Existence of Two Disjoint Long Cycles in Graphs

Yoshimi Egawa
Department of Mathematical Information Science
Science University of Tokyo
1-3 Kagurazaka, Shinjuku-ku, Tokyo, 162-8601 Japan

Shinya Fujita
Department of Mathematics
Keio University
3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223-8522 Japan

Ken-ichi Kawarabayashi
Graduate School of Information Sciences(GSIS)
Tohoku University
Aramaki aza Aoba 09, Aoba-ku Sendai, Miyagi, 980-8579, Japan

Hong Wang
Department of Mathematics
The University of Idaho
Moscow, Idaho 83844, USA

Abstract

Let n, h be integers with $n \geq 6$ and $h \geq 7$. We prove that if G is a graph of order n with $\sigma_2(G) \geq h$, then G contains two disjoint cycles C_1 and C_2 such that $|V(C_1)| + |V(C_2)| \geq \min\{h, n\}$.

1 Introduction

In this paper, we consider only finite, simple, undirected graphs with no loops and no multiple edges. For a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. For a vertex x of a graph G, the neighborhood of x in G is denoted by $N_G(x)$, and we let $d_G(x) := |N_G(x)|$. Also let $\delta(G) := \min\{d_G(x) |\ x \in V(G)\}$. For a noncomplete graph G, let $\sigma_2(G)$:=min$\{d_G(x) + d_G(y) |\ xy \notin E(G)\}$; if G is a complete graph, let $\sigma_2(G) := \infty$. For an integer $n \geq 1$, we let K_n denote the complete graph of order n. In this paper, "disjoint" means "vertex-disjoint".

This paper is concerned with the existence of disjoint cycles. The following theorem was proved by H.Enomoto in [6] and by H.Wang in [9]:

Theorem 1. ([6, 9]) *Let k be an integer with $k \geq 2$. Let G be a graph of order at least $3k$, and suppose that $\sigma_2(G) \geq 4k-1$. Then G contains k disjoint cycles.*

We remark that complete bipartite graphs $K_{2k-1,m}$ with $m \geq 2k-1$ show that in Theorem 1, the condition $\sigma_2(G) \geq 4k-1$ is sharp. On the other hand, the following theorem had been proved by J.-C.Bermond in [1]:

Theorem 2. ([1]) *Let n, h be integers with $n \geq 3$ and $h \geq 4$. Let G be a 2-connected graph of order n, and suppose that $\sigma_2(G) \geq h$. Then G contains a cycle C with $|V(C)| \geq \min\{h, n\}$.*

In [8], combining Theorems 1 and 2, H.Wang made a conjecture that if k, n, d are integers with $k \geq 2, n \geq 3k$, and $d \geq 2k$, and if G is a graph of order n such that $\delta(G) \geq d$, then G contains k disjoint cycles such that $|V(C_1)|+\ldots+|V(C_k)| \geq \min\{2d, n\}$. As can be seen from complete bipartite graphs $K_{d,m}$ with $m \geq d$, in the conclusion of the conjecture, the lower bound $2d$ on $|V(C_1)|+\ldots+|V(C_k)|$ is best possible. For $k=2$, the conjecture was settled in [8]:

Theorem 3. ([8]) *Let n, d be integers with $n \geq 6$ and $d \geq 4$. Let G be a graph of order n, and suppose that $\delta(G) \geq d$. Then G contains two disjoint cycles C_1 and C_2 such that $|V(C_1)|+|V(C_2)| \geq \min\{2d, n\}$.*

For $k \geq 3$, the conjecture was settled in [3]; in fact, the following σ_2-version was proved:

Theorem 4. ([5]) *Let k, n, h be integers with $k \geq 3, n \geq 3k$, and $h \geq 4k-1$. Let G be a graph of order n, and suppose that $\sigma_2(G) \geq h$. Then G contains k disjoint cycles $C_1, \ldots, C_k$ such that $|V(C_1)|+\ldots+|V(C_k)| \geq \min\{h, n\}$.*

The purpose of this paper is to show that Theorem 4 holds for $k=2$ as well; that is to say, we prove the following σ_2-version of Theorem 4:

Main Theorem. Let n, h be integers with $n \geq 6$ and $h \geq 7$. Let G be a graph of order n, and suppose that $\sigma_2(G) \geq h$. Then G contains two disjoint cycles C_1 and C_2 such that $|V(C_1)|+|V(C_2)| \geq \min\{h, n\}$.

In the proof of the Main Theorem, we make use of the following two results (Lemma 5 is proved in [4]; Lemma 6 is essentially proved in [3], and can also be obtained by applying Theorem 2 to an appropriate endblock of the graph under consideration):

Lemma 5. ([4]) *Let $d \geq 3$ be an integer, and let G be a 2-connected graph of order at least 4. Let a, b be distinct vertices of G, and suppose that every vertex in $V(G) - \{a, b\}$, possibly except one, has degree at least d in G. Then G contains a path of length at least d joining a and b.*

Lemma 6. ([3]) *Let $d \geq 2$ be an integer. Let G be a graph of order at least 3, and suppose that every vertex of G, possibly except one, has degree at least d. Then G contains a cycle C such that $|V(C)| \geq d + 1$.*

We add that the case where $h = n$ and $h = n - 1$ of Theorem 4 had already been considered by S.Brandt et al. in [2] and by K.Kawarabayashi in [7], respectively:

Theorem 7. ([2]) *Let k, n be integers with $n \geq 4k - 1$. Let G be a graph of order n, and suppose that $\sigma_2(G) \geq n$. Then G contains k disjoint cycles C_i, $1 \leq i \leq k$, such that $V(C_1) \cup \ldots \cup V(C_k) = V(G)$.*

Theorem 8. ([7]) *Let k, n be integers with $k \geq 2$ and $n \geq 4k$. Let G be a graph of order n, and suppose that $\sigma_2(G) \geq n - 1$. Then one of the following holds:*

(i) *G contains k disjoint cycles C_i, $1 \leq i \leq k$, such that $V(C_1) \cup \ldots \cup V(C_k) = V(G)$;*

(ii) *G has a vertex set $S \subset V(G)$ with $|S| = \frac{n-1}{2}$ such that $G - S$ is independent; or*

(iii) *G is isomorphic to the graph obtained from K_{n-1} by adding a vertex and join it to precisely one vertex of K_{n-1} (i.e., G is isomorphic to $(K_{n-2} \cup K_1) + K_1$).*

Our notation is standard except possibly for the following. Let G be a graph. For a subset L of $V(G)$, the subgraph induced by L is denoted by $\langle L \rangle$. For a subset M of $V(G)$, we let $G - M = \langle V(G) - M \rangle$. For subsets L and M of $V(G)$ with $L \cap M = \emptyset$, we let $E(L, M)$ denote the set of edges of G joining a vertex in L and a vertex in M. A vertex x is often identified with the set $\{x\}$. Thus if $x \in V(G)$, then $G - x$ means $G - \{x\}$, and $E(x, M)$ means $E(\{x\}, M)$ for $M \subset V(G - x)$. Now let $C = x_1 x_2 \ldots x_{|V(C)|} x_1$ be a cycle. For a vertex $x = x_i \in V(C)$, we define $x^{+j} = x_{i+j}$ and $x^{-j} = x_{i-j}$ (indices are to be read modulo $|V(C)|$). For simplicity, we let $x^+ = x^{+1}, x^- = x^{-1}$. Also, for $x_i, x_j \in V(C)$ with $i \leq j < i + |V(C)|$, we let $C[x_i, x_j]$ denote the path $x_i x_{i+1} x_{i+2} \ldots x_j$ (indices are to be read modulo $|V(C)|$).

2 Proof of the Main Theorem

Let n, h, G be as in the Main Theorem. If $n = 6$, then the desired conclusion immediately follows from Theorem 1. Thus we may assume $n \geq 7$. Then in view of the desired conclusion, we may also assume $h \leq n$. Thus, consider $n \geq h \geq 7$. Now if $n = h$ or $h+1$, then applying Theorem 7 or 8 with $k = 2$, we see that G contains cycles with the desired properties. Consequently we may further assume $n \geq h + 2$. First assume that G is disconnected. Let H_1, H_2 be two components of G. Then $\delta(H_1) + \delta(H_2) \geq h$. We may assume $\delta(H_1) \geq \delta(H_2)$. If $\delta(H_2) \leq 1$, then $\delta(H_1) \geq h - 1 > \lceil h/2 \rceil$ and $|V(H_1)| \geq \delta(H_1) + 1 \geq h$, and hence we obtain the desired conclusion by applying Theorem 3 to H_1. Thus we may assume $\delta(H_2) \geq 2$. Then by Lemma 6, H_i contains a cycle C_i with $|V(C_i)| \geq \delta(H_i) + 1$ for each $i = 1, 2$, and C_1 and C_2 are cycles with the desired properties. Next assume that G is connected but not 2-connected. Let B_1, B_2 be two endblocks of G. For each $i = 1, 2$, let z_i be the cut vertex of G lying in B_i(it is possible that $z_1 = z_2$) and set $d_i := \min\{d_G(x) | x \in V(B_i - z_i)\}$. Then $d_1 + d_2 \geq h$. We may assume $d_1 \geq d_2$. If $d_2 \leq 1$, then $d_1 \geq h - 1, |V(B_1)| \geq d_1 + 1 \geq h$ and $\sigma_2(B_1) \geq d_1 + 2 > h$, and hence we obtain the desired conclusion by applying to B_1 the result for the case where G is 2-connected (note that our proof for the case where G is 2-connected does not depend on the result for the case where G is not 2-connected). Thus we may assume $d_2 \geq 2$. Then by Lemma 6 B_2 contains a cycle C_2 with $|V(C_2)| \geq d_2 + 1$. Note also that $\delta(B_1 - z_1) \geq d_1 - 1 \geq h/2 - 1 > 2$. Consequently $B_1 - z_1$ contains a cycle C_1 with $|V(C_1)| \geq d_1$, and C_1 and C_2 are cycles with the desired properties.

We henceforth assume that G is 2-connected. Suppose further that G is a counterexample to the Main Theorem. We distinguish two cases: the case where $n \geq h + 3$ and $h \geq 11$, and the case where $n = h + 2$ or $7 \leq h \leq 10$.

Case 1: $n \geq h + 3$ and $h \geq 11$

Claim 2.1. $\delta(G) \geq 3$.

Proof. Since G is 2-connected, $\delta(G) \geq 2$. Suppose that there exists a $v \in V(G)$ such that $d_G(v) = 2$. Then by the assumption that $\sigma_2(G) \geq h$, $d_{G-N_G(v)-v}(x) \geq d_G(x) - 2 \geq (h-2) - 2 \geq h/2$ for each $x \in V(G - N_G(v) - v)$, and we also have $|V(G - N_G(v) - v)| \geq n - 3 \geq h$. Hence by Theorem 3, $G - N_G(v) - v$ contains cycles with the required properties, which contradicts the assumption that G is a counterexample. □

Let C be a longest cycle in G. Then $|V(C)| \geq h$ by Theorem 2.

Subcase 1.1: $|V(C)| = n$

Write $C = v_1 v_2 \dots v_n v_1$. Set $d = \lceil (h+1)/2 \rceil$. First we prove two claims.

Claim 2.2. *Suppose that $d_G(v_1) \geq d$ and $d_G(v_n) \geq d$, and write $N_{G-E(C)}(v_1) = \{v_{i_1}, \dots, v_{i_p}\}$ $(3 \leq i_1 < i_2 < \dots < i_p \leq n-1)$, $N_{G-E(C)}(v_n) = \{v_{j_1}, \dots, v_{j_q}\}$ $(2 \leq j_1 < j_2 < \dots < j_q \leq n-2)$. Further suppose that $i_1 < j_q$. Then $d_G(v_1) = d_G(v_n) = d$ (hence $p = q = d-2$), and one of the following holds:*

(i) *there exists an integer s with $2 \leq s \leq d-3$ such that $\{i_1, \dots, i_{d-2}\} = \{3, 4, \dots, d-s\} \cup \{n-s, n-s+1, \dots, n-1\}$, $\{j_1, \dots, j_{d-2}\} = \{n-s-d+3, n-s-d+4, \dots, n-s\}$;*

(ii) *there exists an integer r with $2 \leq r \leq d-3$ such that $\{j_1, \dots, j_{d-2}\} = \{2, 3, \dots, r+1\} \cup \{n-d+r+1, n-d+r+2, \dots, n-2\}$, $\{i_1, \dots, i_{d-2}\} = \{r+1, r+2, \dots, d+r-2\}$; or*

(iii) *$\{i_1, \dots, i_{d-3}\} = \{3, 4, \dots, d-1\}$, $i_{d-2} \in \{n-2, n-1\}$, $j_1 \in \{2, 3\}$, $\{j_2, \dots, j_{d-2}\} = \{n-d+2, n-d+3, \dots, n-2\}$.*

Proof. Note that $p = d_G(v_1) - 2 \geq d-2$ and $q = d_G(v_n) - 2 \geq d-2$. Let $I = \{i_1, \dots, i_p\}$, $J = \{j_1, \dots, j_q\}$. Take $k \in I$ and $l \in J$ with $k < l$ so that $l - k$ is as small as possible. Let $L_1 = \{1, 2, \dots, k\}, L_2 = \{l, l+1, \dots, n\}$. Since $C[v_1, v_k]v_1$ and $C[v_l, v_n]v_l$ are disjoint cycles,

$$|L_1| + |L_2| \leq h - 1 \tag{1}$$

by the assumption that G is a counterexample. Since $n \geq h+3$, this implies

$$l - k \geq 5. \tag{2}$$

Note that $I \cup J \subseteq L_1 \cup L_2$ by the minimality of $l-k$. Let $I_1 = I \cap L_1, I_2 = I \cap L_2, J_1 = J \cap L_1$, and $J_2 = J \cap L_2$. Then

$$|I_1| + |I_2| = p \geq d-2,\ |J_1| + |J_2| = q \geq d-2. \tag{3}$$

Note that $k \in I_1$ and $l \in J_2$. For convenience, for a set X of integers and a positive integer t, we let $X^{+t} = \{i+t |\ i \in X\}$ and $X^{-t} = \{i-t |\ i \in X\}$, and write X^+ and X^- for X^{+1} and X^{-1}, respectively. By (2),

$$J_1^{-t} \cap I_1 = \emptyset,\ I_2^{+t} \cap J_2 = \emptyset. \tag{4}$$

for each t with $1 \leq t \leq 4$. Note that $n \notin I_2$ and $1 \notin J_1$ by the definition of I and J. Thus applying (4) with $t = 1$, we get

$$L_1 = I_1 \cup J_1^- \cup (L_1 - I_1 - J_1^-) \text{ (disjoint union)}, \tag{5}$$

$$L_2 = I_2^+ \cup J_2 \cup (L_2 - I_2^+ - J_2) \text{ (disjoint union)}. \tag{6}$$

Since $|I_1|+|J_1|+|I_2|+|J_2| \geq 2(d-2) \geq h-3$ by (3), it follows from (1),(5),(6) that

$$|L_1 - I_1 - J_1^-| + |L_2 - I_2^+ - J_2| \leq 2. \tag{7}$$

We here prove three subclaims.

Subclaim 1. *The following statements hold:*

(i) *Let $i \in I_1$, $j \in J_1$. Then $i \geq j$.*
(ii) *Let $i \in I_2$, $j \in J_2$. Then $i \geq j$.*

Proof. Suppose that there exist $i \in I_1, j \in J_1$ such that $i < j$. We may assume that i and j are chosen so that $j - i$ is as small as possible. By (4), $j - i \geq 5$. Since $\{i+1, i+2, \ldots, j-2\} \subseteq L_1 - I_1 - J_1^-$ by the minimality of $j - i$, this implies $|L_1 - I_1 - J_1^-| \geq 3$, which contradicts (7). Thus (i) is proved, and we can similarly prove (ii). □

Subclaim 2. *The following statements hold:*

(i) *Suppose that $|J_1| \geq 2$ and, in the case where $|J_1| = 2$, suppose further that $|I_1| \geq 2$. Then $I_2 = \emptyset$.*

(ii) *Suppose that $|I_2| \geq 2$ and, in the case where $|I_2| = 2$, suppose further that $|J_2| \geq 2$. Then $J_1 = \emptyset$.*

Proof. Let

$$l_1 = \min J_1,$$
$$l_2 = \begin{cases} \max J_1 & (\max J_1 \neq k) \\ \max (J_1 - \{k\}) & (\max J_1 = k). \end{cases}$$

If $\max J_1 \neq k$, then by the assumption that $|J_1| \geq 2$, $l_2 = \max J_1 > \min J_1 = l_1$; if $\max J_1 = k$, then $I_1 = \{k\}$ by Subclaim 1(i), which implies $|J_1| \geq 3$ by assumption, and hence $l_2 = \max (J_1 - \{k\}) > \min (J_1 - \{k\}) = l_1$. Let $k_1 = \min (I_1 - \{l_2\})$ (recall that $k \in I_1$). Now suppose that $I_2 \neq \emptyset$, and let $k_2 = \max I_2$. If $k_1 \notin J_1$, then we clearly have $\{i | \; l_2 + 1 \leq i \leq k_1 - 1\} \subseteq L_1 - I_1 - J_1^-$; if $k_1 \in J_1$, then ($k_1 = k$ and) $\{i | \; l_2 \leq i \leq k_1 - 2\} \subseteq L_1 - I_1 - J_1^-$ by Subclaim 1(i). In either case, $|\{i | \; l_2 \leq i \leq k_1 - 1\} \cap (L_1 - I_1 - J_1^-)| \geq k_1 - l_2 - 1$. We also have $\{i | \; 1 \leq i \leq l_1 - 2\} \subseteq L_1 - I_1 - J_1^-$, and $\{i | \; k_2 + 2 \leq i \leq n\} \subseteq L_2 - I_2^+ - J_2$ by Subclaim 1. Consequently $(l_1 - 2) + (k_1 - l_2 - 1) + (n - k_2 - 1) \leq 2$ by (7), and hence

$$k_2 - k_1 + l_2 - l_1 \geq n - 6. \tag{8}$$

On the other hand, $v_1 C[v_{k_1}, v_{k_2}] v_1$ and $v_n C[v_{l_1}, v_{l_2}] v_n$ are disjoint cycles. But since $n \geq h+3$, it follows from (8) that $|V(v_1 C[v_{k_1}, v_{k_2}] v_1)| + |V(v_n C[v_{l_1}, v_{l_2}] v_n)| = (k_2 - k_1 + 2) + (l_2 - l_1 + 2) \geq n - 2 > h$, which contradicts the assumption that G is a counterexample. Thus (i) is proved, and we can similarly prove (ii). □

Subclaim 3. (i) *If $|J_1| \geq 2$, then $I_2 = \emptyset$.* (ii) *If $|I_2| \geq 2$, then $J_1 = \emptyset$.*

Proof. Suppose that $|J_1| \geq 2$ and $I_2 \neq \emptyset$. Then by Subclaim 2(i), ($|J_1| = 2$ and) $|I_1| \leq 1$. Since $|I| = p \geq d - 2 = \lceil (h+1)/2 \rceil - 2 \geq 4$, this implies $|I_2| \geq 3$, and hence $J_1 = \emptyset$ by Subclaim 2(ii), which contradicts the assumption that $|J_1| \geq 2$. Thus (i) is proved, and (ii) can be verified in a similar way. $\square$

We are now in a position to complete the proof of Claim 2.2. By Subclaim 3, $|J_1 \cap \{2,3\}| + |I_2 \cap \{n-2, n-1\}| \leq 2$, and hence

$$|J_1^- \cap \{1,2\}| + |I_2^+ \cap \{n-1, n\}| \leq 2. \tag{9}$$

On the other hand, since $\{1,2\} \cap I = \emptyset$ and $\{n-1,n\} \cap J = \emptyset$ by the definition of I and J, we have $\{1,2\} - J_1^- \subseteq L_1 - I_1 - J_1^-$ and $\{n-1,n\} - I_2^+ \subseteq L_2 - I_2^+ - J_2$, and hence it follows from (7) that

$$|\{1,2\} - J_1^-| + |\{n-1,n\} - I_2^+| \leq 2 \tag{10}$$

Consequently equality holds in (9) and (10). The equality in (10) implies that equality holds in (7) and, in view of (5) and (6), it also implies

$$L_1 - \{1,2\} \subseteq I_1 \cup J_1^-,\ L_2 - \{n-1,n\} \subseteq I_2^+ \cup J_2. \tag{11}$$

The equality in (7) together with (1),(3),(5),(6) implies that $p = d-2$ and $q = d-2$. Now if $J_1^- \supseteq \{1,2\}$, then $I_2^+ = \emptyset$ by Subclaim 3(i), and hence it follows from (11) and Subclaim 1(i) that (ii) of the claim holds. Similarly if $I_2^+ \supseteq \{n-1,n\}$, then (i) holds. Finally if $|J_1^- \cap \{1,2\}| = |I_2^+ \cap \{n-1,n\}| = 1$, then $J_1^- \cap (L_1 - \{1,2\}) = I_2^+ \cap (L_2 - \{n-1,n\}) = \emptyset$ by Subclaim 3, and hence it follows from (11) that (iii) holds. This completes the proof of Claim 2.2. $\square$

Claim 2.3. *Let $I, J, p, q, i_1, \ldots, i_p, j_1, \ldots, j_q$ be as in Claim 2.2, and suppose that $d_G(v_1) \geq d, d_G(v_n) \geq d$ and $i_1 \geq j_q$. Then $\min\{d_G(v_1), d_G(v_n)\} = d$, and one of the following holds:*

(i) *$d_G(v_1) = d_G(v_n) = d$, $\{i_1, \ldots, i_p\} = \{i_1, i_1+1, \ldots, i_1+d-3\}$, and $\{j_1, \ldots, j_q\} = \{j_q-d+3, j_q-d+4, \ldots, j_q\}$;*

(ii) *$i_1 = j_q, d_G(v_1) = d+1, d_G(v_n) = d, \{i_1, \ldots, i_p\} = \{i_1, i_1+1, \ldots, i_1+d-2\}$, and $\{j_1, \ldots, j_q\} = \{j_q-d+3, j_q-d+4, \ldots, j_q\}$;*

(iii) *$i_1 = i_q, d_G(v_1) = d, d_G(v_n) = d+1, \{i_1, \ldots, i_p\} = \{i_1, i_1+1, \ldots, i_1+d-3\}$, and $\{j_1, \ldots, j_q\} = \{j_q-d+2, j_q-d+3, \ldots, j_q\}$;*

(iv) *$i_1 = j_q, d_G(v_1) = d_G(v_n) = d$, there exists an integer r with $i_1+1 \leq r \leq i_1+d-3$ such that $\{i_1, \ldots, i_p\} = \{i_1, i_1+1, \ldots, i_1+d-2\} - \{r\}$, and $\{j_1, \ldots, j_q\} = \{j_q-d+3, j_q-d+4, \ldots, j_q\}$;*

(v) *$i_1 = i_q, d_G(v_1) = d_G(v_n) = d$, $\{i_1, \ldots, i_p\} = \{i_1, i_1+1, \ldots, i_1+d-3\}$, and there exists an integer s with $j_q-d+3 \leq s \leq j_q-1$ such that $\{j_1, \ldots, j_q\} = \{j_q-d+2, j_q-d+3, \ldots, j_q\} - \{s\}$; or*

(vi) $i_1 = j_q, d_G(v_1) = d_G(v_n) = d, \{i_1, \ldots, i_p\} = \{i_1, i_1+2, i_1+3, \ldots, i_1 + d-2\}$, *and* $\{j_1, \ldots, j_q\} = \{j_q - d+2, j_q - d+3, \ldots, j_q - 2, j_q\}$.

Proof. As in Claim 2.2, we have

$$p \geq d-2 \text{ and } q \geq d-2. \tag{12}$$

We first consider the case where $i_1 \neq j_q$. In this case, note that $v_1C[v_{i_1}, v_{i_p}]v_1$ and $v_nC[v_{j_1}, v_{j_q}]v_n$ are disjoint cycles, and $|V(v_1C[v_{i_1}, v_{i_p}]v_1)| = i_p - i_1 + 2$ and $|V(v_nC[v_{j_1}, v_{j_q}]v_n)| = j_q - j_1 + 2$. Hence $(i_p - i_1) + (j_q - j_1) + 4 \leq h-1$ by the assumption that G is a counterexample. Since we clearly have $i_p - i_1 \geq p-1$ and $j_q - j_1 \geq q-1$, and since $p+q \geq 2(d-2) \geq h-3$ by (12), this forces $p = q = d-2, i_p - i_1 = p-1$ and $j_q - j_1 = q-1$, and hence (i) holds.

We now consider the case where $i_1 = j_q$. By symmetry, we may assume $i_2 - i_1 \leq j_q - j_{q-1}$. Note that $v_1C[v_{i_2}, v_{i_p}]v_1$ and $v_nC[v_{j_1}, v_{j_q}]v_n$ are disjoint cycles. Hence

$$(i_p - i_2) + (j_q - j_1) + 4 \leq h-1 \tag{13}$$

On the other hand, we clearly have

$$i_p - i_2 \geq p-2, \tag{14}$$

$$j_q - j_1 \geq q-1. \tag{15}$$

Assume for the moment that $i_2 - i_1 \geq 2$. Then $j_q - j_{q-1} \geq 2$, and hence $j_q - j_1 \geq q$. By (14),(12) and (13), this implies $p = q = d-2$, $i_p - i_2 = p-2$ and $j_q - j_1 = q$, and hence $j_q - j_{q-1} = i_2 - i_1 = 2$. Consequently (vi) holds. Thus we may assume $i_2 - i_1 = 1$. If equality holds in both (14) and (15), then by (12) and (13), we have $p = q = d-2$, or $p = d-1$ and $q = d-2$, or $p = d-2$ and $q = d-1$, and hence (i), (ii) or (iii) holds. Thus we may assume $i_p - i_2 \geq p-1$ or $j_q - j_1 \geq q$. Then by (14), (15), (12) and (13), $p = q = d-2$, and we have $i_p - i_2 = p-1$ and $j_q - j_1 = q-1$, or $i_p - i_2 = p-2$ and $j_q - j_1 = q$. Since $i_2 - i_1 = 1$, this implies that (iv) or (v) holds. □

We return to the proof of the Main Theorem. Set $d' = \lfloor (h-1)/2 \rfloor$. Then we have $\delta(G) \leq d'$ by Theorem 3 and the assumption that G is a counterexample. Take a vertex $a \in V(G)$ such that $d_G(a) = \delta(G)$. Since $d_G(a) \leq d'$, there exists a $u \in V(C)$ such that $\{u, u^+\} \cap (N_G(a) \cup \{a\}) = \emptyset$. We may assume $u = v_n$ and $u^+ = v_1$. Since $d_G(a) \leq d'$, we have $\min\{d_G(v_1), d_G(v_n)\} \geq h - d_G(a) \geq h - d' = d$. Hence by Claims 2.2 and 2.3 , $\min\{d_G(v_1), d_G(v_n)\} = d$. This implies $\delta(G) = d_G(a) \geq h - d = d'$, and hence

$$\delta(G) = d' \tag{16}$$

(we do not make use of the vertex a in the rest of the discussion for Subcase 1.1). Let $i_1, \ldots, i_p, j_1, \ldots, j_q$ be as in Claims 2.2 and 2.3.

Subcase 1.1.1. $i_1 < j_q$

We first consider the case where (i) or (ii) of Claim 2.2 holds. By symmetry, we may assume (i) holds. Let s be as in Claim 2.2 (i). We show that

$$V(C[v_4, v_{n-2}] - v_{d-s}) \cap N_G(v_2) = \emptyset. \tag{17}$$

Suppose that there exists a $v_k \in V(C[v_4, v_{n-2}] - v_{d-s}) \cap N_G(v_2)$. Set $C' = C[v_2, v_k]v_2$. If $4 \le k \le d-s-1$ or $n-s \le k \le n-2$, C' and $v_1C[v_{k+1}, v_1]$ are disjoint cycles with $|V(C')| + |V(v_1C[v_{k+1}, v_1])| = n > h$; if $d-s+1 \le k \le n-s-d+2$, C' and $v_nC[v_{n-s-d+3}, v_{n-1}]v_1v_n$ are disjoint cycles with $|V(C')| + |V(v_nC[v_{n-s-d+3}, v_{n-1}]v_1v_n)| = (k-1) + (s+d-1) \ge (d-s) + (s+d-1) = 2d-1 \ge h$; if $n-s-d+3 \le k \le n-s-1$, C' and $v_nC[v_{k+1}, v_{n-1}]v_1v_n$ are disjoint cycles with $|V(C')| + |V(v_nC[v_{k+1}, v_{n-1}]v_1v_n)| = n > h$. In any case, we get a contradiction to the assumption that G is a counterexample. Thus (17) is proved. Further if $v_{n-1}, v_n \in N_G(v_2)$, then $v_2v_{n-1}v_nv_2$ and $v_1C[v_3, v_{n-2}]v_1$ are disjoint cycles with $|V(v_2v_{n-1}v_nv_2)| + |V(v_1C[v_3, v_{n-2}]v_1)| = n > h$, a contradiction. Thus

$$|\{v_{n-1}, v_n\} \cap N_G(v_2)| \le 1. \tag{18}$$

Now by (17) and (18), $d_G(v_2) \le |\{v_{n-1}, v_n\} \cap N_G(v_2)| + |\{v_1, v_3, v_{d-s}\}| \le 4$. But since $d' = \lfloor (h-1)/2 \rfloor \ge 5$, this contradicts (16).

We now consider the case where Claim 2.2 (iii) holds. We show that

$$V(C[v_4, v_{n-3}] - v_{d-1}) \cap N_G(v_2) = \emptyset. \tag{19}$$

Suppose that there exists a $v_k \in V(C[v_4, v_{n-3}] - v_{d-1}) \cap N_G(v_2)$. Set $C' = C[v_2, v_k]v_2$. If $4 \le k \le d-2$, C' and $v_1C[v_{k+1}, v_1]$ are disjoint cycles with $|V(C')| + |V(v_1C[v_{k+1}, v_1])| = n > h$; if $d \le k \le n-d+1$, $v_1C[v_3, v_k]v_2v_1$ and $v_nC[v_{n-d+2}, v_n]$ are disjoint cycles with $|V(v_1C[v_3, v_k]v_2v_1)| + |V(v_nC[v_{n-d+2}, v_n])| \ge k+d-1 \ge 2d-1 \ge h$; if $n-d+2 \le k \le n-3$, C' and $v_nC[v_{k+1}, v_n]$ are disjoint cycles with $|V(C')| + |V(v_nC[v_{k+1}, v_n])| = n-1 > h$. In any case, we get a contradiction. Thus (19) is proved. Further we show that

$$|\{v_{d-1}, v_{n-2}, v_{n-1}\} \cap N_G(v_2)| \le 1. \tag{20}$$

Suppose that $|\{v_{d-1}, v_{n-2}, v_{n-1}\} \cap N_G(v_2)| \ge 2$. If $v_{n-2}, v_{n-1} \in N_G(v_2)$, $v_2v_{n-2}v_{n-1}v_2$ and $v_1C[v_3, v_{n-3}]v_nv_1$ are disjoint cycles with $|V(v_2v_{n-2}v_{n-1}v_2)| + |V(v_1C[v_3, v_{n-3}]v_nv_1)| = n > h$, a contradiction. Thus we have $v_{d-1} \in N_G(v_2)$ and $|\{v_{n-2}, v_{n-1}\} \cap N_G(v_2)| = 1$. Write $\{v_{n-2}, v_{n-1}\} \cap N_G(v_2) = \{v_m\}$. Then since $d = \lceil (h+1)/2 \rceil \ge 6$, $v_2C[v_{d-1}, v_m]v_2$ and $v_1C[v_3, v_{d-2}]v_1$ are disjoint cycles with $|V(v_2C[v_{d-1}, v_m]v_2)| + |V(v_1C[v_3, v_{d-2}]v_1)| \ge n-2 > h$, a contradiction. Thus (20) is proved. Now by (19) and (20), $d_G(v_2) \le |\{v_{d-1}, v_{n-2}, v_{n-1}\} \cap N_G(v_2)| + |\{v_1, v_3, v_n\}| \le 4$, which contradicts (16).

Subcase 1.1.2. $i_1 \ge j_q$

We first consider the case where $j_1 \ge 3$. We show that

$$V(C[v_4, v_{n-1}] - v_{i_p}) \cap N_G(v_2) = \emptyset. \tag{21}$$

Suppose that there exists a $v_k \in V(C[v_4, v_{n-1}] - v_{i_p}) \cap N_G(v_2)$. Set $C' = C[v_2, v_k]v_2$. If $4 \le k \le j_1 - 1$, C' and $v_nC[v_{j_1}, v_n]$ are disjoint cycles with $|V(C')| + |V(v_nC[v_{j_1}, v_n])| \ge 3 + ((p+q-1)+1) \ge 2d-1 \ge h$, a contradiction; if $j_q \le k \le i_1 - 1$, then since $j_1 \ge 3$, $v_2C[v_k, v_{i_p}]v_1v_2$ and $v_nC[v_{j_1}, v_{j_{q-1}}]v_n$ are disjoint cycles with $|V(v_2C[v_k, v_{i_p}]v_1v_2)| + |V(v_nC[v_{j_1}, v_{j_{q-1}}]v_n)| \ge (p+3) + q \ge$

h, a contradiction; if $i_p+1 \leq k \leq n-1$, then since $j_1 \geq 3$, $v_1C[v_{i_1}, v_k]v_2v_1$ and $v_nC[v_{j_1}, v_{j_{q-1}}]v_n$ are disjoint cycles with $|V(v_1C[v_{i_1}, v_k]v_2v_1)|+|V(v_nC[v_{j_1}, v_{j_{q-1}}]v_n)| \geq (p+3)+q \geq h$, a contradiction. Thus we have $j_1 \leq k \leq j_q - 1$ or $i_1 \leq k \leq i_p-1$. First assume $j_1 \leq k \leq j_q-1$. Then by Claim 2.3, $\{v_k, v_{k+1}\} \cap N_G(v_n) \neq \emptyset$. Take $v_l \in \{v_{k+1}, v_{k+2}\} \cap N_G(v_n)$. Then C' and $v_nC[v_l, v_n]$ are disjoint cycles with $|V(C')| + |V(v_nC[v_l, v_n])| \geq n-2 > h$, a contradiction. Next assume $i_1 \leq k \leq i_p - 1$. Then by Claim 2.3, $\{v_{k+1}, v_{k+2}\} \cap N_G(v_1) \neq \emptyset$. Take $v_m \in \{v_{k+1}, v_{k+2}\} \cap N_G(v_1)$. Then C' and $v_1C[v_m, v_1]$ are disjoint cycles with $|V(C')| + |V(v_1C[v_m, v_1])| \geq n-1 > h$, a contradiction. Thus (21) is proved. Now by (21), $d_G(v_2) \leq |\{v_1, v_3, v_{i_p}, v_n\}| \leq 4$, which contradicts (16).

We now consider the case where $j_1 = 2$. If $i_p \leq n-2$, then by considering $N_G(v_{n-1})$ in place of $N_G(v_2)$, we can argue as in the preceding paragraph to get a contradiction. Thus we may assume $i_p = n-1$. This implies that (i) of Claim 2.3 holds (so $j_q = d-1$ and $i_1 = n-d+2$). We show that

$$V(C[v_4, v_{n-2}]) \cap N_G(v_2) = \emptyset. \tag{22}$$

Suppose that there exists a $v_k \in V(C[v_4, v_{n-2}]) \cap N_G(v_2)$. Set $C' = C[v_2, v_k]v_2$. If $4 \leq k \leq j_q - 1$, C' and $v_nC[v_{k+1}, v_n]$ are disjoint cycles with $|V(C')| + |V(v_nC[v_{k+1}, v_n])| = n-1 > h$; if $j_q \leq k \leq i_1 - 2$, then since $q = d-2 \geq 4$, $v_2C[v_k, v_{i_p}]v_1v_2$ and $v_nC[v_{j_2}, v_{j_{q-1}}]v_n$ are disjoint cycles with $|V(v_2C[v_k, v_{i_p}]v_1v_2)| + |V(v_nC[v_{j_2}, v_{j_{q-1}}]v_n)| \geq (p+4)+(q-1) \geq h$; if $i_1 - 1 \leq k \leq n-2$, C' and $v_1C[v_{k+1}, v_1]$ are disjoint cycles with $|V(C')| + |V(v_1C[v_{k+1}, v_1])| = n > h$. In any case, we get a contradiction. Thus (22) is proved. Now by (22), $d_G(v_2) \leq |\{v_1, v_3, v_{n-1}, v_n\}| = 4$, which contradicts (16). This completes the proof for Subcase 1.1.

Subcase 1.2: $|V(C)| < n$

Write $C = v_1v_2 \ldots v_{n'}v_1 (n' = |V(C)|)$. If $G - V(C)$ contains a cycle D, then since $|V(C)| \geq h$, C and D satisfy the properties required in the Main Theorem, which contradicts the assumption that G is a counterexample. Thus $G - V(C)$ is a forest. We divide the proof further into two cases according as $G - V(C)$ is isomorphic to K_1 or K_2, or not.

Subcase 1.2.1: $G - V(C)$ is isomorphic to neither K_1 nor K_2.

In this case, by the assumption that $\sigma_2(G) \geq h$, there exist $a, b \in V(G) - V(C)$ with $a \neq b$ such that

$$d_{G-V(C)}(a) \leq 1, d_{G-V(C)}(b) \leq 1 \text{ and } d_G(a) + d_G(b) \geq h. \tag{23}$$

If possible, we choose a and b so that they lie in the same component of $G - V(C)$ (we allow $ab \in E(G)$). Since $\min\{|N_G(a) \cap V(C)|, |N_G(b) \cap V(C)|\} \geq \delta(G) - 1 \geq 2$ by Claim 2.1, and $\max\{|N_G(a) \cap V(C)|, |N_G(b) \cap V(C)|\} \geq \lceil h/2 \rceil - 1 \geq 5$ by (23) and the assumption of Case 1, there exist four distinct vertices $u, v, u', v' \in V(C)$ such that u, v, u', v' occur on C in this order, $u, v \in N_G(a)$, and $u', v' \in N_G(b)$. We may assume that we have chosen u, v, u', v' so that $|V(C[v, u'])| + |V(C[v', u])|$ is as small as possible. Then $(N_G(a) \cup N_G(b)) \cap ((V(C[v, u']) - \{v, u'\}) \cup (V(C[v', u]) - \{v', u\})) = \emptyset$. If $|E(C[u, v])| + |E(C[u', v'])| \geq h - 4$, then the two cycles $aC[u, v]a$ and $bC[u', v']b$ satisfy the required properties. Thus

$$|E(C[u, v])| + |E(C[u', v'])| \leq h - 5. \tag{24}$$

We first consider the case where a and b lie in the same component of $G - V(C)$. Write $(N_G(a) \cup N_G(b)) \cap (V(C[u, v])) = \{v_{i_1}, \ldots, v_{i_s}\}$ and $(N_G(a) \cup N_G(b)) \cap (V(C[u', v'])) = \{v_{i_{s+1}}, \ldots, v_{i_{s+t}}\}$. We may assume $i_1 < i_2 < \ldots < i_{s+t}$ (thus $u = v_{i_1}, v = v_{i_s}, u' = v_{i_{s+1}}, v' = v_{i_{s+t}}$). By the maximality of $|V(C)|$, $i_{l+1} - i_l \geq 2$ for each $1 \leq l \leq s+t-1$, and hence $|E(C[u, v])| + |E(C[u', v'])| \geq 2(s-1) + 2(t-1)$. By (24), this implies

$$s + t \leq (h - 1)/2. \tag{25}$$

On the other hand, $s + t = |(N_G(a) \cup N_G(b)) \cap V(C)| = |N_G(a) \cap V(C)| + |N_G(b) \cap V(C)| - |N_G(a) \cap N_G(b) \cap V(C)| \geq h - 2 - |N_G(a) \cap N_G(b) \cap V(C)|$ by (23), and hence

$$|N_G(a) \cap N_G(b) \cap V(C)| \geq (h - 3)/2 \tag{26}$$

by (25). It follows from (25) and (26) that for each $1 \leq l \leq s+t-1$, we have $v_{i_l} \in N_G(a) \cap N_G(b)$ or $v_{i_{l+1}} \in N_G(a) \cap N_G(b)$, and hence $i_{l+1} - i_l \geq 3$ by the maximality of $|V(C)|$. Since $s+t = |(N_G(a) \cup N_G(b)) \cap V(C)| \geq \max\{|N_G(a) \cap V(C)|, |N_G(b) \cap V(C)|\} \geq h/2 - 1$ by (23), this implies that $|E(C[u, v])| + |E(C[u', v'])| \geq 3(s-1) + 3(t-1) \geq 3(h/2 - 3) > h - 4$, which contradicts (24).

We now consider the case where a and b lie in distinct components of $G - V(C)$. In this case, our choice of a and b implies that each component of $G - V(C)$ is isomorphic to K_1 or K_2. Our choice of a and b also implies that each component of $G - V(C)$ which is isomorphic to K_2 contains a vertex whose degree in G is strictly less than $h/2$. Consequently there exists a component F of $G - V(C)$ such that $F \cong K_1$ and the unique vertex x of F satisfies $d_G(x) \geq h/2$. We may assume $a = x$. Then by the maximality of $|V(C)|$, $|E(C[u, v])| + |E(C[u', v'])| \geq 2(|N_G(a) \cap V(C[u, v])| - 1) + 2(|N_G(a) \cap V(C[u', v'])| - 1) = 2d_G(a) - 4 \geq h - 4$, which contradicts (24).

Subcase 1.2.2. $G - V(C)$ is isomorphic to K_1 or K_2.

If $G - V(C) \cong K_1$, let $V(G) - V(C) = \{a\}$, and write $N_G(a) \cap V(C) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_p}\}$ with $i_1 < i_2 < \ldots < i_p$; if $G - V(C) \cong K_2$, let $V(G) - V(C) = \{a, b\}$, and write $(N_G(a) \cup N_G(b)) \cap V(C) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_p}\}$ with $i_1 < i_2 < \ldots < i_p$. By Claim 2.1 , $p \geq d_G(a) - 1 \geq 2$. Set $I = \{i_1, \ldots, i_p\}$. We may assume $i_1 = 1$. Then $n' \notin I$ by the maximality of $|V(C)|$. Write $N_G(v_{n'}) \cap V(C) = \{v_1, v_{j_2}, \ldots, v_{j_{q-1}}, v_{n'-1}\}$ with $1 < j_2 < j_3 < \ldots < j_{q-1} < n'$, and set $J = \{1, j_2, \ldots, j_{q-1}, n' - 1\}$. Since $n' \notin I, q = d_G(v_{n'}) \geq 3$ by

Claim 2.1. By the assumption that $\sigma_2(G) \geq h$, we also get

$$|I| + |J| = p + q \geq d_G(a) - 1 + d_G(v_{n'}) \geq h - 1. \qquad (27)$$

As in Subcase 1.1, for a set X of integers, we let $X^+ = \{i+1 | i \in X\}$ and $X^- = \{i-1 | i \in X\}$. By the maximality of $|V(C)|$,

$$I \cap J^+ = \emptyset. \qquad (28)$$

We first consider the case where $i_2 < j_{q-1}$. Choose $k \in I$ and $l \in J$ with $i_2 \leq k < l \leq j_{q-1}$ so that $l - k$ is as small as possible. Set $L_1 = \{1, 2, \dots, k\}$, $L_2 = \{l, l+1, \dots, n'\}$. Then $I \cup J \subseteq L_1 \cup L_2$ by the minimality of $l - k$. By (28), $(I \cap L_1)^- \cap (J \cap L_1) = \emptyset$, and hence $|L_1| \geq |(I \cap L_1)^- - \{0\}| + |J \cap L_1| \geq |I \cap L_1| + |J \cap L_1| - 1$. Again by (28), $(I \cap L_2) \cap (J \cap L_2)^+ = \emptyset$. Since $n' \notin J$ by the definition of J, this implies $|L_2| \geq |I \cap L_2| + |J \cap L_2|$. Consequently

$$|L_1| + |L_2| \geq |I| + |J| - 1 \geq h - 2. \qquad (29)$$

On the other hand, since $1, k \in I$ and $l \in J$, $\langle V(C[v_1, v_k]) \cup (V(G) - V(C)) \rangle$ contains a cycle of order at least $|L_1| + 1$ and $v_{n'}C[v_l, v_{n'}]$ is a cycle of order $|L_2|$, and hence $|L_1| + 1 + |L_2| \leq h - 1$ by the assumption that G is a counterexample. Thus equality holds in (29), and hence equality holds in (27). In particular, $p = d_G(a) - 1 = h - d_G(v_{n'}) - 1$. Since $n' \notin I$, this implies that $G - V(C) \cong K_2$ and $v_i \in N_G(a) \cap N_G(b)$ for all $i \in I$. Therefore $\langle V(C[v_1, v_k]) \cup (V(G) - V(C)) \rangle$ contains a cycle of order $|L_1| + 2$, and hence $|L_1| + 2 + |L_2| \leq h - 1$, which contradicts (29). This concludes the discussion for the case where $i_2 < i_{q-1}$.

We now consider the case where $i_2 \geq j_{q-1}$. Note that by the maximality of $|V(C)|$, $i_{r+1} - i_r \geq 2$ for each $1 \leq r \leq p - 1$. First assume $p \geq 4$. Then $|V(C[v_{i_3}, v_{i_p}])| \geq 2p - 5$, and hence $\langle V(C[v_{i_3}, v_{i_p}]) \cup (V(G) - V(C)) \rangle$ contains a cycle C_1 of order at least $2p - 4$. Also $C[v_{n'}, v_{j_{q-1}}]v_{n'}$ is a cycle C_2 of order at least q, and since C_1 and C_2 are disjoint we have $|V(C_1)| + |V(C_2)| \geq 2p - 4 + q \geq p + q$. Consequently by the assumption that G is a counterexample, equality holds in (27). As before, this implies that $G - V(C) \cong K_2$ and $v_i \in N_G(a) \cap N_G(b)$ for all $i \in I$. Therefore $\langle V(C[v_{i_3}, v_{i_p}])) \cup (V(G) - V(C)) \rangle$ contains a cycle of order at least $2p - 3$, which contradicts the assumption that G is a counterexample. Next assume $p = 3$. Then $\langle V(C[v_{i_2}, v_{i_3}]) \cup (V(G) - V(C)) \rangle$ contains a cycle of order at least 4. Also since $q \geq h - 4 > 4$ by (27), $C[v_{n'}, v_{j_{q-2}}]v_{n'}$ is a cycle of order at least $q - 1$, and we have $4 + q - 1 = p + q$. Consequently equality holds in (27). This implies that $G - V(C) \cong K_2$ and $v_i \in N_G(a) \cap N_G(b)$ for all $i \in I$, and hence $\langle V(C[v_{i_2}, v_{i_3}]) \cup (V(G) - V(C)) \rangle$ contains a cycle of order at least 5, a contradiction. Finally assume $p = 2$. Then by Claim 2.1, $G - V(C) \cong K_2$ and $v_i \in N_G(a) \cap N_G(b)$ for each $i \in I$. By (27), we also have $q \geq h - 3 > 4$. If $i_2 > j_{q-1}$, then $abv_{i_2}a$ and $C[v_{n'}, v_{j_{q-1}}]v_{n'}$ are disjoint cycles with $|V(abv_{i_2}a)| + |V(C[v_{n'}, v_{j_{q-1}}]v_{n'})| \geq 3 + q \geq h$, a contradiction. Thus $i_2 = j_{q-1}$. By (28), this implies $j_{q-1} - 1 \notin J$, and hence $|V(C[v_{j_2}, v_{j_{q-1}}])| \geq q - 1$. Consequently abv_1a and $v_{n'}C[v_{j_2}, v_{j_{q-1}}]v_{n'}$

are disjoint cycles and $|V(abv_1a)|+|V(v_{n'}C[v_{j_2},v_{j_{q-1}}]v_{n'})| \geq 3+q \geq h$, which is again a contradiction. This completes the proof for Case 1.

Case 2: $n = h+2$ or $7 \leq h \leq 10$

By Theorem 1, G contains two disjoint cycles C_1 and C_2. We may assume that C_1 and C_2 are chosen so that $|V(C_1)|+|V(C_2)|$ is as large as possible, and so that $\omega(G-V(C_1)-V(C_2))$ is as large as possible, subject to the condition that $|V(C_1)|+|V(C_2)|$ is maximum (here $\omega(G-V(C_1)-V(C_2))$ denotes the number of components of $G-V(C_1)-V(C_2)$). By the assumption that G is a counterexample,

$$|V(C_1)|+|V(C_2)| \leq h-1. \tag{30}$$

Let $H = G-V(C_1)-V(C_2)$. Since $n \geq h+2$, we have $|V(H)| \geq 3$ by (30).

Claim 2.4. *Let $\alpha = 1$ or 2. Let H' be a component of H, and let $a \in V(H')$. Then the following hold.*

(i) *If $v \in V(C_\alpha)$ and $av \in E(G)$, then $E(v^+, V(H')) = \emptyset$ and $E(v^{+2}, V(H'-a)) = \emptyset$.*

(ii) (a) *$|E(a, V(C_\alpha))| \leq |V(C_\alpha)|/2$.*

(b) *If $|E(a, V(C_\alpha))| = |V(C_\alpha)|/2$, then $E(V(H'-a), V(C_\alpha)) = \emptyset$.*

Proof. Statement (i) follows immediately from the maximality of $|V(C_1)|+|V(C_2)|$, and (i) implies (ii). □

We now consider two subcases separately according as H is connected or not.

Subcase 2.1. H is disconnected.

In this subcase, we divide the proof further into two cases.
Subcase 2.1.1. $n = h+2$.

Let H_1, H_2 be two components of H. Let $a_1 \in V(H_1)$ and $a_2 \in V(H_2)$. By the assumption that $\sigma_2(G) \geq h$ and by Claim 2.4(ii)(a), $n-2 = h \leq d_G(a_1)+d_G(a_2) \leq |V(C_1)|+|V(C_2)|+|V(H_1)|-1+|V(H_2)|-1 \leq n-2$. Thus equality holds, which means that $|V(H)| = |V(H_1)|+|V(H_2)|$ and $|E(a_i, V(C_\alpha))| = |V(C_\alpha)|/2$ for each $i = 1,2$ and each $\alpha = 1,2$. Hence by Claim 2.4(ii)(b), $E(V(H_i-a_i), V(C_1 \cup C_2)) = \emptyset$ for each $i = 1,2$. Since G is 2-connected, this forces $|V(H_1)| = |V(H_2)| = 1$, and hence $|V(H)| = 2$. This contradicts the earlier assertion that $|V(H)| \geq 3$.

Subcase 2.1.2. $7 \leq h \leq 10$

By the assumption that $\sigma_2(G) \geq h$, there exists a component H' of H such that

$$d_G(x) \geq \lceil h/2 \rceil \text{ for every } x \in V(H'). \tag{31}$$

Since $|E(x, V(C_1) \cup V(C_2))| \leq (h-1)/2$ for each $x \in V(H')$ by Claim 2.4(ii)(a) and (30), we have $|V(H')| \geq 2$.

Claim 2.5. *Every vertex of H', possibly except one, has degree at least 2 in H'.*

Proof. Suppose that there exist $a, b \in V(H')$ with $a \neq b$ such that $d_H(a) = d_H(b) = 1$. Then by (31), $|E(x, V(C_1) \cup V(C_2))| \geq \lceil h/2 \rceil - 1$ for each $x \in \{a, b\}$. In view of Claim 2.4(ii)(a) and (30), this implies that at least one of $|V(C_1)|$ and $|V(C_2)|$ is even, and that $|E(x, V(C_\alpha))| = \lfloor |V(C_\alpha)|/2 \rfloor$ for each $\alpha \in \{1, 2\}$ and each $x \in \{a, b\}$. But this contradicts Claim 2.4(ii)(b). □

Note that Claim 2.5 implies that the order of H' is at least 3, and that if H' is not 2-connected, then H' has an endblock of order at least 3. Now if H' is 2-connected, let $B = H'$; if H' is not 2-connected, let B be an endblock of H' with $|V(B)| \geq 3$. Fix $z \in V(B)$. In the case where H' is not 2-connected, we let z be the unique cutvertex of H' which lies in B. We derive a contradiction by proving a series of claims.

Claim 2.6. *Let $\alpha = 1$ or 2, and let $a, b \in V(B - z)$ with $a \neq b$. Suppose that there exist $u, v \in V(C_\alpha)$ such that $au, bv \in E(G)$. Then $u = v$.*

Proof. Suppose that there exist $u, v \in V(C_\alpha)$ with $u \neq v$ such that $au, bv \in E(G)$. Since $|V(B)| \geq 3$, B contains a path P of order at least 3 joining a and b. Also since $|V(C_\alpha)| \leq h - 1 - 3 \leq 6$ by (30), $C_\alpha[u, v]$ or $C_\alpha[v, u]$ has order at least $|V(C_\alpha)| - 2$ by maximality of C_α. Consequently $\langle V(C_\alpha) \cup V(P) \rangle$ contains a cycle of order at least $|V(C_\alpha)| + 1$, a contradiction. □

Claim 2.7. *There exists an $a \in V(B - z)$ such that $|E(a, V(C_1))| \geq 2$ or $|E(a, V(C_2))| \geq 2$.*

Proof. We may assume $|V(C_1)| \leq |V(C_2)|$. Suppose that for every $x \in V(B - z)$, we have $|E(x, V(C_1))| \leq 1$ and $|E(x, V(C_2))| \leq 1$. Then by (31),

$$d_B(x) \geq \lceil h/2 \rceil - 2 \text{ for every } x \in V(B - z), \tag{32}$$

and hence

$$|V(B)| \geq \lceil h/2 \rceil - 1. \tag{33}$$

Let C be a longest cycle in B. Since (32) in particular implies $\sigma_2(B) \geq \lceil h/2 \rceil$, it follows from Theorem 2 that

$$|V(C)| \geq \min\{\lceil h/2 \rceil, |V(B)|\}. \tag{34}$$

On the other hand, (30) together with the maximality of $|V(C_1)| + |V(C_2)|$ implies that $|V(C)| \leq |V(C_1)| \leq \lfloor (h-1)/2 \rfloor = \lceil h/2 \rceil - 1$, and hence $|V(B)| = |V(C)| = |V(C_1)| = \lceil h/2 \rceil - 1$ by (33) and (34). This implies that B is a complete graph and $|E(x, V(C_1))| = |E(x, V(C_2))| = 1$ for every $x \in V(B - z)$. By Claim 2.6, there exists a $v \in V(C_1)$ such that $xv \in E(G)$ for every $x \in V(B - z)$. But then $\langle V(B) \cup \{v\} \rangle$ contains a cycle of order $|V(B)| + 1 > |V(C_1)|$, a contradiction. □

Let a be as in Claim 2.7. We may assume $|E(a, V(C_1))| \geq 2$.

Claim 2.8. *There exists a $b \in V(B) - \{z, a\}$ such that $|E(b, V(C_2))| \geq 2$.*

Proof. Suppose that $|E(x, V(C_2))| \leq 1$ for every $x \in V(B) - \{z, a\}$. Then for every $x \in V(B) - \{z, a\}$, we have $d_B(x) \geq \lceil h/2 \rceil - 1$ because $E(x, V(C_1)) = \emptyset$ by Claim 2.6. Hence $|V(B)| \geq \lceil h/2 \rceil \geq 4$. Take $ay \in E(B)$. Then by Lemma 5, B contains a path P of length at least $\lceil h/2 \rceil - 1$ joining a and y, and hence we obtain a cycle of length at least $\lceil h/2 \rceil$ by adding ay to P. But since $\min\{|V(C_1)|, |V(C_2)|\} \leq (h-1)/2$ by (30), this contradicts the maximality of $|V(C_1)| + |V(C_2)|$. □

We are now in a position to derive a final contradiction. Let b be as in Claim 2.8. By Claim 2.6,

$$E(x, V(C_1)) = \emptyset \text{ for every } x \in V(B) - \{z, a\}, \tag{35}$$

and

$$E(x, V(C_2)) = \emptyset \text{ for every } x \in V(B) - \{z, b\}. \tag{36}$$

By symmetry, we may assume $|V(C_1)| \leq |V(C_2)|$. By Claim 2.4(ii)(a), $|V(C_1)| \geq 4$, and hence $h = 9$ or 10 and $4 = |V(C_1)| \leq |V(C_2)| \leq 5$ by (30). By Claim 2.4(ii)(a), this implies $|E(a, V(C_1))| = |E(b, V(C_2))| = 2$, and hence $d_B(x) \geq \lceil h/2 \rceil - 2 = 3$ for each $x \in \{a, b\}$ by (35) and (36). This implies $|V(B)| \geq 4$, and hence $V(B) - \{z, a, b\} \neq \emptyset$. Since $d_B(x) \geq 5$ for every $x \in V(B) - \{z, a, b\}$ by (35) and (36), this implies $|V(B)| \geq 6$. Since $\sigma_2(B) \geq 2 + \min\{d_B(x) |\ x \in V(B - z)\} \geq 5$, it now follows from Theorem 2 that B contains a cycle of order at least 5. Since $|V(C_1)| = 4$, this contradicts the maximality of $|V(C_1)| + |V(C_2)|$. This concludes the discussion for Subcase 2.1.

Subcase 2.2. H is connected.

We may assume $|V(C_1)| \leq |V(C_2)|$. We start with the following refinement of Claim 2.4.

Claim 2.9. *Let $\alpha = 1$ or 2. Let $a \in V(H)$ and $v \in V(C_\alpha)$, and suppose that $av \in E(G)$. Then the following hold.*

(i) $E(\{v^+, v^{+2}\}, V(H)) = \emptyset$.

(ii) $E(v^{+3}, V(H - a)) = \emptyset$.

Proof. Recall that $|V(H)| \geq 3$. By Claim 2.4, $E(v^+, V(H)) = \emptyset$ and $E(v^{+2}, V(H - a)) = \emptyset$. If $v^{+2}a \in E(G)$, then since $E(v^+, V(H)) = \emptyset$, we get a contradiction to the maximality of $\omega(H)$ by replacing C_α by the cycle $aC_\alpha[v^{+2}, v]a$. Thus $v^{+2}a \notin E(G)$, which proves (i). To prove (ii), suppose that $E(v^{+3}, V(H - a)) \neq \emptyset$ (it is possible that $v^{+3} = v$). Take $b \in N_G(v^{+3}) \cap V(H - a)$, and let P be a path in H joining a and b. By (i), $E(\{v^+, v^{+2}\}, V(H)) = \emptyset$. Hence by replacing C_α by the cycle $PC_\alpha[v^{+3}, v]a$, we get a contradiction to the maximality of $|V(C_1)| + |V(C_2)|$ or the maximality of $\omega(H)$. □

Claim 2.10. *Let $\alpha = 1$ or 2. Let $a, b \in V(H)$ with $a \neq b$. Then $|E(\{a, b\}, V(C_\alpha))| \leq |V(C_\alpha)|/2$.*

Proof. Write $(N_G(a) \cup N_G(b)) \cap V(C_\alpha) = \{w_1, \dots, w_t\}$ so that $w_1, \dots, w_t$ occur on C_α in this order. We may assume $t \geq 1$. For each $1 \leq i \leq t$, let $X_i = V(C[w_i, w_{i+1}^-])$ (in the case where $i = t$, we take $w_{i+1} = w_1$). Let $1 \leq i \leq t$. If $|E(w_i, \{a, b\})| = 1$, then since $|X_i| \geq 3$ by Claim 2.9, we have $|E(\{a, b\}, X_i)| = 1 \leq |X_i|/3 < |X_i|/2$; if $|E(w_i, \{a, b\})| = 2$, then $|X_i| \geq 4$ by Claim 2.9, and hence $|E(\{a, b\}, X_i)| = 2 \leq |X_i|/2$. Since i is arbitrary, we obtain $|E(\{a, b\}, V(C_\alpha)| = \sum_{1 \leq i \leq t} |E(\{a, b\}, X_i)| \leq \sum_{1 \leq i \leq t} |X_i|/2 = |V(C_\alpha)|/2$. □

Returning to the proof of the Main Theorem, we first show that H contains a cycle. Suppose that H is a tree, and let $a, b \in V(H)$ be distinct vertices with $d_H(a) = d_H(b) = 1$. Since $|V(H)| \geq 3$, a and b are non-adjacent, and hence $d_G(a) + d_G(b)\sigma_2(G) \geq h$. On the other hand, it follows from Claim 2.10 and (30) that $d_G(a) + d_G(b) = |E(\{a, b\}, V(C_1) \cup V(C_2))| + 2 \leq (|V(C_1)| + |V(C_2)|)/2 + 2 \leq (h-1)/2 + 2 < h$, which is a contradiction. Thus H contains a cycle. We now distinguish two cases.

Subcase 2.2.1. $n = h + 2$.

Before proving this subcase, we define the following notation. Let $C = x_1 x_2 \dots x_{|V(C)|} x_1$ be a cycle and let Y be a subset of $V(C)$. We define $Y^+ = \{x_i^+ \mid x_i \in Y\}$ and $Y^{+2} = \{x_i^{+2} \mid x_i \in Y\}$.

By Claim 2.10, there exists an $a \in V(H)$ such that $|E(a, V(C_1) \cup V(C_2))| \leq (|V(C_1)| + |V(C_2)|)/4$. Then $d_G(a) \leq |V(H)| - 1 + (|V(C_1)| + |V(C_2)|)/4$. By Claim 2.9, there exists a $u \in V(C_1)$ such that $E(\{u, u^+\}, V(H)) = \emptyset$. Then $|N_G(u) \cap V(C_2))| \geq d_G(u) - (|V(C_1)| - 1) \geq \sigma_2(G) - d_G(a) - (|V(C_1)| - 1) \geq h - d_G(a) - (|V(C_1)| - 1) \geq n - 2 - (|V(H)| - 1 + (|V(C_1)| + |V(C_2)|)/4) - (|V(C_1)| - 1) = (3|V(C_2)| - |V(C_1)|)/4 \geq |V(C_2)|/2$. Similarly $|N_G(u^+) \cap V(C_2))| \geq |V(C_2)|/2$, which clearly implies $|(N_G(u^+) \cap V(C_2))^+ \cup (N_G(u^+) \cap V(C_2))^{+2}| \geq |V(C_2)|/2 + 1$. Consequently $(N_G(u) \cap V(C_2)) \cap ((N_G(u^+) \cap V(C_2))^+ \cup (N_G(u^+) \cap V(C_2))^{+2}) \neq \emptyset$, which implies that $\langle V(C_1) \cup V(C_2) \rangle$ contains a cycle C with $|V(C)| \geq |V(C_1)| + |V(C_2)| - 1$. Now let D be a cycle in H. Then C and D are disjoint cycles with $|V(C)| + |V(D)| \geq |V(C_1)| + |V(C_2)| + 2$, which contradicts the maximality of $|V(C_1)| + |V(C_2)|$.

Subcase 2.2.2. $7 \leq h \leq 10$.

Let D be a longest cycle in H. By the maximality of $|V(C_1)| + |V(C_2)|$,

$$|V(D)| \leq |V(C_1)|. \tag{37}$$

By Lemma 6, $\delta(H) \leq |V(D)| - 1$. Take $a \in V(H)$ such that $d_H(a) = \delta(H)$. Then by Claim 2.9, $d_G(a) \leq |V(D)| - 1 + \lfloor |V(C_1)|/3 \rfloor + \lfloor |V(C_2)|/3 \rfloor$. By Claim 2.9, there exists a $u \in V(C_1)$ such that $E(\{u, u^+\}, V(H)) = \emptyset$. Then $d_G(x) \geq \sigma_2(G) - d_G(a) \geq h - d_G(a)$ for each $x \in \{u, u^+\}$. Thereby and because of $d_G(x) \leq |V(C_1)| - 1 + |E(x, V(C_2))|$, we have

$$\begin{aligned} |E(x, V(C_2))| \geq h - (|V(D)| - 1 + \lfloor |V(C_1)|/3 \rfloor + \lfloor |V(C_2)|/3 \rfloor) \\ -(|V(C_1)| - 1) \text{ for each } x \in \{u, u^+\}. \end{aligned} \tag{38}$$

Suppose that $|V(D)| + |V(C_1)| + \lfloor |V(C_1)|/3 \rfloor + \lfloor |V(C_2)|/3 \rfloor \leq h$. Then $|E(x, V(C_2))| \geq 2$ for each $x \in \{u, u^+\}$ by (38), and hence there exist $v, w \in V(C_2)$ with $v \neq w$ such that $uv, u^+w \in E(G)$. Since $|V(C_2)| \leq 6$ by (30), this implies that $\langle V(C_1) \cup V(C_2) \rangle$ contains a cycle C with $|V(C)| \geq |V(C_1)| + |V(C_2)| - 2$. But then $|V(C)| + |V(D)| \geq |V(C_1)| + |V(C_2)| + 1$, which contradicts the maximality of $|V(C_1)| + |V(C_2)|$. Thus

$$|V(D)| + |V(C_1)| + \lfloor |V(C_1)|/3 \rfloor + \lfloor |V(C_2)|/3 \rfloor \geq h + 1. \tag{39}$$

Since $\lfloor |V(C_1)|/3 \rfloor + \lfloor |V(C_2)|/3 \rfloor \leq 3$ by (30), this together with (37) implies that $|V(C_1)| \geq \lceil (h-2)/2 \rceil = \lfloor (h-1)/2 \rfloor$, and hence $|V(C_1)| = \lfloor (h-1)/2 \rfloor$ and $|V(C_2)| \leq \lceil (h-1)/2 \rceil \leq 5$ by (30). This in turn implies $\lfloor |V(C_1)|/3 \rfloor = \lfloor |V(C_2)|/3 \rfloor = 1$, and hence it follows from (30),(39) and (37) that $h - 1$ is even and $|V(D)| = |V(C_1)| = |V(C_2)| = (h-1)/2$. It now follows from (38) that $E(x, V(C_2)) \neq \emptyset$ for each $x \in \{u, u^+\}$, and hence $\langle V(C_1) \cup V(C_2) \rangle$ contains a cycle C with $|V(C)| \geq |V(C_1)| + 1$. But then $|V(C)| + |V(D)| \geq |V(C_1)| + |V(D)| + 1 = |V(C_1)| + |V(C_2)| + 1$, which again contradicts the maximality of $|V(C_1)| + |V(C_2)|$. This completes the proof of the Main Theorem.

Acknowledgments

This research was partially supported by the 21st Century COE Program; Integrative Mathematical Science: Progress in Mathematics Motivated by Social and Natural Sciences (to S. F.) and Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research, by Sumitomo Foundation and by Inoue Research Award for Young Scientists (to K. K.).

References

[1] J.-C.Bermond, On Hamilton walks, in "Proceedings of the Fifth British Combinatorial Conference, Aberdeen,1975", 41-51, Congressus Numerantiaum No. XV, Utilitas Math, Winnipeg, 1976.

[2] S.Brandt, G.Chen, R.J.Gould and L.Lesniak, Degree conditions for 2-factors, J.Graph Theory **24**(1997) 165-173.

[3] G.A.Dirac, Some theorems on abstract graphs, Proc. London Math. Soc. **3**(1952) 69-81.

[4] Y.Egawa, R.Glas, S.C.Locke, Cycles and paths through specified vertices in k-connected graphs, J. Combin. Theory, Ser. **B 52**(1991) 20-29.

[5] Y.Egawa, M.Hagita, K.Kawarabayashi,and H.Wang, Covering vertices of a graph by k disjoint cycles, Discrete Math. **270**(2003) 114-124.

[6] H.Enomoto, On the existence of disjoint cycles in a graph, Combinatorica **18**(1998) 487-492.

[7] K.Kawarabayashi, Degree sum conditions and graphs which are not covered by k cycles, submitted.

[8] H.Wang, On the maximal number of vertices covered by disjoint cycles. Australas. J. Combin. **21**(2000) 179-186.

[9] H.Wang, On the maximum number of independent cycles in a graph, Discrete Math. **205**(1999) 183-190.