2-Factors in r-Connected $\{K_{1,k}, P_4\}$-Free Graphs

Yoshimi Egawa
Department of Mathematical Information Science,
Tokyo University of Science
Kagurazaka 1-3, Shinjuku-ku, Tokyo, 162-8601, Japan

Jun Fujisawa*
Department of Computer Science,
Nihon University
Sakurajosui 3-25-40, Setagaya-Ku, Tokyo, 156-8550, Japan
E-mail: fujisawa@cs.chs.nihon-u.ac.jp

Shinya Fujita†
Department of Mathematics,
Gunma National College of Technology
Toribamachi 580, Maebashi, Gunma, 371-8530, Japan
E-mail: fujita@nat.gunma-ct.ac.jp

Katsuhiro Ota
Department of Mathematics,
Keio University
Hiyoshi 3-14-1, Kouhoku-ku, Yokohama, 223-8522, Japan

Abstract

In [3], Faudree et.al. considered the proposition "Every $\{X, Y\}$-free graph of sufficiently large order has a 2-factor," and they determined those pairs $\{X, Y\}$ which make this proposition true. Their result says that one of them is $\{X, Y\} = \{K_{1,4}, P_4\}$. In this paper, we investigate the existence of 2-factors in r-connected $\{K_{1,k}, P_4\}$-free graphs. We prove that if $r \geq 1$ and $k \geq 2$, and if G is an r-connected $\{K_{1,k}, P_4\}$-free graph with minimum degree at least $k-1$, then G has a 2-factor with at most $\max\{k-r, 1\}$ components unless $(k-1)K_2 + (k-2)K_1 \subseteq G \subseteq (k-1)K_2 + K_{k-2}$. The bound on the minimum degree is best possible.

Key Words 2-factor, forbidden subgraph, minimum degree

1 Introduction

In this paper, all graphs considered are finite, undirected, and without loops or multiple edges. For a graph G, $V(G)$, $E(G)$ and $\delta(G)$ denote the set of vertices and the set of edges

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and the minimum degree of G, respectively. Also we let $\alpha(G)$ denote the independence number of G and let $\kappa(G)$ denote the (vertex-)connectivity of G. For a subset M of $V(G)$, we let $G[M]$ denote the subgraph induced by M in G. Let $\mathcal{H}$ be a set of connected graphs, each of which has three or more vertices. A graph G is said to be $\mathcal{H}$-free if no graph in $\mathcal{H}$ is an induced subgraph of G. When $|\mathcal{H}| = 1$, say, $\mathcal{H} = \{X\}$, we use the term "X-free" to mean "$\mathcal{H}$-free".

In this paper, we study the relationship between forbidden subgraphs and the existence of a 2-factor with few components. In the research field concerning forbidden subgraphs for the existence of a 2-factor with one component, that is, the existence of a hamiltonian cycle, there is a famous conjecture due to Matthews and Sumner [5].

Conjecture 1 (Matthews and Sumner [5]). *Every 4-connected $K_{1,3}$-free graph has a hamiltonian cycle.*

In [1], Broersma et.al. showed that the above conjecture is true if we replace the assumption "$K_{1,3}$-free" by "$\{K_{1,3}, K_1 + 2K_2\}$-free." Along a slightly different line, there are some results concerning minimum degree conditions for the existence of a hamiltonian cycle in $K_{1,3}$-free graphs. For example, Lai et.al. ([4]) proved that if G is a 3-connected $K_{1,3}$-free graph of order $n \geq 196$ with $\delta(G) > (n+6)/10$, then G has a hamiltonian cycle. Apart from the existence of a hamiltonian cycle, there are many results concerning forbidden subgraphs for the existence of 2-factors. It seems that most of the research has been done from the following viewpoints:

- Consider the proposition "Every $\mathcal{H}$-free graph of sufficiently large order has a 2-factor", and determine those families $\mathcal{H}$ which make the proposition true.
- For a given family $\mathcal{H}$, determine the sharp degree condition for the existence of 2-factors in $\mathcal{H}$-free graphs.
- What if we consider the above problems in highly connected graphs?

As an illustration of research done in the above directions, we mention some known results. In [6], Ota and Tokuda showed that every connected $K_{1,n}$-free graph G ($n \geq 3$) with $\delta(G) \geq 2(n-1)$ has a 2-factor. Actually, they obtained a more general result, that is, they determined the sharp degree condition for the existence of r-factors in $K_{1,n}$-free graphs. In [3], Faudree et. al. considered the proposition "Every $\{X, Y\}$-free graph of sufficiently large order has a 2-factor," and they determined the pairs $\{X, Y\}$ which make this proposition true.Their result says that one of them is $\{X, Y\} = \{K_{1,4}, P_4\}$. In connection with this result, they also obtained the following theorem.

Theorem 1 (Faudree et. al. [3]). *If G is a 2-connected $\{K_{1,4}, P_4\}$-free graph of order at least 9, then G has a 2-factor with at most 2 components.*

In this paper, we focus on the existence of 2-factors with few components in $\{K_{1,k}, P_4\}$-free graphs. Our purpose is to extend Theorem 1 to $\{K_{1,k}, P_4\}$-free graphs from the above viewpoints. Our first result involves a degree condition:

Theorem 2. *Let $r \geq 1$ and $k \geq 2$, and let G be an r-connected $\{K_{1,k}, P_4\}$-free graph with $\delta(G) \geq k-1$. Then either*

a) G contains a 2-factor with at most $\max\{k-r, 1\}$ components, or

b) G is a graph which satisfies $(k-1)K_2 + (k-2)K_1 \subseteq G \subseteq (k-1)K_2 + K_{k-2}$ (so $|V(G)| = 3k-4$ and $\delta(G) = k-1$).

We here discuss the sharpness of bounds in Theorem 2. For that purpose, assume that $1 \leq r \leq k-2$. Then the graph $(k-1)K_m + K_r$ show that in the conclusion of the theorem, the upper bound $k-r$ on the number of components of a 2-factor of G is best possible in the sense that there exists an r-connected $\{K_{1,k}, P_4\}$-free graph G with arbitrary large minimum degree such that G has no 2-factor with strictly fewer than $k-r$ components. We now turn our attention to the lower bound $k-1$ on $\delta(G)$ in the assumption. Note that the graph $((k-2)K_2 \cup K_m) + K_{k-3}$ show that there exists a $(k-3)$-connected $\{K_{1,k}, P_4\}$-free graph G with arbitrary large order such that $\delta(G) = k-2$ and G has no 2-factor. Thus if $1 \leq r \leq k-3$, the bound $k-1$ is best possible. But if $r = k-2 \geq 2$, the situation is different (if $r = k-2 = 1$, the bound $k-1$ is clearly best possible). In fact, the following theorem holds:

Theorem 3. *Let $r \geq 2$ and $k \geq 2$ be integers with $r \geq k-2$. Let G be an r-connected $\{K_{1,k}, P_4\}$-free graph. Then either*

a) G contains a 2-factor with at most $\max\{k-r, 1\}$ components, or

b) $k \geq 4$, and G is a graph which satisfies $(qK_1 \cup (k-1-q)K_2) + (k-2)K_1 \subseteq G \subseteq (qK_1 \cup (k-1-q)K_2) + K_{k-2}$ for some q with $0 \leq q \leq k-1$ (so $|V(G)| \leq 3k-4$ and $\kappa(G) = k-2$).

Note that if we let $r = 2$ and $k = 4$ in Theorem 3, then we obtain Theorem 1.

In the proof of these theorems, we use the following theorem.

Theorem 4 (Chvátal and Erdős [2]). *Let G be an r-connected graph with at least three vertices. If $r \geq \alpha(G)$, then G contains a hamiltonian cycle.*

Also we use the following lemma.

Lemma 1. *Let G be a non-complete P_4-free graph and let S be a minimum cutset of G. Then for every two vertices u, v with $u \in S$ and $v \in V(G) \setminus S$, $uv \in E(G)$.*

The proof of this lemma is implicit in [3, Theorem 3]. The following lemma immediately follows from Lemma 1.

Lemma 2. *Let $k \geq 2$, and let G be a connected P_4-free graph. Then G is $K_{1,k}$-free if and only if $\alpha(G) \leq k-1$.*

2 Proof of Theorem 2

Note that in view of Lemma 2, the assumption that G is $\{K_{1,k}, P_4\}$-free is equivalent to the statement that G is P_4-free and $\alpha(G) \leq k-1$.

Now we proceed by induction on k. First let $k = 2$. Then G is a complete graph. If $|V(G)| \geq 3$, then G contains a hamiltonian cycle, and hence a) holds. Otherwise, G must be K_2, which satisfies b). Let now $k \geq 3$, and assume that the theorem holds for smaller value of k. We may assume that G is not a complete graph, because otherwise a) holds.

Note that $|V(G)| \geq 3$ because $\delta(G) \geq k-1 \geq 2$. If $\kappa(G) \geq k-1$, then since $\alpha(G) \leq k-1$, Theorem 4 implies that G contains a hamiltonian cycle, and hence a) holds. Thus we may assume that $\kappa(G) \leq k-2$.

Let S be a minimum cutset of G. Since G is r-connected, $k-2 \geq |S| = \kappa(G) \geq r$. Let $H_1, H_2, \ldots, H_l$ be the components of $G - S$, and let $\alpha_i = \alpha(H_i)$ for every i with $1 \leq i \leq l$. By Lemma 1,

$$uv \in E(G) \text{ for every } u \in S \text{ and } v \in V(G) \setminus S. \tag{1}$$

Moreover, for every $v \in V(G) \setminus S$, $d_{G-S}(v) = d_G(v) - |S| \geq k-1-|S| \geq 1$. Hence $|H_i| \geq 2$ for every i with $1 \leq i \leq l$.

If $\alpha(G) \leq k-2$, then by the induction hypothesis, G contains a 2-factor with at most $\max\{k-1-r, 1\}$ components (note that if G satisfies b) for $k-1$, then by the parenthetic remark in the statement of b), we have $\delta(G) = (k-1)-1$, which contradicts the assumption that $\delta(G) \geq k-1$), and hence a) holds. Thus we may assume that $\alpha(G) = k-1$. Let I be a maximum independent subset of $V(G)$ with $|I| = \alpha(G) = k-1$. Then by (1), $I \subseteq S$ or $I \subseteq V(G) \setminus S$. Since $|S| \leq k-2$, it follows that $I \subseteq V(G) \setminus S$, which implies $\sum_{i=1}^{l} \alpha_i = k-1$.

We consider two cases.

Case 1. There exists i with $1 \leq i \leq l$ such that $\alpha_i \leq |S|$.

Take i so that $|H_i| = \max\{|H_j| \mid 1 \leq j \leq l,\ \alpha_j \leq |S|\}$. Note that $k - \alpha_i \geq k - |S| \geq 2$. Let S' be a subset of S with cardinality $\alpha_i - 1$. Let $S^* = S \setminus S'$ and $H^* = G - (S \cup V(H_i))$. Moreover, let $G' = G[S' \cup V(H_i)]$ and $G^* = G[S^* \cup V(H^*)]$.

Now $|S^*| = |S| - |S'| \leq k-2-(\alpha_i - 1) = k - \alpha_i - 1$ and $\alpha(H^*) = k - \alpha_i - 1$. Hence it follows from (1) that $\alpha(G^*) = k - \alpha_i - 1$. Further $|H^*| \geq \alpha(H^*) + 1 \geq k - \alpha_i$ because $|H_j| \geq 2$ and H_j is connected for every j with $1 \leq j \leq l$ and $j \neq i$. Hence for every $v \in S^*$, we have $d_{G^*}(v) \geq |H^*| \geq k - \alpha_i$ by (1). On the other hand, for every $v \in H^*$, $d_{G^*}(v) = d_G(v) - |S'| \geq k-1-(\alpha_i - 1) = k - \alpha_i$. Therefore $\delta(G^*) \geq k - \alpha_i$. Moreover, it follows from (1) that $\kappa(G^*) \geq \min\{|S^*|, |H^*|\} = |S^*| = |S| - \alpha_i + 1$. Since G^* is an induced subgraph of G, G^* is P_4-free. Consequently, by the induction hypothesis, G^* contains a 2-factor F^* with at most $\max\{k - \alpha_i - (|S| - \alpha_i + 1), 1\} = k - |S| - 1$ components (see the parenthetic remark in b) of the statement of the theorem).

Assume for the moment that $|H_i| \geq 3$. Since $|S'| = \alpha_i - 1$ and $\alpha(H_i) = \alpha_i$, it follows from (1) that $\alpha(G') = \alpha_i$. Moreover, by (1) and the fact that H_i is connected, we have $\kappa(G') \geq |S'| + 1 = \alpha_i$. Therefore we obtain a hamiltonian cycle F' of G' by Theorem 4. Now $F' \cup F^*$ is a 2-factor with at most $k - |S| \leq k - r$ components, and hence a) holds.

Thus we may assume that $|H_i| = 2$. Since $d_G(v) \geq k-1$ for every $v \in V(H_i)$, we have $|S| = k-2$. Now for every j with $1 \leq j \leq l$, $\alpha_j \leq \sum_{h=1}^{l} \alpha_h - 1 \leq k-2 = |S|$. Hence for every j with $1 \leq j \leq l$, we obtain $H_j = K_2$ by the choice of i and the fact that $|H_j| \geq 2$. Since $\sum_{j=1}^{l} \alpha_j = k-1$, $l = k-1$. With (1) and the assumption that $|S| = k-2$, we see that b) holds.

Case 2. For every i with $1 \leq i \leq l$, $\alpha_i \geq |S|+1$.

Let i, j be distinct integers with $1 \leq i, j \leq l$. Then $\alpha_i \leq k-1-\alpha_j \leq k-1-(|S|+1) = k-2-|S|$. Hence $\delta(H_i) \geq \delta(G) - |S| \geq k-1-|S| \geq \alpha_i + 1$. Note that H_i is P_4-free and $\kappa(H_i) \geq 1$. Consequently, by the induction hypothesis, H_i contains a 2-factor with at most $\alpha_i + 1 - 1 = \alpha_i$ components.

Applying the above argument to every component of $G - S$, we see that $G - S$ contains a 2-factor F with at most $\sum_{i=1}^{l} \alpha_i = k-1$ components. Let $C_1, C_2, \ldots, C_m$ be the components of F. For every i with $1 \leq i \leq m$, take $u_i v_i \in E(C_i)$ and let $P_i = C_i - u_i v_i$.

Write $S = \{w_1, w_2, \ldots, w_s\}$ ($s = \kappa(G)$). Recall that we have (1). In the case where $m \geq s$, let $C = v_1 P_1 u_1 w_1 v_2 P_2 u_2 w_2 v_3 P_3 u_3 \cdots\ v_s P_s u_s w_s v_1$. Then $(\bigcup_{i=s+1}^{m} C_i) \cup C$ is a 2-factor of G with $m - s + 1 \leq k - s \leq k - r$ components, and hence a) holds. In the case where $m < s$, let $C = v_1 P_1 u_1 w_1 v_2 P_2 u_2 w_2 v_3 P_3 u_3 \cdots\ v_m P_m u_m w_m v_1$. Since $|V(G-S)| \geq \alpha(G-S) = k-1$, F has at least $k-1$ edges. Hence there are at least $k-1-m$ edges in $E(C) \cap E(F)$. Choose $s - m$ edges from $E(C) \cap E(F)$, say $u'_1 v'_1, u'_2 v'_2, \ldots, u'_{s-m} v'_{s-m}$ (note that $s - m \leq k-2-m$). Let C' be the cycle obtained from C by replacing $u'_i v'_i$ by $u'_i w_{m+i} v'_i$ for every $1 \leq i \leq s-m$. Then C' is a hamiltonian cycle of G, and hence a) holds. This completes the proof of Theorem 2. $\square$

3 Proof of Theorem 3

First, note that G has at least three vertices because G is 2-connected. As in the proof of Theorem 2, we may assume that $k \geq 3$ and G is not a complete graph.

As in the proof of Theorem 2, we may also assume $\kappa(G) \leq k-2$. Then $\kappa(G) = r = k-2$. Since $r \geq 2$, this implies $k \geq 4$. Let S be a cutset with $|S| = k-2$. Let $H_1, H_2, \dots, H_l$ be the components of $G - S$, and let $\alpha_i = \alpha(H_i)$ for every i with $1 \leq i \leq l$. By Lemma 1,

$$uv \in E(G) \text{ for every } u \in S \text{ and } v \in V(G) \setminus S. \tag{2}$$

If $\alpha(G) \leq k-2$, then by Theorem 4, G contains a hamiltonian cycle and hence a) holds. Thus we may assume that $\alpha(G) = k-1$. As in the proof of Theorem 2, this implies $\sum_{i=1}^{l} \alpha_i = k-1$. Since $\alpha_i \geq 1$ for every i with $1 \leq i \leq l$, it follows that $\alpha_j \leq k-1-1 = k-2$ for every j with $1 \leq j \leq l$.

Take i so that $|H_i|$ is as large as possible. Note that $k - \alpha_i - 1 > 0$. Let S' be a subset of S with cardinality $\alpha_i - 1$. Let $S^* = S \setminus S'$ and $H^* = G - (S \cup V(H_i))$. Moreover, let $G' = G[S' \cup V(H_i)]$ and $G^* = G[S^* \cup V(H^*)]$. Now $|S^*| = |S| - |S'| = k - 2 - (\alpha_i - 1) = k - \alpha_i - 1$ and $\alpha(H^*) = k - \alpha_i - 1$. Hence by (2), we obtain $\alpha(G^*) = k - \alpha_i - 1$ and $\kappa(G^*) \geq \min\{|S^*|, |H^*|\} = |S^*| = k - \alpha_i - 1$. Consequently it follows from Theorem 4 that G^* contains a Hamiltonian cycle F^* or $G^* \simeq K_2$.

We first consider the case where $|H_i| \geq 3$. Since $|S'| = \alpha_i - 1$ and $\alpha(H_i) = \alpha_i$, it follows from (2) that $\alpha(G') = \alpha_i$, Moreover, by (2) and the fact that H_i is connected, we have $\kappa(G') \geq |S'| + 1 = \alpha_i$. Hence we obtain a hamiltonian cycle F' of G' by Theorem 4. If G^* contains a hamiltonian cycle, then $F' \cup F^*$ is a 2-factor with $2 = k - r$ components, and hence a) holds. Thus we may assume $G^* \simeq K_2$. Then $|S^*| = |H^*| = 1$. Write $S^* = \{w_0\}$ and $V(H^*) = \{v_0\}$. Since $|S| = r \geq 2$, we have $S' \neq \emptyset$, and hence F' contains an edge wv with $w \in S'$ and $v \in V(H_i)$. In view of (2), we can replace wv by wv_0w_0v, to get a hamiltoninan cycle of G, which implies a).

We now consider the case where $|H_i| \leq 2$. By the choice of i, $|H_j| \leq 2$ for every j with $1 \leq j \leq l$. This implies $l = \sum_{j=1}^{l} \alpha_j = k - 1$. With the fact $|S| = k - 2$ and (2), we see that b) holds. This completes the proof of Theorem 3. □

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