle \cup_{i=1}^{k_3} V(U_i) \rangle$ and $H := G - S - T - U$. We choose $S_1, \ldots, S_{k_1}, T_1, \ldots, T_{k_2}, U_1, \ldots, U_{k_3}$ so that
(a) $k_1 + k_2$ is maximum, and subject to the condition (a),
(b) $k_1$ is maximum, and subject to the condition (b),
(c) $|\{i \mid 1 \leq i \leq k_1, S_i \supset K_1 + (K_2 \cup K_2)\}|$ is maximum, and subject to the condition (c),
(d) $|E(H)|$ is maximum.

If $k_3 = 0$, then $k_1 + k_2 = k$, which means that the desired conclusion holds. Hence we may assume that $k_3 \geq 1$. Let $Q := \{v \in V(H) \mid E(v, V(T)) = \emptyset\}$ and put $q = |Q|$. Now we prove basic lemmas.

**Lemma 2.1.** *The following statements hold:*

(i) $E(V(S), V(U)) = \emptyset$.

(ii) $E(V(H), V(U)) = \emptyset$.

**Proof.** Suppose that there exist $S_i, U_j$ with $1 \le i \le k_1$, $1 \le j \le k_3$ such that $E(V(S_i), V(U_j)) \ne \emptyset$. Suppose that $S_i \cong K_1 + (K_1 \cup K_3)$. Let $a$ be a vertex in $S_i$ such that $d_{S_i}(a) = 4$ and $S_i - a \supset K_1 \cup K_3$. Suppose that there exists $x \in V(S_i - a)$ such that $E(x, V(U_j)) \ne \emptyset$. Then we can find two vertex-disjoint subgraphs $A_1, A_2$ in $\langle V(S_i) \cup V(U_j)\rangle$ such that $A_i$ contains $F$ as a spanning subgraph for $i = 1, 2$. Then by replacing $S_i, U_j$ by $A_1, A_2$, we get a contradiction to the maximality of $k_1 + k_2$. Thus we have $E(V(S_i - a), V(U_j)) = \emptyset$. Hence we may assume that $E(a, V(U_j)) \ne \emptyset$. Let $z$ be a vertex in $U_j$ such that $az \in E(G)$, and take $x, y \in V(S_i - a)$ so that $xy \notin E(G)$. Then $\langle\{a, x, y, z\}\rangle \cong K_{1,3}$, which contradicts the assumption that $G$ is claw-free. Hence we may assume that $S_i$ contains $K_1 + (K_2 \cup K_2)$ as a spanning subgraph. Again, let $a$ be a vertex in $S_i$ such that $d_{S_i}(a) = 4$ and $S_i - a \supset K_2 \cup K_2$. Suppose that there exists $x \in V(S_i - a)$ such that $E(x, V(U_j)) \ne \emptyset$. Then by replacing $S_i, U_j$ by $S_i - x, \langle V(U_j) \cup \{x\}\rangle$, respectively, we get a contradiction to the maximality of $k_1 + k_2$. Thus we may assume that $E(V(S_i - a), V(U_j)) = \emptyset$ and $E(a, V(U_j)) \ne \emptyset$. Since $d_{S_i}(a) = 4$ and $G$ is claw-free, this implies that $S_i - a \cong K_4$. Then by replacing $S_i, U_j$ by $S_i - a, \langle V(U_j) \cup \{a\}\rangle$, respectively, we get a contradiction to the maximality of $k_1 + k_2$. Thus (i) holds. Suppose that there exists $U_j$ with $1 \le j \le k_3$ such that $E(V(H), V(U_j)) \ne \emptyset$. Let $v$ be a vertex in $H$ such that $E(v, V(U_j)) \ne \emptyset$. Then by replacing $U_j$ by $\langle V(U_j) \cup \{v\}\rangle$, we get a contradiction to the maximality of $k_1 + k_2$. Thus (ii) holds. □

We see from Lemma 2.1 that $k_2 \ge 1$ because $\delta(G) \ge 3$ and $k_3 \ge 1$.

**Lemma 2.2.** *Let $1 \le i \le k_1$, and suppose that $S_i \supset K_1 + (K_2 \cup K_2)$. Let $V(S_i) = \{a, b, c, d, e\}$, and suppose that $bc, de \in E(G)$ and $d_{S_i}(a) = 4$. Let $v$ be a vertex in $H$ such that $E(v, V(S_i)) \ne \emptyset$. Then there exists $x \in V(S_i)$ such that $xv \in E(G)$ and $S_i - x \supset F$.*

**Proof.** Suppose that $E(v, \{b, c, d, e\}) \ne \emptyset$. Take $x \in \{b, c, d, e\}$ such that $xv \in E(G)$. Then obviously $S_i - x \supset F$ because $bc, de \in E(G)$ and $d_{S_i}(a) = 4$. Thus we may assume that $av \in E(G)$ and $E(v, \{b, c, d, e\}) = \emptyset$. Since $G$ is claw-free, this forces $\langle\{b, c, d, e\}\rangle \cong K_4$. Then $av \in E(G)$ and $S_i - a \supset F$. □

**Lemma 2.3.** *Let $1 \le i \le k_1$, and suppose that $S_i \cong K_1 + (K_1 \cup K_3)$. Let $V(S_i) = \{a, b, c, d, e\}$, and let $d_{S_i}(a) = 4, d_{S_i}(b) = 3, d_{S_i}(c) = 3, d_{S_i}(d) = 3$ and $d_{S_i}(e) = 1$. Then the following statements hold:*

(i) $E(a, V(H)) = \emptyset$.

(ii) *For every $v \in V(H)$, $|E(v, \{b, c, d\})| \le 1$.*

(iii) *Let $v$ be a vertex in $H$ such that $E(v, V(S_i)) \ne \emptyset$. Then there exists $x \in V(S_i)$ such that $vx \in E(G)$ and $S_i - x \supset F$.*

**Proof.** Suppose that there exists $x \in V(H)$ such that $ax \in E(G)$. Suppose that $E(x, V(S_i - a)) \neq \emptyset$. Then there exists a subgraph $X$ in $\langle \{x\} \cup V(S_i) \rangle$ such that $X$ contains $K_1 + (K_2 \cup K_2)$ as a spanning subgraph. Then by replacing $S_i$ by $X$, we get a contradiction to the maximality of $|\{i \mid 1 \leq i \leq k_1, S_i \supset K_1 + (K_2 \cup K_2)\}|$. Thus we have $E(x, V(S_i - a)) = \emptyset$. Then $\langle \{x, a, b, e\} \rangle \cong K_{1,3}$ because $be \notin E(G)$. This contradicts the assumption that $G$ is claw-free. Thus (i) holds. To prove (ii), suppose that there exists $v \in V(H)$ such that $|E(v, \{b, c, d\})| \geq 2$. Then $\langle V(S_i - e) \cup \{v\} \rangle$ contains $K_1 + (K_2 \cup K_2)$ as a spanning subgraph. Hence by replacing $S_i$ by $\langle V(S_i - e) \cup \{v\} \rangle$, we get a contradiction to the maximality of $|\{i \mid 1 \leq i \leq k_1, S_i \supset K_1 + (K_2 \cup K_2)\}|$. Thus (ii) holds. Let $v$ be a vertex in $H$ such that $E(v, V(S_i)) \neq \emptyset$. Then by (i), $E(v, \{b, c, d, e\}) \neq \emptyset$. Since $S_i - x \supset F$ for every $x \in \{b, c, d, e\}$, the assertion of (iii) obviously holds. □

**Lemma 2.4.** *For each $i$ with $1 \leq i \leq k_1$, $|\{v \in V(H) \mid |E(v, V(S_i))| \geq 2\}| \leq 2$.*

**Proof.** By way of contradiction, we may assume that there exists $S_i$ with $1 \leq i \leq k_1$ such that $|\{v \in V(H) \mid |E(v, V(S_i))| \geq 2\}| \geq 3$. Take $v_1, v_2, v_3 \in V(H)$ such that $|E(v_j, V(S_i))| \geq 2$ for each $j$ with $1 \leq j \leq 3$. First suppose that $S_i \cong K_1 + (K_1 \cup K_3)$. Let $e$ be a vertex in $S_i$ such that $d_{S_i}(e) = 1$. In view of Lemma 2.3(i) and (ii), we see that $v_j e \in E(G)$ for each $v_j$ with $1 \leq j \leq 3$. Since $G$ is claw-free, $\langle \{v_1, v_2, v_3, e\} \rangle \supset F$. Then by replacing $S_i, U_1$ by $S_i - e, \langle \{v_1, v_2, v_3, e\} \rangle$, respectively, we get a contradiction to the maximality of $k_1 + k_2$. Next suppose that $S_i$ contains $K_1 + (K_2 \cup K_2)$ as a spanning subgraph. Let $a$ be a vertex in $S_i$ such that $d_{S_i}(a) = 4$ and $S_i - a \supset K_2 \cup K_2$. Suppose that there exists $x \in V(S_i - a)$ such that $|E(x, \{v_1, v_2, v_3\})| \geq 2$. By symmetry, we may assume that $xv_1, xv_2 \in E(G)$. Take $y \in V(S_i) - \{a, x\}$ such that $yx \in E(G)$ and $\langle V(S_i) - \{x, y\} \rangle \cong K_3$. Since $G$ is claw-free, $\langle \{x, y, v_1, v_2\} \rangle \supset F$. If $E(v_3, V(S_i) - \{x, y\}) \neq \emptyset$, then by replacing $S_i$ and $U_1$ by $\langle (V(S_i) - \{x, y\}) \cup \{v_3\} \rangle$ and $\langle \{x, y, v_1, v_2\} \rangle$, respectively, we get a contradiction to the maximality of $k_1 + k_2$. Thus $E(v_3, V(S_i) - \{x, y\}) = \emptyset$. Since $|E(v_3, V(S_i))| \geq 2$, this implies that $xv_3 \in E(G)$. Note that $\langle \{x, v_1, v_2, v_3\} \rangle \supset F$ because $G$ is claw-free. Then by replacing $S_i$ and $U_1$ by $\langle \{x, v_1, v_2, v_3\} \rangle$ and $S_i - x$, respectively, we get a contradiction to the maximality of $k_1 + k_2$. Hence it follows that

$$|E(x, \{v_1, v_2, v_3\})| \leq 1 \text{ for every } x \in V(S_i - a). \qquad \text{(A)}$$

By (A) and by symmetry, we may assume that $av_1, av_2 \in E(G)$. Let $b, c, d, e$ be vertices in $S_i - a$ such that $bc, de \in E(G)$. Suppose that either $|E(v_3, \{b, c\})| = 2$ or $|E(v_3, \{d, e\})| = 2$. By symmetry, we may assume that $|E(v_3, \{b, c\})| = 2$. By (A), this forces either $E(\{v_1, v_2\}, \{b, c, d, e\}) = \{v_1 d, v_2 e\}$ or $E(\{v_1, v_2\}, \{b, c, d, e\}) = \{v_1 e, v_2 d\}$. By symmetry, we may assume $E(\{v_1, v_2\}, \{b, c, d, e\}) = \{v_1 d, v_2 e\}$. Since $\langle \{a, v_2, b, d\} \rangle \not\cong K_{1,3}$, this forces $bd \in E(G)$. Then by replacing $S_i$ and $U_1$ by $\langle \{v_3, b, c, d\} \rangle$ and $\langle \{v_1, v_2, a, e\} \rangle$, respectively, we get a contradiction to the maximality of $k_1 + k_2$. Thus we have

$$|E(v_3, \{b, c\})| \leq 1 \text{ and } |E(v_3, \{d, e\})| \leq 1. \qquad \text{(B)}$$

Next suppose that $|E(v_3, \{b, c, d, e\})| = 2$. By (B) and by symmetry, we may assume that $N_G(v_3) \cap \{b, c, d, e\} = \{b, d\}$. This together with (A) forces either $E(\{v_1, v_2\}, \{b, c, d, e\}) = \{v_1 c, v_2 e\}$ or $E(\{v_1, v_2\}, \{b, c, d, e\}) = \{v_1 e, v_2 c\}$. By the symmetry of the roles of $v_1$ and $v_2$, we may assume that $E(\{v_1, v_2\}, \{b, c, d, e\}) = \{v_1 c, v_2 e\}$. Since $\langle \{a, v_1, b, d\} \rangle \not\cong K_{1,3}$, this forces $bd \in E(G)$. Then by replacing $S_i$ and $U_1$ by $\langle \{a, e, v_1, v_2\} \rangle$ and $\langle \{v_3, b, c, d\} \rangle$, respectively, we get a contradiction to the maximality of $k_1 + k_2$. Thus we have

$$|E(v_3, \{b, c, d, e\})| = 1 \text{ and } av_3 \in E(G). \qquad \text{(C)}$$

We see from (A) and (C) that there exists $x \in \{b, c, d, e\}$ such that $E(x, \{v_1, v_2, v_3\}) = \emptyset$ and for each $y \in \{b, c, d, e\} - \{x\}$, $E(y, \{v_1, v_2, v_3\}) \neq \emptyset$. Since $|E(a, \{v_1, v_2, v_3\})| = 3$, $E(x, \{v_1, v_2, v_3\}) = \emptyset$ and $G$ is claw-free, it follows that $\langle \{v_1, v_2, v_3\} \rangle \cong K_3$. Take $y \in \{b, c, d, e\} - \{x\}$. Then by replacing $S_i$ and $U_1$ by $\langle \{v_1, v_2, v_3, y\} \rangle$ and $S_i - y$, we we get a contradiction to the maximality of $k_1 + k_2$. □

**Lemma 2.5.** *Let $u \in Q$. Then there exists $i$ with $1 \leq i \leq k_1$ such that $|E(u, V(S_i))| \geq 2$.*

**Proof.** Suppose that $|E(u, V(S_i))| \leq 1$ for each $i$ with $1 \leq i \leq k_1$. Let $L = \{j \mid 1 \leq j \leq k_1, E(u, V(S_j)) \neq \emptyset\}$. Then by Lemma 2.2 and Lemma 2.3(iii), $F \subset S_l - N_G(u)$ for each $l \in L$. Since $\delta(G) \geq 3$ and $G$ is claw-free, $\langle V(H) \cup (N_G(u) \cap (\cup_{l \in L} V(S_l)) \rangle \supset F$. Hence we can take $|L| + 1$ vertex-disjoint subgraphs $S'_1, \ldots, S'_{|L|+1}$ such that $S'_i \supset F$ for each $1 \leq i \leq |L| + 1$ in $\langle V(H) \cup (\cup_{l \in L} V(S_l)) \rangle$. Then by replacing $\cup_{l \in L} S_l$ and $U_1$ by $\cup_{1 \leq j \leq |L|+1} S'_j$, we get a contradiction to the maximality of $k_1 + k_2$. □

**Lemma 2.6.** *Let $i$ be an integer with $1 \leq i \leq k_2$, and let $a, b, c, d$ be vertices in $T_i$ such that $d_{T_i}(a) \leq d_{T_i}(b) \leq d_{T_i}(c) \leq d_{T_i}(d)$. Then the following statements hold:*

(i) $E(d, V(H)) = \emptyset$.

(ii) *For each $x \in \{b, c\}$, $|E(x, V(H))| \leq 1$.*

(iii) $|E(a, V(H))| \leq 2$ *and equality holds only if $T_i \cong F$ and $\langle (N_G(a) \cap V(H)) \cup \{a\} \rangle \cong K_3$.*

**Proof.** Suppose that $E(d, V(H)) \neq \emptyset$. Let $v$ be a vertex in $H$ such that $vd \in E(G)$. If $av \in E(G)$, then by replacing $T_i$ by $\langle \{v\} \cup V(T_i) \rangle$, we get a contradiction to the maximality of $k_1$ because $\langle \{v\} \cup V(T_i) \rangle \supset K_1 + (K_2 \cup K_2)$. Thus $av \notin E(G)$. Since $\langle \{d, a, v, b\} \rangle \ncong K_{1,3}$ and $\langle \{d, a, v, c\} \rangle \ncong K_{1,3}$, it follows that $|\{ab, bv\} \cap E(G)| \geq 1$ and $|\{ac, cv\} \cap E(G)| \geq 1$. Then by replacing $T_i$ by $\langle \{v\} \cup V(T_i) \rangle$, we get a contradiction to the maximality of $k_1$ because $\langle \{v\} \cup V(T_i) \rangle \supset K_1 + (K_1 \cup K_3)$ or $K_1 + (K_2 \cup K_2)$. Thus (i) holds. Suppose that there exists $x \in \{b, c\}$ such that $|E(x, V(H))| \geq 2$. Let $u, v \in N_G(x) \cap V(H)$. By (i), $E(d, \{u, v\}) = \emptyset$. Since $\langle \{x, d, u, v\} \rangle \ncong K_{1,3}$, $uv \in E(G)$. Then by replacing $T_i$ by $\langle V(T_i - a) \cup \{u, v\} \rangle$, we get a contradiction to the maximality of $k_1$ because $\langle V(T_i - a) \cup \{u, v\} \rangle \supset K_1 + (K_2 \cup K_2)$. Thus (ii) holds. To prove (iii), suppose that there exist $u, v \in V(H)$ such that $au, av \in E(G)$. Since $\langle \{a, u, v, d\} \rangle \ncong K_{1,3}$, we see from (i) that $uv \in E(G)$. If $ab \in E(G)$ or $ac \in E(G)$, say, $ab \in E(G)$, then by replacing $T_i$ by $\langle \{u, v, a, b, d\} \rangle$, we get a contradiction to the maximality of $k_1$. Thus $T_i \cong F$ holds. Also, if $|E(a, V(H))| = 2$, then $\langle (N_G(a) \cap V(H)) \cup \{a\} \rangle \cong K_3$. Since $u$ and $v$ are taken arbitrarily, if $|E(a, V(H))| \geq 3$, then $\langle V(H) \cup \{a, d\} \rangle$ contains a subgraph $T$ such that $T \supset K_1 + (K_1 \cup K_3)$ and $|V(T)| = 4$. Then by replacing $T_i$ by $T$, we get a contradiction to the maximality of $k_1$. Thus (iii) holds. □

The following two lemmas are used in the case where $k_3 = 1$ only. In view of Lemma 2.1, note that if $k_3 = 1$, then there exists $i$ with $1 \leq i \leq k_2$ such that $E(V(T_i), V(U_1)) \neq \emptyset$ because $\delta(G) \geq 3$. Throughout the rest of the proof of the Main Theorem, we may assume that $E(V(T_1), V(U_1)\ ) \neq \emptyset$ when $k_3 = 1$.

**Lemma 2.7.** *Suppose that $k_3 = 1$. Then there exists $x \in V(T_1)$ such that $E(x, V(U_1)) \neq \emptyset$ and $E(y, V(T_1 - x)) \neq \emptyset$ for every $y \in \{v \in V(H) \mid E(v, V(T_1)) \neq \emptyset\}$.*

**Proof.** Let $a, b, c, d$ be vertices in $T_1$ such that $d_{T_1}(a) \leq d_{T_1}(b) \leq d_{T_1}(c) \leq d_{T_1}(d)$. If $E(d, V(U_1)) \neq \emptyset$, then the assertion follows from Lemma 2.6(i) as $x = d$. Thus we may assume that $E(d, V(U_1)) = \emptyset$. Take $x \in \{a, b, c\}$ such that $E(x, V(U_1)) \neq \emptyset$. Let $z$ be a vertex in $U_1$ such that $xz \in E(G)$. Suppose that there exists $y \in V(H)$ such that $xy \in E(G)$. Then by Lemma 2.1 and Lemma 2.6(i), $\langle\{x, y, z, d\}\rangle \cong K_{1,3}$, which contradicts the assumption that $G$ is claw-free. Hence for every $y \in \{v \in V(H) \mid E(v, V(T_1)) \neq \emptyset\}$, $xy \notin E(G)$, which implies that $E(y, V(T_1 - x)) = E(y, V(T_1)) \neq \emptyset$. □

**Lemma 2.8.** *Suppose that $k_3 = 1$. Suppose that there exists $v \in V(H)$ such that $E(v, V(T_1)) \neq \emptyset$. Then one of the following holds:*

(i) *There exists $S_i$ with $1 \leq i \leq k_1$ such that $|E(v, V(S_i))| \geq 2$.*

(ii) *There exists $T_j$ with $2 \leq j \leq k_2$ such that $E(v, V(T_j)) \neq \emptyset$.*

**Proof.** Suppose not. Then for each $S_i$ with $1 \leq i \leq k_1$, $|E(v, V(S_i))| \leq 1$ and $E(v, \cup_{2 \leq j \leq k_2} V(T_j)) = \emptyset$. Take $y \in V(T_1)$ such that $vy \in E(G)$. Let $L = \{j \mid 1 \leq j \leq k_1, E(v, V(S_j)) \neq \emptyset\}$. Then by Lemma 2.2 and Lemma 2.3(iii), $F \subset S_l - N_G(v)$ for each $l \in L$. Since $\delta(G) \geq 3$ and $G$ is claw-free, $\langle V(H) \cup (N_G(v) \cap (\cup_{l \in L} V(S_l))) \cup \{y\}\rangle \supset F$. Hence we can take $|L| + 1$ vertex-disjoint subgraphs $S'_1, \ldots, S'_{|L|+1}$ such that $S'_i$ contains $F$ as a spanning subgraph for each $1 \leq i \leq |L| + 1$ in $\langle V(H) \cup (\cup_{l \in L} V(S_l)) \cup \{y\}\rangle$. In view of Lemma 2.7, we can take a subgraph $U'$ such that $U'$ contains $F$ as a spanning subgraph in $\langle V(T_1 - y) \cup V(U_1)\rangle$. Then by replacing $\cup_{l \in L} S_l$, $T_1$ and $U_1$ by $\cup_{1 \leq j \leq |L|+1} S'_j$ and $U'$, respectively, we get a contradiction to the maximality of $k_1 + k_2$. □

# 3 Proof of the Main Theorem

We prove the following two claims.

**Claim 1.** $q \leq 2k_1$.
**Proof.** The claim follows from Lemma 2.4 and Lemma 2.5. □

**Claim 2.** For each $i$ with $1 \leq i \leq k_2$, $|\{v \in V(H) \mid E(v, V(T_i)) \neq \emptyset\}| \leq 3$.
**Proof.** Suppose that there exists $T_i$ with $1 \leq i \leq k_2$ such that $|\{v \in V(H) \mid E(v, V(T_i) \neq \emptyset\}| \geq 4$. Let $v_1, v_2, v_3, v_4$ be vertices in $\{v \in V(H) \mid E(v, V(T_i) \neq \emptyset\}$. Let $a, b, c, d$ be vertices in $T_i$ such that $d_{T_i}(a) \leq d_{T_i}(b) \leq d_{T_i}(c) \leq d_{T_i}(d)$. Then in view of Lemma 2.6, we have $T_i \cong F$, $|E(a, V(H))| = 2$, $|E(b, V(H))| = 1$, $|E(c, V(H))| = 1$ and $E(d, V(H)) = \emptyset$. Again by Lemma 2.6, we may assume that $av_1, av_2, v_1v_2, bv_3, cv_4 \in E(G)$. Suppose that $E(\{v_1, v_2\}, V(H) - \{v_1, v_2\}) \neq \emptyset$. Then we can find two vertex-disjoint subgraphs $A_1, A_2$ such that $A_i$ contains $F$ as a spanning subgraph for $i = 1, 2$ in $\langle V(T_i) \cup V(H)\rangle$. Consequently, by replacing $T_i$ and $U_1$ by $A_1$ and $A_2$, respectively, we get a contradiction to the maximality of $k_1 + k_2$. Hence we may assume that $E(\{v_1, v_2\}, V(H) - \{v_1, v_2\}) = \emptyset$. Then by replacing $T_i$ by $\langle\{v_1, v_2, a, d\}\rangle$, we get a contradiction to the maximality of $|E(H)|$.
□

Now we devide the proof into two cases.

Case 1: $k_3 \geq 2$

By Lemma 2.1(ii), Claim 1 and Claim2, we have $|V(G)| \leq 5k_1 + 4k_2 + 3k_3 + q + 3k_2 \leq 7k_1 + 7k_2 + 3k_3 = 7(k_1 + k_2 + k_3) - 4k_3 \leq 7k - 8$, which contradicts the assumption that $|V(G)| \geq 7k - 6$.

Case 2: $k_3 = 1$

In this case, recall that $k_2 \geq 1$ and that we have assumed $E(V(T_1), V(U_1)) \neq \emptyset$.

**Claim 3.** $q + |\{v \in V(H) \mid E(v, V(T)) \neq \emptyset\}| \leq 2k_1 + 3k_2 - 3$.
**Proof.** If $\{v \in V(H) \mid E(v, V(T_1)) \neq \emptyset\} = \emptyset$, then the claim immediately follows from Claims 1 and 2. Thus we may assume that $\{v \in V(H) \mid E(v, V(T_1)) \neq \emptyset\} \neq \emptyset$. In view of Lemma 2.8, we may assume that $\{v \in V(H) \mid E(v, V(T_1)) \neq \emptyset\}$ can be partitioned into two sets $A$ and $B$ (i.e., $A \cap B = \emptyset$ and $|A| + |B| = |\{v \in V(H) \mid E(v, V(T_1)) \neq \emptyset\}|$ ) such that

(i) there exists $S_i$ with $1 \leq i \leq k_1$ such that $|E(v, V(S_i))| \geq 2$ for each $v \in A$, and

(ii) there exists $T_j$ with $2 \leq j \leq k_2$ such that $E(v, V(T_j)) \neq \emptyset$ for each $v \in B$.

Then by Claims 1 and 2, it follows that

$$\begin{gathered} q + |\{v \in V(H) \mid E(v, V(T)) \neq \emptyset\}| \\ \leq (2k_1 - |A|) + \{3(k_2 - 1) - |B|\} + |\{v \in V(H) \mid E(v, V(T_1)) \neq \emptyset\}| \\ \leq 2k_1 + 3k_2 - 3. \end{gathered}$$

□

Consequently by Claim 3,

$$\begin{gathered} |V(G)| \leq 5k_1 + 4k_2 + 3k_3 + |V(H)| \\ \leq 5k_1 + 4k_2 + 3k_3 + q + |\{v \in V(H) \mid E(v, V(T)) \neq \emptyset\}| \\ \leq 5k_1 + 4k_2 + 3k_3 + 2k_1 + 3k_2 - 3 \\ = 7(k_1 + k_2 + k_3) - 4k_3 - 3 \leq 7k - 7, \end{gathered}$$

which contradicts the assumption that $|V(G)| \geq 7k - 6$.

This completes the proof of the Main Theorem. □

# References


[1] S.Fujita, Vertex-disjoint copies of $K_1 + (K_1 \cup K_2)$ in graphs, SUT Journal of Mathematics, **38** (2002) 201-209.

[2] S.Fujita, Disjoint stars and forbidden subgraphs, Hiroshima Mathematical Journal, in press.

[3] K.Kawarabayashi, $F$-factor and vertex-disjoint $F$ in a graph, Ars Combinatoria **62**(2002) 183-187.

[4] D.P.Sumner, 1-Factors and antifactor sets, J.London Math. Soc., **13** (1976) 351-359.

[5] H.Wang, Vertex-disjoint triangles in claw-free graphs with minimum degree at least three, Conbinatorica **18**(3) (1998) 441-447.