On the Isomorphism Problem of Coxeter Groups and Related Topics

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Introduction

A *Coxeter group* is a group which admits a group presentation of a certain typical type. One of the most active fields in the recent researches on Coxeter groups is the *isomorphism problem* of Coxeter groups; that is the problem of deciding which Coxeter groups are isomorphic as abstract groups, and of studying further properties of isomorphisms between Coxeter groups. Several results and observations for this problem have been given in this decade, particularly in the case of finitely-generated Coxeter groups. However, only a few observations have been given for non-finitely-generated cases. The aim of this dissertation is to give a breakthrough for the isomorphism problem of Coxeter groups without the assumption of the finiteness of generators. Moreover, we also give some more results on abstract groups and on Coxeter groups, which are of independent importance and interest.

This dissertation consists of two parts, each containing an independent paper discussing topics related to the isomorphism problem of Coxeter groups. In Part I, entitled "On the direct indecomposability of infinite irreducible Coxeter groups and the Isomorphism Problem of Coxeter groups", we study a relationship between the isomorphism problem and irreducible components of Coxeter groups; our result reduces the problem to the case of infinite irreducible Coxeter groups. Moreover, an analogue of the Krull-Remak-Schmidt Theorem on indecomposable decompositions of abstract groups is provided, and the automorphism groups of Coxeter groups and the centralizers of normal subgroups in Coxeter groups generated by involutions are also described.

In Part II, entitled "Almost central involutions in split extensions of Coxeter groups by graph automorphisms", we give a sufficient condition for an isomorphism between two Coxeter groups to be reflection-preserving. This is an important step of the study of the isomorphism problem of Coxeter groups. More detailed examination of this condition will be done in a forthcoming paper of the author which is now in preparation. We also study a split extension of any Coxeter group by Coxeter graph automorphisms, determining the involutions in such an extension whose centralizer has finite index. Moreover, certain special elements of Coxeter groups and the fixed-point subgroups in Coxeter groups by Coxeter graph automorphisms are also studied.

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Part I

On the direct indecomposability of infinite irreducible Coxeter groups and the Isomorphism Problem of Coxeter groups

ON THE DIRECT INDECOMPOSABILITY OF INFINITE IRREDUCIBLE COXETER GROUPS AND THE ISOMORPHISM PROBLEM OF COXETER GROUPS

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In this paper we prove that any irreducible Coxeter group of infinite order, which is possibly of infinite rank, is directly indecomposable as an abstract group. The key ingredient of the proof is that we can determine, for an irreducible Coxeter group W, the centralizers in W of the normal subgroups of W that are generated by involutions. As a consequence, the problem of deciding whether two general Coxeter groups are isomorphic is reduced to the case of irreducible ones. We also describes the automorphism group of a general Coxeter group in terms of those of its irreducible components.

key words: Coxeter groups; indecomposability; Isomorphism Problem; automorphism groups; centralizers.
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1 Introduction

In this paper, we prove that all infinite irreducible Coxeter groups are directly indecomposable as abstract groups (Theorem 3.3).

Regarding direct indecomposability of Coxeter groups, it is well known that there exist finite irreducible Coxeter groups which are directly decomposable (such as the Weyl group G_2). On the other hand, for infinite irreducible Coxeter groups, no general result has been known until recently. In a recent paper [9], L. Paris proved the direct indecomposability of all infinite irreducible Coxeter groups of finite rank, by using certain special elements called essential elements which are used also in [6]. However, by definition, a Coxeter group of infinite rank never possesses an essential element, so that the proof cannot be applied directly to the case of infinite ranks.

Our result here is obtained by a different approach. Let W be an irreducible Coxeter group whose order is infinite, possibly of infinite rank. We give a complete description of the centralizer C of any normal subgroup N of W which is generated by involutions (Theorem 3.1). From the description it follows that, unless $N = \{1\}$ or $C = \{1\}$, there is a subgroup $H \subsetneq W$ which contains both N and C. Once this is proved, the direct indecomposability of W is clear, since any direct factor of W is a normal subgroup and is generated by involutions (since it is a quotient of W), and its centralizer contains the complementary factor.

As a consequence of the direct indecomposability of infinite irreducible Coxeter groups, we give results on the isomorphisms between two Coxeter groups (Theorem 3.4). Since we also know how each finite irreducible Coxeter group decomposes into directly indecomposable factors, our results imply that we can determine whether or not two given Coxeter groups are isomorphic if we can determine which infinite irreducible Coxeter groups are isomorphic. In addition, our results also give certain decompositions of an automorphism of a general Coxeter group W (Theorem 3.10). One decomposition describes its form from the viewpoint of the directly indecomposable decomposition of W; another decomposition describes its form from the viewpoint of the decomposition $W = W_{\mathrm{fin}} \times W_{\mathrm{inf}}$, where W_{fin} (resp. W_{inf}) is the product of the finite (resp. infinite) irreducible components of W in the given Coxeter system. Note that these results can also be deduced from the Krull-Remak-Schmidt Theorem in group theory, if the Coxeter group has a composition series. Theorem 3.4 is also a generalization of Theorem 2.1 of [9]; our proof here is similar to, but slightly more delicate than that in [9], by the lack of finiteness of the ranks. Note also that, in another recent paper [7], M. Mihalik, J. Ratcliffe and S. Tschantz also examined the "Isomorphism Problem" (namely, the problem of deciding which Coxeter groups are isomorphic) for the case of finite ranks, by a highly different approach.

Contents. Section 2 collects the preliminary facts and results. In Section 2.1, we give some remarks on general groups, especially on the definition and properties of the core subgroups. Sections 2.2 and 2.3 summarize definitions, notations and properties of Coxeter systems, Coxeter graphs and root systems of Coxeter groups. In Section 2.4, we recall a method, given by V. V. Deodhar [2], for decomposing the longest element of any finite parabolic subgroup into pairwise commuting reflections. Owing to this decomposition, we can compute easily the action of the longest element on a root, even if it is not contained in the root system of the parabolic subgroup. As an application, in Section 2.5, we determine all irreducible Coxeter groups of which the center is a nontrivial direct factor. (This is not a new result, but is included there since the result is used in the following sections.) Some properties of normalizers of parabolic subgroups are summarized as Section 2.6.

Our main results are stated and proved in Section 3. The direct indecomposability of infinite irreducible Coxeter groups is shown in Section 3.1 (Theorem 3.3). Note that the theorem also includes the description, which has been known, of all nontrivial direct product decompositions of finite irreducible Coxeter groups. In Section 3.2, we reduce the Isomorphism Problem of general Coxeter groups to the case of infinite irreducible ones (Theorem 3.4). In the proof, we give a result on such a problem in a slightly wider context (Theorem 3.9), by which our result is deduced. Moreover, another result in Section 3.3 describes the automorphism group of a general Coxeter group in terms of those of the irreducible components (Theorem 3.10 (ii)). Note that a Coxeter group possesses some 'natural' automorphisms, which arise from automorphisms of the irreducible components together with a permutation of isomorphic components. We also give a characterization of Coxeter groups for which the group of the 'natural' automorphisms has finite index in the whole automorphism group (Theorem 3.10 (iii)).

Our proof of Theorem 3.3 is based on our description of the centralizers of the normal subgroups, which are generated by involutions, in irreducible Coxeter groups (Theorem 3.1). This theorem is proved in Section 4.1 by using a description, given in Sections 4.2–4.4, of the core subgroups of normalizers of parabolic subgroups.

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2 Preliminaries

2.1 Notes on general groups

In this paper, we treat two kinds of direct products of groups G_λ with (possibly infinite) index set Λ; the *complete* direct product (whose elements $(g_\lambda)_\lambda$ are all the maps $\Lambda \to \bigsqcup_{\mu\in\Lambda} G_\mu$, $\lambda \mapsto g_\lambda$ such that $g_\lambda \in G_\lambda$) and the *restricted* direct product (consisting of all the elements $(g_\lambda)_\lambda$ such that g_λ is the unit element of G_λ for all but finitely many $\lambda \in \Lambda$). Note that these two products coincide if $|\Lambda| < \infty$. Since here we treat mainly the latter type rather than the former one, we let *the term "direct product" alone and the symbol* $\prod$ *mean the restricted direct product* throughout this paper. (The complete one also appears in this paper, always together with notification.)

For two groups G, G', let $\mathrm{Hom}(G, G')$, $\mathrm{Isom}(G, G')$ denote the sets of all homomorphisms, isomorphisms $G \to G'$ respectively. Put $\mathrm{End}(G) = \mathrm{Hom}(G, G)$ and $\mathrm{Aut}(G) = \mathrm{Isom}(G, G)$. The following lemma is easy, but will be referred later.

Lemma 2.1. *Assume that the center $Z(G)$ of a group G is either trivial or a cyclic group of prime order. Then the following three conditions are equivalent:*
(I) *$Z(G) = 1$ or $Z(G)$ is not a direct factor of G.*
(II) *If $f \in \mathrm{Hom}(G, Z(G))$, then $f(Z(G)) = 1$.*
(III) *If G' is a direct product of (arbitrarily many) cyclic groups of prime order and $f \in \mathrm{Hom}(G, G')$, then $f(Z(G)) = 1$.*

Proof. This is trivial if $Z(G) = 1$, so that we assume that $Z(G)$ is a cyclic group of prime order. Note that the implication (III) $\Rightarrow$ (II) is obvious.
(I) $\Leftrightarrow$ (II): If (I) is not satisfied, and $G = Z(G) \times H$, then the projection $G \to Z(G)$ does not satisfy the conclusion of (II). Conversely, if $f \in \mathrm{Hom}(G, Z(G))$ and $f(Z(G)) \neq 1$, then $f(Z(G)) = Z(G)$, $\ker f \cap Z(G) = 1$ (since $Z(G)$ is simple) and so we have $G = Z(G) \times \ker f$.
(II) $\Rightarrow$ (III): This is clear if G' itself is a cyclic group of prime order (by noting that $\mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/\ell\mathbb{Z}) = 1$ for distinct primes p, ℓ). For a general case, apply it to the composite map $\pi \circ f$ for every projection π from G' to one of its factors. $\square$

Here we define the following multiplication for the set $\mathrm{Hom}(G, Z(G))$ by which it forms a monoid. First, we define a map $\mathrm{Hom}(G, Z(G)) \to \mathrm{End}(G)$, $f \mapsto f^\flat$ by

$$f^\flat(w) = wf(w)^{-1} \text{ for all } w \in G.$$

This is well defined since $Z(G)$ is central in G. The image of $H \subset \mathrm{Hom}(G, Z(G))$ by the map is denoted by $H^\flat$. Now define the product $f * g$ of two elements $f, g \in \mathrm{Hom}(G, Z(G))$ by

$$(f * g)(w) = f(w)g(w)(f \circ g)(w)^{-1} \text{ for all } w \in G.$$

This is also well defined, and then $\mathrm{Hom}(G, Z(G))$ forms a monoid with the trivial map (denoted by 1) as the unit element (for example, we have the associativity

$$\begin{aligned}\big((f*g)*h\big)(w) &= \big(f*(g*h)\big)(w)\\ &= f(w)g(w)h(w)(f\circ g)(w)^{-1}(f\circ h)(w)^{-1}(g\circ h)(w)^{-1}(f\circ g\circ h)(w) \qquad (2.1)\\ &= (f*h)(w)(f^\flat\circ g\circ h^\flat)(w)\end{aligned}$$

for $f, g, h \in \mathrm{Hom}(G, Z(G))$). Let $\mathrm{Hom}(G, Z(G))^\times$ denote the group of invertible elements of $\mathrm{Hom}(G, Z(G))$ with respect to the multiplication $*$. On the other hand, $\mathrm{End}(G)$ also forms a monoid with composition of maps as multiplication; then the group of invertible elements in the monoid $\mathrm{End}(G)$ is precisely the group $\mathrm{Aut}(G)$.

Moreover, $\mathrm{Aut}(G)$ acts on the monoids $\mathrm{Hom}(G, Z(G))$ and $\mathrm{End}(G)$ by

$$h\cdot f = h\circ f\circ h^{-1} \quad \text{for } h\in \mathrm{Aut}(G),\ f\in \mathrm{Hom}(G, Z(G)) \text{ or } \mathrm{End}(G).$$

Lemma 2.2. ***(i)*** *The map $f\mapsto f^\flat$ is an injective homomorphism $\mathrm{Hom}(G, Z(G))\to \mathrm{End}(G)$ of monoids compatible with the action of $\mathrm{Aut}(G)$.*
(ii) *For $f\in \mathrm{Hom}(G, Z(G))$, the following three conditions are equivalent:*
(I) *$f\in \mathrm{Hom}(G, Z(G))^\times$.* ***(II)*** *$f^\flat\in \mathrm{Aut}(G)$.*
(III) *The restriction $f^\flat|_{Z(G)}$ is an automorphism of $Z(G)$.*
(iii) *If $H\subset \mathrm{Hom}(G, Z(G))^\times$ is a subgroup invariant under the action of $\mathrm{Aut}(G)$, then its image $H^\flat$ is a normal subgroup of $\mathrm{Aut}(G)$.*

Proof. The claim (i) is straightforward, while (iii) follows from (i), (ii) and definition of the action of $\mathrm{Aut}(G)$. From now, we prove (ii). The implication (I) $\Rightarrow$ (II) is obvious. On the other hand, (II) implies (III) since any automorphism preserves the center. Moreover, if (III) is satisfied, then we can construct the inverse element f' of $f\in \mathrm{Hom}(G, Z(G))$ by $f'(w) = (f^\flat|_{Z(G)})^{-1}\big(f(w)\big)^{-1}$ ($w\in G$); we have

$$\begin{aligned}(f'*f)(w) = f'(w)f(w)f'(f(w))^{-1} &= f'(wf(w)^{-1})f(w)\\ &= (f^\flat|_{Z(G)})^{-1}\big(f(wf(w)^{-1})\big)^{-1}f(w)\\ &= (f^\flat|_{Z(G)})^{-1}\big(f^\flat(f(w))\big)^{-1}f(w)\\ &= f(w)^{-1}f(w) = 1,\end{aligned}$$

so that $f'*f = 1$. Similarly, we have $f*f' = 1$. Hence the claim holds. $\square$

Lemma 2.3. *If a group G is abelian, then the embedding $\mathrm{Hom}(G, Z(G))\to \mathrm{End}(G)$, $f\mapsto f^\flat$, is an isomorphism with inverse map $f\mapsto f^\flat$. Moreover, its restriction is an isomorphism $\mathrm{Hom}(G, Z(G))^\times\to \mathrm{Aut}(G)$.*

Proof. Note that $Z(G) = G$, so that $\mathrm{Hom}(G, Z(G)) = \mathrm{End}(G)$ as sets. Thus the map $\mathrm{End}(G)\to \mathrm{Hom}(G, Z(G))$, $f\mapsto f^\flat$ is well defined. Now we have $(f^\flat)^\flat(w) = wf^\flat(w)^{-1} = f(w)$ for all $f\in \mathrm{End}(G)$ and $w\in G$, so that $(f^\flat)^\flat = f$. Thus the first claim holds. Now the second one follows from Lemma 2.2 (ii). $\square$

Note that, if $G = G_1\times G_2$, then the sets $\mathrm{Hom}(G_i, Z(G))$ ($i = 1, 2$) are embedded into $\mathrm{Hom}(G, Z(G))$ via the map $f\mapsto f\circ\pi_i$ (where π_i is the projection $G\to G_i$). It is easily checked that, if $f, g\in \mathrm{Hom}(G_i, Z(G))$, then $f*g\in$

$\mathrm{Hom}(G_i, Z(G))$, thus $\mathrm{Hom}(G_i, Z(G))$ forms a submonoid of $\mathrm{Hom}(G, Z(G))$. Moreover, the above formula of the inverse element f' of $f \in \mathrm{Hom}(G, Z(G))$ implies that, if $f \in \mathrm{Hom}(G_i, Z(G))$ is invertible as an element of $\mathrm{Hom}(G, Z(G))$, then its inverse belongs to $\mathrm{Hom}(G_i, Z(G))$. Thus the notation $\mathrm{Hom}(G_i, Z(G))^\times$ is unambiguous.

Lemma 2.4. ***(i)*** *Let $f, g \in \mathrm{Hom}(G, Z(G))$ such that $f(Z(G)) = g(Z(G)) = 1$. Then $f, g \in \mathrm{Hom}(G, Z(G))^\times$ and $(f * g)(w) = f(w)g(w)$ for all $w \in G$ (so that $f * g = g * f$ by symmetry). Moreover, the map $w \mapsto f(w)^{-1}$ is the inverse element of f in $\mathrm{Hom}(G, Z(G))^\times$.*
(ii) *Suppose that $G = G_1 \times G_2$ and $Z(G_2) = 1$. Then $\mathrm{Hom}(G, Z(G))^\times = H_1 \rtimes H_2$ where $H_1 = \mathrm{Hom}(G_2, Z(G))$, $H_2 = \mathrm{Hom}(G_1, Z(G_1))^\times$. Moreover, H_1 is abelian, $(f * g)(w) = f(w)g(w)$ for $f, g \in H_1$ and $f * g * f' = f^\flat \circ g \circ (f^\flat)^{-1}$ for $f \in H_2$ and $g \in H_1$, where f' is the inverse element of $f \in H_2$.*

Proof. **(i)** By the hypothesis, $f^\flat$ is identity on $Z(G)$, so that f is invertible by Lemma 2.2 (ii) (and g is so). The other claims follow from definition (note that now $f \circ g = 1$).
(ii) Note that $Z(G) = Z(G_1)$ by the hypothesis. Then by (i), H_1 is an abelian subgroup of $\mathrm{Hom}(G, Z(G))^\times$ in which the multiplication is as in the statement.

For $f \in H_2$ and $g \in H_1$, the formula (2.1) implies that $f * g * f'$ is as in the statement (note that $f * f' = 1$ and ${f'}^\flat = (f^\flat)^{-1}$). In particular, we have $f * g * f'(G_1) \subset f^\flat \circ g(G_1) = 1$, since $f' \in H_2$ and so ${f'}^\flat(G_1) \subset G_1$. This means that $f * g * f' \in H_1$. Since obviously $H_1 \cap H_2 = 1$, we have $H_1 H_2 = H_1 \rtimes H_2$.

Finally, let $f \in \mathrm{Hom}(G, Z(G))^\times$. Take $g \in H_1$ such that $g(w) = f \circ \pi_2(w)^{-1}$ where π_2 is the projection $G \to G_2$ (this is the inverse element of $f \circ \pi_2 \in H_1$). Then for $w \in G_2$, we have

$$(g * f)(w) = g(w)f(w)g\big(f(w)\big)^{-1} = f(w)^{-1}f(w) = 1$$

since $g(Z(G)) = 1$. This means that $g * f \in \mathrm{Hom}(G_1, Z(G_1))$, while it is invertible since both f and g are so. Thus we have $g * f \in H_2$ and $f = (f \circ \pi_2) * g * f \in H_1 H_2$. Hence $\mathrm{Hom}(G, Z(G))^\times = H_1 \rtimes H_2$. □

In the proof of our results, we use the following notion. For a group G, we write $H \le G$, $H \lhd G$ if H is a subgroup, normal subgroup of G, respectively.

Definition 2.5. *For $H \le G$, define the* core $\mathrm{Core}_G(H)$ *of H in G to be the unique maximal normal subgroup of G contained in H (namely, $\bigcap_{w \in G} wHw^{-1}$).*

The following properties are deduced immediately from definition:

$$\text{If } H_1 \le H_2 \le G, \text{ then } \mathrm{Core}_G(H_1) \subset \mathrm{Core}_G(H_2). \tag{2.2}$$

$$\text{If } \mathrm{Core}_G(H) \le H_1 \le G, \text{ then } \mathrm{Core}_G(H) \subset \mathrm{Core}_G(H_1). \tag{2.3}$$

$$\text{If } H_\lambda \le G \ (\lambda \in \Lambda), \text{ then } \mathrm{Core}_G\Big(\bigcap_{\lambda \in \Lambda} H_\lambda\Big) = \bigcap_{\lambda \in \Lambda} \mathrm{Core}_G(H_\lambda). \tag{2.4}$$

$$\text{If } H_1 \le H_2 \le G, \ w \in G \text{ and } wH_1w^{-1} \cap H_2 = 1, \text{ then } H_1 \cap \mathrm{Core}_G(H_2) = 1. \tag{2.5}$$

Lemma 2.6. *Let $G_1 \le G_2 \le \cdots$, $H_1 \le H_2 \le \cdots$ be two infinite chains of subgroups of the same group such that $G_i \cap H_j = H_i$ for all $i \le j$. Put $G = \bigcup_{i=1}^\infty G_i$ and $H = \bigcup_{i=1}^\infty H_i$. Then $\mathrm{Core}_G(H) \subset \bigcup_{i=1}^\infty \mathrm{Core}_{G_i}(H_i)$.*

Proof. It is enough to show that $\mathrm{Core}_G(H) \cap H_i \subset \mathrm{Core}_{G_i}(H_i)$ (or more strongly, $\mathrm{Core}_G(H) \cap H_i \lhd G_i$) for all i. Note that the hypothesis implies $G_i \cap H = H_i$. Then for $g \in G_i$ and $h \in \mathrm{Core}_G(H) \cap H_i$, we have $ghg^{-1} \in \mathrm{Core}_G(H)$ and $ghg^{-1} \in G_i$, so that $ghg^{-1} \in G_i \cap H = H_i$. Thus the claim holds. □

The next lemma describes the centralizers of normal subgroups in terms of the cores of certain subgroups. Before stating this, note the following easy facts:

$$\text{If } H \lhd G, \text{ then the centralizer } Z_G(H) \text{ of } H \text{ is also normal in } G. \tag{2.6}$$

$$\text{If } X_1, X_2 \subset G \text{ are subsets and } X_1 \subset Z_G(X_2), \text{ then } X_2 \subset Z_G(X_1). \tag{2.7}$$

Lemma 2.7. *Let H be the smallest normal subgroup of G containing a subset $X \subset G$. Then $Z_G(H) = \mathrm{Core}_G(Z_G(X)) = \bigcap_{x \in X} \mathrm{Core}_G(Z_G(x))$.*

Proof. The second equality follows from (2.4). For the first one, the inclusion $\subset$ is deduced from (2.6) (since $Z_G(H) \subset Z_G(X)$). For the other inclusion, the centralizer of $\mathrm{Core}_G(Z_G(X))$ in G is normal in G (by (2.6)) and contains X, so that it also contains H. Thus the claim follows from (2.7). □

2.2 Coxeter groups and Coxeter graphs

Here we refer to [5] for basic definitions and properties. A pair (W, S) of a group W and its generating set S is a *Coxeter system* (and W itself is a *Coxeter group*) if W has the presentation

$$W = \langle S \mid (st)^{m(s,t)} = 1 \text{ if } s, t \in S \text{ and } m(s,t) < \infty \rangle$$

where $m : S \times S \to \{1, 2, \dots\} \cup \{\infty\}$ is a symmetric map such that $m(s,t) = 1$ if and only if $s = t$. (W, S) is said to be *finite* (*infinite*) if the group W is finite (infinite, respectively). The cardinality of S is called the *rank* of (W, S) (or even of W). Throughout this paper, *we do not assume, unless otherwise noticed, that the rank of* (W, S) *is finite (or even countable).* Note that, owing to the well-known fact that the element $st \in W$ above has precisely order $m(s,t)$ in W, this map m can be recovered uniquely from the Coxeter system (W, S).

Two Coxeter systems (W, S) and (W', S') are said to be *isomorphic* if there is some $f \in \mathrm{Isom}(W, W')$ such that $f(S) = S'$. Then there is a one-to-one correspondence (up to isomorphism) between Coxeter systems and the *Coxeter graphs*; which are simple (loopless), undirected, edge-labelled graphs with labels in $\{3, 4, \dots\} \cup \{\infty\}$. The Coxeter graph $\mathit{\Gamma}$ corresponding to (W, S) has the vertex set S, and two vertices $s, t \in S$ are joined in $\mathit{\Gamma}$ by an edge with label $m(s,t)$ if and only if $m(s,t) \geq 3$ (by convention, the labels '3' are usually omitted). $\mathit{\Gamma}$ (or (W, S)) is said to be of *finite type* if W is finite. It is also well known that, when W_I denotes the parabolic subgroup of W generated by a subset $I \subset S$, (W_I, I) is also a Coxeter system with Coxeter graph $\mathit{\Gamma}_I$ which is the full subgraph of $\mathit{\Gamma}$ on the vertex set I.

A Coxeter system (W, S) is called *irreducible* if the corresponding Coxeter graph $\mathit{\Gamma}$ is connected. In this case, W is also said to be *irreducible.* As is well known, W is decomposed as the direct product of its *irreducible components*, which are the parabolic subgroups W_I of W corresponding to the connected components $\mathit{\Gamma}_I$ of $\mathit{\Gamma}$ (in this case, each subset I is also said to be an *irreducible component* of S). A parabolic subgroup $W_I \subset W$ is said to be *irreducible* if the Coxeter system (W_I, I) is irreducible. As we mentioned in Introduction, an irreducible Coxeter group may be directly decomposable (as an abstract group) in general. Our main result determines which irreducible Coxeter group is indeed directly indecomposable.

In this paper, we use the following notations for some Coxeter graphs.

Definition 2.8. *We use the notations in Fig. 1. For each of the Coxeter graphs, let s_i denote the vertex having label i. Moreover, for each Coxeter graph $\Gamma(\mathcal{T}_n)$ in Fig. 1 ($\mathcal{T}$ = A, B, D, E, F, H), let $\Gamma(\mathcal{T}_k)$ ($k < n$) be the full subgraph of $\Gamma(\mathcal{T}_n)$ on vertex set $\{s_i \mid 1 \le i \le k\}$. For any $\mathcal{T}$, let $(W(\mathcal{T}), S(\mathcal{T}))$ be the Coxeter system corresponding to the Coxeter graph $\Gamma(\mathcal{T})$.*

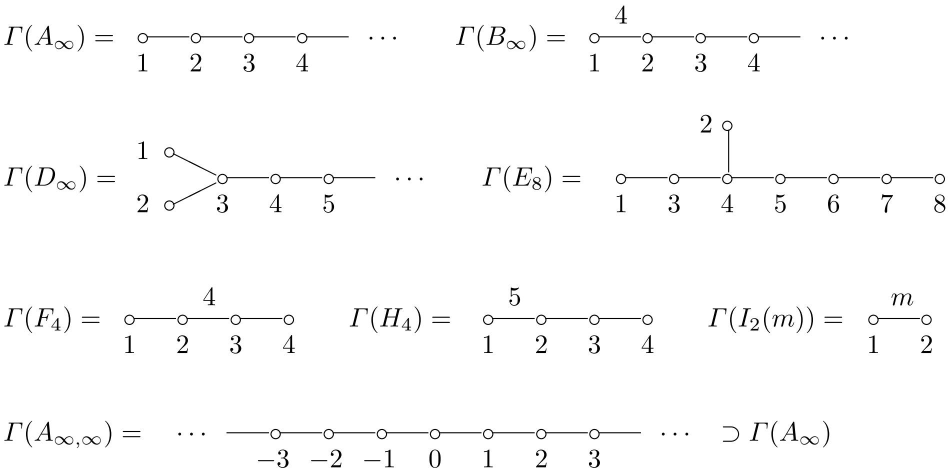

Figure 1: Some connected Coxeter graphs

By definition, $\Gamma(\mathcal{T}_\infty)$ ($\mathcal{T} = A, B, D$) and $\Gamma(A_{\infty,\infty})$ are Coxeter graphs with countable (infinite) vertex sets. On the other hand, it is well known that the Coxeter graphs $\Gamma(A_n)$ ($1 \le n < \infty$), $\Gamma(B_n)$ ($2 \le n < \infty$), $\Gamma(D_n)$ ($4 \le n < \infty$), $\Gamma(E_6)$, $\Gamma(E_7)$, $\Gamma(E_8)$, $\Gamma(F_4)$, $\Gamma(H_3)$, $\Gamma(H_4)$ and $\Gamma(I_2(m))$ ($5 \le m < \infty$) are all the connected Coxeter graphs of finite type (up to isomorphism). Note that $\Gamma(B_1) = \Gamma(D_1) = \Gamma(A_1)$, while $\Gamma(D_2) \simeq \Gamma(A_1 \times A_1)$ and $\Gamma(D_3) \simeq \Gamma(A_3)$ (but the vertex labels are different).

2.3 Root systems of Coxeter groups

For a Coxeter system (W, S), let Π be the set of symbols α_s ($s \in S$) and V the vector space over $\mathbb{R}$ containing the set Π as a basis. We define the symmetric bilinear form $\langle\, ,\, \rangle$ on V for the basis by

$$\langle \alpha_s, \alpha_t \rangle = -\cos(\pi/m(s,t)) \text{ if } m(s,t) < \infty,\ \langle \alpha_s, \alpha_t \rangle = -1 \text{ if } m(s,t) = \infty.$$

Then W acts faithfully on the space V by $s \cdot v = v - 2\langle \alpha_s, v \rangle \alpha_s$ ($s \in S$, $v \in V$). Let $\Phi = W \cdot \Pi$, the *root system* of (W, S). The above rule implies that the action of W preserves the bilinear form; as a consequence, any element (*root*) of Φ is a unit vector. It is a crucial fact that Φ is a disjoint union of the set Φ^+

of positive roots (i.e. roots in which the coefficient of every $\alpha_s \in \Pi$ is ≥ 0) and the set $\Phi^- = -\Phi^+$ of negative roots. It is well known that the cardinality of the set $\Phi[w] = \{\gamma \in \Phi^+ \mid w \cdot \gamma \in \Phi^-\}$ is (finite and) equal to the length $\ell(w)$ of $w \in W$ with respect to the generating set S. From the fact it follows easily that the set $\Phi[w]$ characterizes the element $w \in W$; namely,

$$\text{if } w, u \in W \text{ and } \Phi[w] = \Phi[u]\,, \text{ then } w = u \tag{2.8}$$

(observe $\Phi\left[wu^{-1}\right] = \emptyset$ and so $wu^{-1} = 1$).

The *reflection* along a root $\gamma = w \cdot \alpha_s \in \Phi$ is defined by $s_\gamma = wsw^{-1} \in W$. This definition does not depend on the choice of w and s, and s_γ indeed acts as a reflection on the space V; $s_\gamma \cdot v = v - 2\langle \gamma, v\rangle\gamma$ for $v \in V$. Note that $s_{\alpha_s} = s$ for $s \in S$. The following fact is easy to show (by the fact that $\Phi = \Phi^+ \sqcup \Phi^-$):

$$\text{if } s \in S, \gamma \in \Phi^+ \text{ and } \langle \alpha_s, \gamma\rangle > 0, \text{ then } s_\gamma \cdot \alpha_s \in \Phi^-. \tag{2.9}$$

For $v \in V$, put

$$v = \sum_{s \in S} ([\alpha_s]\, v)\alpha_s \text{ and } \mathrm{supp}(v) = \{s \in S \mid [\alpha_s]\, v \neq 0\}.$$

For $I \subset S$, let V_I be the subspace of V spanned by the set $\Pi_I = \{\alpha_s \mid s \in I\}$ and $\Phi_I = \Phi \cap V_I$ (namely, the set of all $\gamma \in \Phi$ such that $\mathrm{supp}(\gamma) \subset I$). Then it is well known that Φ_I coincides with the root system $W_I \cdot \Pi_I$ of the Coxeter system (W_I, I) (cf. Lemma 4 of [3], etc. for the proof). This fact yields the following:

$$\text{If } \gamma \in \Phi, \text{ then } (\gamma \in \Phi_{\mathrm{supp}(\gamma)} \text{ and so) the set } \mathrm{supp}(\gamma) \text{ is connected in } \mathit{\Gamma}. \tag{2.10}$$

Moreover, it is well known (cf. [5], Section 5.8, Exercise 4, etc.) that:

$$\text{If } I \subset S \text{ and } \gamma \in \Phi, \text{ then } s_\gamma \in W_I \text{ if and only if } \gamma \in \Phi_I. \tag{2.11}$$

For $I \subset S$, let

$$\begin{aligned} I^\perp &= \{s \in S \smallsetminus I \mid st = ts \text{ for all } t \in I\} \\ &= \{s \in S \smallsetminus I \mid s \text{ is adjacent in } \mathit{\Gamma} \text{ to no element of } I\} \\ &= \{s \in S \mid \alpha_s \text{ is orthogonal to every } \alpha_t \in \Pi_I\}. \end{aligned}$$

Then we have the following properties:

$$\text{If } \gamma \in \Phi^+ \text{ and } \mathrm{supp}(\gamma) \not\subset I \subset S, \text{ then } w \cdot \gamma \in \Phi^+ \text{ for all } w \in W_I. \tag{2.12}$$

$$\text{If } \gamma \in \Phi, I = \mathrm{supp}(\gamma) \text{ and } s \in S \smallsetminus (I \cup I^\perp), \text{ then } \mathrm{supp}(s \cdot \gamma) = I \cup \{s\}. \tag{2.13}$$

(For (2.12), take some $t \in \mathrm{supp}(\gamma) \smallsetminus I$, then $w \cdot \gamma$ has the same (positive) coefficient of α_t as γ. For (2.13), note that $\langle \alpha_s, \gamma\rangle < 0$ by the hypothesis.)

For $I \subset S$ and $w \in W$, let $\Phi_I^+ = \Phi_I \cap \Phi^+$, $\Phi_I^- = \Phi_I \cap \Phi^-$ and $\Phi_I[w] = \Phi_I \cap \Phi[w]$.

Lemma 2.9. *Let $w \in W$, $I, J \subset S$ and suppose that $I \cap J = \emptyset$, $w \cdot \Pi_I = \Pi_I$ and $w \cdot \Pi_J \subset \Phi^-$. Then $\Phi_{I \cup J}[w] = \Phi_{I \cup J}^+ \smallsetminus \Phi_I$.*

Proof. Let $\gamma \in \Phi^+_{I \cup J}$ such that $[\alpha_s]\,\gamma > 0$ for at least one $s \in J$ (note that $w \cdot \alpha_s \in \Phi^-$). Now if $w \cdot \alpha_s \in \Phi^-_I$, then $\alpha_s = w^{-1} \cdot (w \cdot \alpha_s)$ must be a linear combination of Π_I (since $w \cdot \Pi_I = \Pi_I$), but this is impossible. Thus we have $[\alpha_t]\,(w \cdot \alpha_s) < 0$ for some $t \in S \smallsetminus I$. Moreover, the hypothesis implies that $[\alpha_t]\,(w \cdot \alpha_{s'}) = 0$ for all $s' \in I$ and $[\alpha_t]\,(w \cdot \alpha_{s'}) \leq 0$ for all $s' \in J$. Thus we have

$$\begin{aligned}[\alpha_t]\,(w \cdot \gamma) &= [\alpha_t] \left(w \cdot \sum_{s' \in I \cup J} ([\alpha_{s'}]\,\gamma)\alpha_{s'} \right) \\ &= \sum_{s' \in I \cup J} ([\alpha_{s'}]\,\gamma)\,[\alpha_t]\,(w \cdot \alpha_{s'}) \leq [\alpha_s]\,\gamma\,[\alpha_t]\,(w \cdot \alpha_s) < 0.\end{aligned}$$

Hence the claim holds, since $w \cdot \Phi^+_I \subset \Phi^+$ by the hypothesis. □

Definition 2.10. *For a Coxeter system (W, S), we define the* odd Coxeter graph *Γ^{odd} of (W, S) to be the subgraph of Γ obtained by removing all edges labelled by an even number or ∞.*

It is well known (cf. [5], Section 5.3, Exercise, etc.) that, for $s, t \in S$,

$$\alpha_t \in W \cdot \alpha_s \text{ if and only if } s, t \text{ are in the same connected component of } \Gamma^{\mathrm{odd}}. \tag{2.14}$$

Moreover, the following lemma is deduced immediately from the definition that all fundamental relations of W are of the form $(st)^{m(s,t)} = 1$ $(s, t \in S)$.

Lemma 2.11. *Any $f \in \mathrm{Hom}(W, \{\pm 1\})$ assigns the same value to every vertex $s \in S$ of a connected component of Γ^{odd}. Conversely, any mapping $S \to \{\pm 1\}$ having this property extends uniquely to a homomorphism $W \to \{\pm 1\}$.*

2.4 Reflection decompositions of longest elements

If W_I is a finite parabolic subgroup of a Coxeter group W, then let $w_0(I)$ denote the *longest element* of W_I. This element is an involution and maps the set Π_I onto $-\Pi_I$, so that there is an involutive graph automorphism σ_I of the Coxeter graph Γ_I such that

$$w_0(I) \cdot \alpha_s = -\alpha_{\sigma_I(s)} \text{ for all } s \in I.$$

It is well known that, for an irreducible Coxeter system (W, S), we have $Z(W) \neq 1$ if and only if $W \simeq W(\mathcal{T})$ for one of $\mathcal{T} = A_1$, B_n ($n < \infty$), D_k ($k \geq 4$ even), E_7, E_8, F_4, H_3, H_4 and $I_2(m)$ ($m \geq 6$ even). This condition is also equivalent to that $|W| < \infty$ and $\sigma_S = \mathrm{id}_S$. Moreover, $Z(W) = \{1, w_0(S)\}$ if $Z(W) \neq 1$, while σ_S is determined as the unique non-identical automorphism of Γ whenever W is finite, irreducible and $Z(W) = 1$. Note that any automorphism $\tau \in \mathrm{Aut}(\Gamma)$ induces naturally an automorphism of W, which maps each element $w_0(I)$ to $w_0(\tau(I))$.

The next lemma introduces certain normal subgroups G_{B_n}, G_{D_n} of $W(B_n)$, $W(D_n)$, respectively, which play an important role in later sections.

Lemma 2.12. *(See Definition 2.8 for notations.)*
(i) *Let $1 \leq n \leq \infty$. Then the subgroup G_{B_n} of $W(B_n)$ generated by all $w_0(S(B_i))$ ($1 \leq i \leq n$, $i < \infty$) is normal in $W(B_n)$.*

(ii) *Let $1 \le n \le \infty$. Then the smallest normal subgroup G_{D_n} of $W(D_n)$ containing all $w_0(S(D_{2k}))$ ($1 \le k < \infty$, $2k \le n$) is the subgroup generated by all $w_0(S(D_i))$ ($2 \le i \le n$, $i < \infty$).*
(iii) *Moreover, each of the above normal subgroups is an elementary abelian 2-group with the generating set given there as the basis.*

Proof. For the case $n < \infty$, we refer to Section 2.10 of [5] the following realizations of $W(B_n)$ and $W(D_n)$. Namely, $W(B_n)$ is isomorphic to the semidirect product $K_n \rtimes \mathrm{Sym}_n$ of an elementary abelian 2-group $K_n \simeq \{\pm 1\}^n$, where the i-th element of the basis is denoted by κ_i, by the group Sym_n of permutations of the κ_i's. The isomorphism sends s_1 to κ_1 and s_i ($i \ge 2$) to $(i-1\ i) \in \mathrm{Sym}_n$. Now it is easily checked that $w_0(S(B_i)) = \kappa_1\kappa_2\cdots\kappa_i$, so that we have $G_{B_n} = K_n$. Moreover, $W(D_n)$ is isomorphic to $K_n^+ \rtimes \mathrm{Sym}_n$ where K_n^+ is the subgroup of K_n generated by $\kappa_i\kappa_{i+1}$ ($1 \le i \le n-1$), and the isomorphism sends s_1 to $\kappa_1\kappa_2 \cdot (1\ 2)$ and s_i ($i \ge 2$) to $(i-1\ i)$. Now $w_0(S(D_i)) = \kappa_1\kappa_2\cdots\kappa_i$ if $i \ge 2$ is even and $w_0(S(D_i)) = \kappa_2\kappa_3\cdots\kappa_i$ if $i \ge 2$ is odd, so that we have $G_{D_n} = K_n^+$. By those observations, the claims (i)–(iii) are deduced immediately.

For the remaining case $n = \infty$, note that $W(B_\infty)$ is the direct limit of the sequence $W(B_1) \subset W(B_2) \subset \cdots$, and similarly for $W(D_\infty)$. Thus the claims are also deduced in this case by the argument in the previous paragraph. □

In the paper [2], Deodhar established a method (in the proof of Theorem 5.4) for decomposing any involution $w \in W$ as a product of commuting reflections. From now, we apply this method and then obtain a decomposition of any longest element $w_0(I)$, which we call here a *reflection decomposition*. Among the various expressions of the elements $w_0(I)$, this decomposition possesses an advantage in a computation of the action on the space V. First, to each finite irreducible Coxeter system $(W,S) = (W(\mathcal{T}), S(\mathcal{T}))$ of type $\mathcal{T}$, we associate a (or two) positive root(s) $\widetilde{\alpha}_{\mathcal{T}} = \widetilde{\alpha}_{\mathcal{T}}^{(1)}$ (and $\widetilde{\alpha}_{\mathcal{T}}^{(2)}$), as follows (where we abbreviate $c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n \in V$ to $(c_1, c_2, \dots, c_n)$ in some cases):

$$\widetilde{\alpha}_{A_n} = \sum_{i=1}^{n} \alpha_i \quad (1 \le n < \infty),\ \widetilde{\alpha}_{D_n} = \alpha_1 + \alpha_2 + \sum_{i=3}^{n-1} 2\alpha_i + \alpha_n \quad (4 \le n < \infty),$$
$$\widetilde{\alpha}_{B_n}^{(1)} = \alpha_1 + \sum_{i=2}^{n} \sqrt{2}\,\alpha_i,\ \widetilde{\alpha}_{B_n}^{(2)} = \sqrt{2}\,\alpha_1 + \sum_{i=2}^{n-1} 2\alpha_i + \alpha_n \quad (2 \le n < \infty),$$
$$\widetilde{\alpha}_{E_6} = (1,2,2,3,2,1),\ \widetilde{\alpha}_{E_7} = (2,2,3,4,3,2,1),\ \widetilde{\alpha}_{E_8} = (2,3,4,6,5,4,3,2),$$
$$\widetilde{\alpha}_{F_4}^{(1)} = (2,3,2\sqrt{2},\sqrt{2}),\ \widetilde{\alpha}_{F_4}^{(2)} = (\sqrt{2},2\sqrt{2},3,2),$$
$$\widetilde{\alpha}_{H_3} = (c+1,2c,c),\ \widetilde{\alpha}_{H_4} = (3c+2,4c+2,3c+1,2c) \quad (\text{where } c = 2\cos\frac{\pi}{5}),$$
$$\widetilde{\alpha}_{I_2(m)} = \frac{1}{2\sin(\pi/2m)}\alpha_1 + \frac{1}{2\sin(\pi/2m)}\alpha_2 \quad (m \ge 5 \text{ odd}),$$
$$\widetilde{\alpha}_{I_2(m)}^{(i)} = \frac{\cos(\pi/m)}{\sin(\pi/m)}\alpha_i + \frac{1}{\sin(\pi/m)}\alpha_{3-i} \quad (m \ge 5 \text{ even}, i = 1,2).$$

To check that each of these is actually a root of $(W(\mathcal{T}), S(\mathcal{T}))$, note the equality

$c^2 = c + 1$ and the following formula for the root system of type $I_2(m)$:

If $w = (\cdots s_2 s_1 s_2) \in W(I_2(m))$ (k elements), then

$$w \cdot \alpha_1 = \begin{cases} \frac{\sin(k\pi/m)}{\sin(\pi/m)}\alpha_1 + \frac{\sin((k+1)\pi/m)}{\sin(\pi/m)}\alpha_2 & \text{if } k \text{ is odd,} \\ \frac{\sin((k+1)\pi/m)}{\sin(\pi/m)}\alpha_1 + \frac{\sin(k\pi/m)}{\sin(\pi/m)}\alpha_2 & \text{if } k \text{ is even.} \end{cases}$$

For example, we have

$$\widetilde{\alpha}_{F_4}^{(1)} = s_1 s_2 s_3 s_4 s_2 s_3 s_2 \cdot \alpha_1, \quad \widetilde{\alpha}_{F_4}^{(2)} = s_4 s_3 s_2 s_1 s_3 s_2 s_3 \cdot \alpha_4,$$
$$\widetilde{\alpha}_{H_3} = s_2 s_1 s_2 s_1 s_3 s_2 \cdot \alpha_1, \quad \widetilde{\alpha}_{H_4} = s_4 s_3 s_2 s_1 s_2 s_1 s_3 s_2 s_1 s_4 s_3 s_2 s_1 s_2 s_3 s_4 \cdot \widetilde{\alpha}_{H_3},$$
$$\widetilde{\alpha}_{I_2(2k+1)} = (\cdots s_2 s_1 s_2) \cdot \alpha_1 \ (k \text{ elements}), \ \widetilde{\alpha}_{I_2(4k)}^{(i)} = (s_{3-i} s_i)^{k-1} s_{3-i} \cdot \alpha_i.$$

By (2.14), if $\mathcal{T} \neq B_n$, F_4, $I_2(m)$ (m even), then Φ consists of a single orbit $W(\mathcal{T}) \cdot \alpha_1$ (and so it contains $\widetilde{\alpha}_{\mathcal{T}}$). On the other hand, if $\mathcal{T} = B_n$, F_4 or $I_2(4k)$, then (2.14) implies that Φ consists of two orbits (namely, $W \cdot \alpha_1$ and $W \cdot \alpha_2$ if $\mathcal{T} = B_n$, $I_2(4k)$, and $W \cdot \alpha_1$ and $W \cdot \alpha_4$ if $\mathcal{T} = F_4$). In these case, $\widetilde{\alpha}_{\mathcal{T}}^{(1)}$ lies in the orbit $W \cdot \alpha_1$ and $\widetilde{\alpha}_{\mathcal{T}}^{(2)}$ lies in the other one.

In contrast with the above cases, if $\mathcal{T} = I_2(4k+2)$, then Φ consists of two orbits $W(\mathcal{T}) \cdot \alpha_1$ and $W(\mathcal{T}) \cdot \alpha_2$, and now we have $\widetilde{\alpha}_{\mathcal{T}}^{(1)} \in W(\mathcal{T}) \cdot \alpha_2$ (and $\widetilde{\alpha}_{\mathcal{T}}^{(2)}$ lies in the other orbit). In fact, we have $\widetilde{\alpha}_{I_2(4k+2)}^{(i)} = (s_{3-i} s_i)^k \cdot \alpha_{3-i}$ for $i = 1, 2$.

To simplify the description, we denote the reflection along the root $\widetilde{\alpha}_{\mathcal{T}}^{(i)}$ by $\widetilde{r}(\mathcal{T}, i)$. If we have only one root $\widetilde{\alpha}_{\mathcal{T}}^{(i)}$, namely $\mathcal{T} \neq B_n$, F_4, $I_2(m)$ (m even), then we also write $\widetilde{r}(\mathcal{T}) = \widetilde{r}(\mathcal{T}, 1)$.

Remark 2.13. *By the above observation, if $\mathcal{T} = B_n$, F_4 or $I_2(4k)$, then $\widetilde{r}(\mathcal{T}, 1)$ is conjugate to s_1, and $\widetilde{r}(\mathcal{T}, 2)$ is conjugate to s_2 (if $\mathcal{T} = B_n$ or $I_2(4k)$) or to s_4 (if $\mathcal{T} = F_4$). On the other hand, if $\mathcal{T} = I_2(4k+2)$, then $\widetilde{r}(\mathcal{T}, 1)$, $\widetilde{r}(\mathcal{T}, 2)$ are conjugate to s_2, s_1, respectively.*

Lemma 2.14. ***(i)*** *If $\mathcal{T} \neq A_n$ ($n \geq 2$), $I_2(m)$ (m odd), then for the root $\widetilde{\alpha}_{\mathcal{T}}^{(i)}$, there is an index $N(\mathcal{T}, i)$ such that $\langle \widetilde{\alpha}_{\mathcal{T}}^{(i)}, \alpha_j \rangle = 0$ for all $j \neq N(\mathcal{T}, i)$. Moreover, we have $\langle \widetilde{\alpha}_{\mathcal{T}}^{(i)}, \alpha_{N(\mathcal{T},i)} \rangle > 0$ and $\Phi[\widetilde{r}(\mathcal{T}, i)] = \Phi^+ \smallsetminus \Phi_{S(\mathcal{T}) \smallsetminus \{s_{N(\mathcal{T},i)}\}}$. (If we have only one root $\widetilde{\alpha}_{\mathcal{T}}^{(i)}$, then we also write $N(\mathcal{T}) = N(\mathcal{T}, 1)$.)*
(ii) *If $\mathcal{T} = A_n$ ($n \geq 2$) or $I_2(m)$ (m odd), then there are two indices $N_1(\mathcal{T}), N_2(\mathcal{T})$ such that $\langle \widetilde{\alpha}_{\mathcal{T}}, \alpha_{N_j(\mathcal{T})} \rangle > 0$ for $j = 1, 2$ and $\langle \widetilde{\alpha}_{\mathcal{T}}, \alpha_j \rangle = 0$ for all $j \neq N_1(\mathcal{T}), N_2(\mathcal{T})$. Moreover, we have $\Phi[\widetilde{r}(\mathcal{T})] = \Phi^+ \smallsetminus \Phi_{S(\mathcal{T}) \smallsetminus \{s_{N_1(\mathcal{T})}, s_{N_2(\mathcal{T})}\}}$.*

Proof. **(i)** The first claim follows from a direct computation, by putting

$$N(A_1) = 1, \quad N(B_n, 1) = n, \quad N(B_n, 2) = n-1, \quad N(D_n) = n-1,$$
$$N(E_6) = 2, \quad N(E_7) = 1, \quad N(E_8) = 8, \quad N(F_4, 1) = 1, \quad N(F_4, 2) = 4,$$
$$N(H_3) = 2, \quad N(H_4) = 4, \quad N(I_2(2k), 1) = 2, \quad N(I_2(2k), 2) = 1.$$

For the second one, expand the equality $\langle \widetilde{\alpha}_{\mathcal{T}}^{(i)}, \widetilde{\alpha}_{\mathcal{T}}^{(i)} \rangle = 1$ and use the first claim. Now the third one follows from (2.9) and Lemma 2.9.
(ii) The former claim also follows from a direct computation, by putting

$$N_1(A_n) = 1, \ N_2(A_n) = n, \quad N_1(I_2(2k+1)) = 1, \ N_2(I_2(2k+1)) = 2.$$

The remaining proof is similar to (i). $\square$

Now our method for obtaining a reflection decomposition of $w_0(I)$, which is the same as a decomposition given by Deodhar's method ([2], proof of Theorem 5.4), is summarized as follows:

(I) If $I = \emptyset$, then this algorithm finishes with the (trivial) decomposition $w_0(I) = 1$. If $I \neq \emptyset$, choose an irreducible component J of I. Let $J = S(\mathcal{T})$.

(II) If $\mathcal{T} \neq A_n$ ($n \geq 2$), $I_2(m)$ (m odd), take the (or one of the two) root(s) $\widetilde{\alpha}_{\mathcal{T}}^{(i)}$. By Lemma 2.14 (i), $\widetilde{r}(\mathcal{T}, i)$ commutes with all elements of $K = I \smallsetminus \{s_{N(\mathcal{T},i)}\}$, and we have $w_0(I) = \widetilde{r}(\mathcal{T}, i)w_0(K)$ (since $\Phi\left[w_0(I)\right] = \Phi\left[\widetilde{r}(\mathcal{T}, i)w_0(K)\right] = \Phi_I^+$; cf. (2.8)). Then apply this algorithm inductively to the (smaller) set K.

(III) If $\mathcal{T} = A_n$ ($n \geq 2$) or $I_2(m)$ (m odd), then similarly, $\widetilde{r}(\mathcal{T})$ commutes with all elements of $K = I \smallsetminus \{s_{N_1(\mathcal{T})}, s_{N_2(\mathcal{T})}\}$ and $w_0(I) = \widetilde{r}(\mathcal{T})w_0(K)$ by Lemma 2.14 (ii). Then apply this algorithm inductively to the (smaller) set K.

By collecting the subset $K \subset I$ appearing in the step (II) or (III) of every turn, we obtain a decreasing sequence ($K_0 = I$,) $K_1, \dots, K_{r-1}, K_r = \emptyset$. We call this a *generator sequence* (of length r) for the set I.

2.5 Direct product decompositions of finite Coxeter groups

As an application of the reflection decomposition introduced in Section 2.4, we determine easily which finite irreducible Coxeter groups have the center as a nontrivial direct factor. Although this result itself is not a new one, we restate it here since the result is used in later sections.

For a Coxeter system (W, S), let W^+ denote the normal subgroup of W (of index two) consisting of elements of even length. This coincides with the kernel of the map $\mathrm{sgn} \in \mathrm{Hom}(W, \{\pm 1\})$ such that $\mathrm{sgn}(w) = (-1)^{\ell(w)}$. Since any reflection in W has odd length, the following lemma follows from (the proof of) Lemma 2.1:

Lemma 2.15. *If (W, S) is a finite irreducible Coxeter system and $Z(W) \neq 1$, then we have $W = Z(W) \times W^+$ if and only if some (or equivalently, any) generator sequence for S (cf. Section 2.4) has odd length.*

Theorem 2.16. *Let (W, S) be an irreducible Coxeter system such that $Z(W) \neq 1$ (so that $|W| < \infty$). Then $Z(W)$ ($\simeq W(A_1)$) is a proper direct factor of W if and only if $W \simeq W(\mathcal{T})$ for $\mathcal{T} = B_{2k+1}$, $I_2(4k+2)$ ($k \geq 1$), E_7 or H_3. In the first two cases, W is isomorphic to $W(A_1) \times W(D_{2k+1})$, $W(A_1) \times W(I_2(2k+1))$ respectively. In the last two cases, we have $W = Z(W) \times W^+$.*

Proof. Note that $Z(W) \simeq \{\pm 1\}$ by the hypothesis. Since $Z(W(A_1)) = W(A_1)$, we may assume $W \neq W(A_1)$.

Case 1. $W = W(B_n)$ **(**$n \geq 2$**):** First, we have $\mathrm{Hom}(W, \{\pm 1\}) = \{1, \mathrm{sgn}, \varepsilon_1, \varepsilon_2\}$ by Lemma 2.11, where 1 denotes the trivial map, $\varepsilon_1(s_1) = -1$, $\varepsilon_1(s_i) = 1$, $\varepsilon_2(s_1) = 1$ and $\varepsilon_2(s_i) = -1$ ($i \neq 1$). Now we consider the following reflection decomposition:

$$w_0(S) = \widetilde{r}(B_n, 1)\widetilde{r}(B_{n-1}, 1) \cdots \widetilde{r}(B_2, 1)s_1.$$

By Remark 2.13, each reflection $\widetilde{r}(B_k, 1)$ is conjugate to s_1. This implies that any expression of $\widetilde{r}(B_k, 1)$ as a product of generators contains an odd number of s_1 and an even number of s_i ($i \neq 1$). Thus we have

$$\mathrm{sgn}(\widetilde{r}(B_k, 1)) = \varepsilon_1(\widetilde{r}(B_k, 1)) = -1 \text{ and } \varepsilon_2(\widetilde{r}(B_k, 1)) = 1.$$

If n is even, then all $f \in \mathrm{Hom}(W, \{\pm 1\})$ maps $w_0(S)$ to 1 by the above property. Thus by Lemma 2.1, $Z(W)$ is not a direct factor.

On the other hand, if n is odd, then we have $\varepsilon_1(w_0(S)) = -1$ and so $W = Z(W) \times \ker \varepsilon_1$ by the proof of Lemma 2.1. Note that $\ker \varepsilon_1$ consists of elements in which s_1 appears an even number of times. Since s_1 commutes with all s_i ($3 \leq i \leq n$), it can be deduced directly that $\ker \varepsilon_1$ is generated by $s'_1 = s_1 s_2 s_1$ and all $s'_i = s_i$ ($2 \leq i \leq n$). Moreover, $\ker \varepsilon_1$ forms a Coxeter group of type D_n; in fact, $s'_1, \dots, s'_n$ satisfy the fundamental relations of type D_n (so that $\ker \varepsilon_1$ is a quotient of $W(D_n)$), while the order $|W(B_n)|/2$ of $\ker \varepsilon_1$ coincides with $|W(D_n)|$. Hence the claim holds in this case.

Case 2. $W = W(\mathcal{T})$ **for** $\mathcal{T} = D_{2k}$ **(**$k \geq 2$**),** E_7**,** E_8**,** H_3**,** H_4**:** Since Γ^{odd} is connected in this case, we have $\mathrm{Hom}(W, \{\pm 1\}) = \{1, \mathrm{sgn}\}$ by Lemma 2.11. Thus the claim follows from Lemmas 2.1 and 2.15, by taking the following generator sequence for S (where we abbreviate the set $\{s_{i_1}, s_{i_2}, \dots, s_{i_r}\}$ to $i_1 i_2 \cdots i_r$):

$$\begin{cases} S(D_{2k-2}) \cup \{s_{2k}\},\ S(D_{2k-2}), \dots,\ S(D_4),\ 124,\ 12,\ 1,\ \emptyset & \text{if } \mathcal{T} = D_{2k}, \\ S(E_7),\ 234567,\ 23457,\ 2345,\ 235,\ 23,\ 2,\ \emptyset & \text{if } \mathcal{T} = E_8, \\ 234567,\ 23457,\ 2345,\ 235,\ 23,\ 2,\ \emptyset & \text{if } \mathcal{T} = E_7, \\ S(H_3),\ 13,\ 1,\ \emptyset & \text{if } \mathcal{T} = H_4, \\ 13,\ 1,\ \emptyset & \text{if } \mathcal{T} = H_3 \end{cases}$$

(note that the first sequence consists of $2k$ terms).

Case 3. $W = W(F_4)$**:** We have a generator sequence 234, 23, 2, $\emptyset$ for S and the corresponding decomposition of $w_0(S)$ into four reflections, all of which are conjugate to s_1 and s_2 (cf. Remark 2.13). This (and Lemma 2.11) implies that any $f \in \mathrm{Hom}(W, \{\pm 1\})$ maps all the four reflections to the same element $f(s_1)$, so that $f(w_0(S)) = 1$. Hence the claim follows from Lemma 2.1.

Case 4. $W = W(I_2(2k))$ **(**$k \geq 3$**):** We have a reflection decomposition $w_0(S) = \widetilde{r}(I_2(2k), 1) s_1$. If k is even, then $\widetilde{r}(I_2(2k), 1)$ is conjugate to s_1 (cf. Remark 2.13). Now by a similar argument to the previous case, any $f \in \mathrm{Hom}(W, \{\pm 1\})$ maps $w_0(S)$ to 1. Thus $Z(W)$ is not a direct factor by Lemma 2.1.

On the other hand, if k is odd, then $\widetilde{r}(I_2(2k), 1)$ is conjugate to s_2 (cf. Remark 2.13). Thus $\varepsilon_1 \in \mathrm{Hom}(W, \{\pm 1\})$ ($\varepsilon(s_1) = -1$, $\varepsilon(s_2) = 1$) sends $w_0(S)$ to -1, so that $W = Z(W) \times \ker \varepsilon_1$ by the proof of Lemma 2.1. Moreover, $\ker \varepsilon_1$ is generated by two reflections $s_1 s_2 s_1$ and s_2, and so $\ker \varepsilon_1$ is a Coxeter system of type $I_2(k)$ (since $s_1 s_2 s_1 s_2$ has order k). Hence the claim holds in all cases. $\square$

Since the groups $W(E_7)^+$ and $W(H_3)^+$ are known to be (isomorphic to) the well-examined simple groups $S_6(2)$ and A_5 respectively (cf. [5], Sections 2.12–13, etc.), we omit the proof of the following properties of these groups. Note that these properties can also be proved by using Theorems 2.16 and 3.3 below.

Lemma 2.17. *Let* $G = W(\mathcal{T})^+$, $\mathcal{T} \in \{E_7, H_3\}$. *Then* G *has trivial center, is directly indecomposable and is generated by involutions. Moreover,* G *is not isomorphic to a Coxeter group.*

2.6 Notes on normalizers in Coxeter groups

In this subsection, we summarize some properties of normalizers $N_W(W_I)$ of parabolic subgroups W_I in Coxeter groups W. Note that the explicit structure of $N_W(W_I)$ is well-described in [1] (or in [4], when $|W| < \infty$) and also in [8]; however, such a strong result is not required for our purpose.

The following result is due to D. Krammer [6]. The first part appears in Proposition 3.1.9 (a) of [6], while the second one is easily deduced by the argument of Section 3.1 of [6], particularly by Corollary 3.1.5.

Proposition 2.18 (cf. [6]). ***(i)*** *If $I \subset S$, then $N_W(W_I)$ is the semidirect product $W_I \rtimes G_I$ of W_I by the group $G_I = \{w \in W \mid w \cdot \Pi_I = \Pi_I\}$.*
(ii) *If $I \subset J \subset S$ and W_I is an infinite irreducible component of W_J, then $N_W(W_J) \subset W_{I \cup I^\perp}$.*

We also require the following result. This was originally given by Deodhar [2], in the proof of Proposition 4.2, for the case $|S| < \infty$ only. Here we give a proof covering the case $|S| = \infty$ as well for the sake of completeness.

Proposition 2.19 (cf. [2], Proposition 4.2). *If (W, S) is irreducible and $|W| = \infty$, then $|\Phi \smallsetminus \Phi_I| = \infty$ for all proper subsets $I \subset S$.*

Proof. We consider the case $|S| < \infty$ first. Put $\Psi_J = \{\gamma \in \Phi \mid \mathrm{supp}(\gamma) = J\}$ for $J \subset S$. Our aim is to show $|\Psi_S| = \infty$. If this fails, and $J \subset S$ is maximal subject to $|\Psi_J| = \infty$ (note that some of Ψ_K's must be infinite since their finite union Φ is so), then $J \subsetneq S$ and $s \in S \smallsetminus J$ maps Ψ_J injectively into $\Psi_{J \cup \{s\}}$ when s is adjacent to J in Γ. Since Γ is connected by the hypothesis, such an element s indeed exists, contradicting the maximality of J. Hence the claim holds.

Secondly, suppose $|S| = \infty$. Since Γ is connected, there are infinitely many finite subsets $J \subset S$ such that $J \not\subset I$ and Γ_J is connected. Now the claim holds, since each Φ_J contains a root γ such that $\mathrm{supp}(\gamma) = J$. □

By those properties, we can prove the following corollary.

Corollary 2.20. *Let $s \in S$ and $I = S \smallsetminus \{s\}$.*
(i) *If $1 \neq w \in G_I$, then $\Phi[w] = \Phi^+ \smallsetminus \Phi_I$. Hence by (2.8), such an element w is unique if it exists.*
(ii) *If $|W| < \infty$ and $w_0(S) \in N_W(W_I)$, then $N_W(W_I) = W_I \rtimes \{1, w_0(S)\}$.*
(iii) *If (W, S) is irreducible and $|W| = \infty$, then $G_I = 1$ and $N_W(W_I) = W_I$.*

Proof. **(i)** In this case, we have $w \cdot \alpha_s \in \Phi^-$ (otherwise, we have $w \cdot \Phi^+ \subset \Phi^+$ but this is a contradiction). Now the claim follows from Lemma 2.9.
(ii) Note that $w_0(S) \notin W_I$, while $|G_I| \leq 2$ by (i). Thus by Proposion 2.18 (i), $N_W(W_I)$ is generated by W_I and $w_0(S)$. Now the claim holds, since $w_0(S)^2 = 1$.
(iii) In this case, we have $|\Phi^+ \smallsetminus \Phi_I| = \infty$ by Proposition 2.19. Thus we have $G_I = 1$ by (i), since the set $\Phi[w]$ is always finite. Hence the claim holds. □

Owing to this description, we have the following:

Corollary 2.21. ***(i)*** *If $W = W(B_n)$, $2 \leq n < \infty$, then $\bigcap_{i=1}^{n-1} N_W(W_{S(B_i)}) = G_{B_n}$.*
(ii) *If $W = W(D_n)$, $3 \leq n < \infty$, then $\bigcap_{i=2}^{n-1} N_W(W_{S(D_i)}) = G_{D_n} \rtimes \langle s_1 \rangle$.*

Proof. Note that, by Lemma 2.12, G_{B_n} is generated by all $w_0(S(B_k))$ ($1 \leq k \leq n$). On the other hand, by Lemma 2.12 again, the product $G_{D_n}\langle s_1 \rangle$ is a semidirect product with G_{D_n} normal, and it is generated by all $w_0(S(D_k))$ ($1 \leq k \leq n$).

We prove the two claims in parallel. Let $\mathcal{T} = B$ and $L = 1$ (for (i)), $\mathcal{T} = D$ and $L = 2$ (for (ii)), respectively. By the above remark, it is enough to show that the group in the left side is generated by all $w_0(S(\mathcal{T}_k))$ $(1 \leq k \leq n)$. We use induction on n. First, note that $w_0(S(\mathcal{T}_n)) \in N_W(W_{S(\mathcal{T}_i)})$ for all $L \leq i \leq n-1$. Put $W' = W_{S(\mathcal{T}_{n-1})}$. Then by Corollary 2.20 (ii), we have $N_W(W') = W' \rtimes \langle w_0(S(\mathcal{T}_n))\rangle$. Thus the claim holds if $n = L+1$; in fact, in this case, $W' = W_{S(\mathcal{T}_L)}$ is generated by all $w_0(S(\mathcal{T}_i))$ $(1 \leq i \leq L)$.

If $n > L+1$, then the above equality implies that

$$\begin{aligned}\bigcap_{i=L}^{n-1} N_W(W_{S(\mathcal{T}_i)}) &= \left(\bigcap_{i=L}^{n-2} N_W(W_{S(\mathcal{T}_i)})\right) \cap \Big(W' \rtimes \langle w_0(S(\mathcal{T}_n))\rangle\Big)\\ &= \left(\bigcap_{i=L}^{n-2} N_{W'}(W_{S(\mathcal{T}_i)})\right) \rtimes \langle w_0(S(\mathcal{T}_n))\rangle\end{aligned}$$

since $w_0(S(\mathcal{T}_n)) \in \bigcap_{i=L}^{n-2} N_W(W_{S(\mathcal{T}_i)})$. By the induction, the first factor of the semidirect product is generated by all $w_0(S(\mathcal{T}_i))$ $(1 \leq i \leq n-1)$. Thus the claim also holds in this case. Hence the proof is concluded. $\square$

We summarize some more properties of the normalizers. First, we have:

$$\text{If } I, J \subset S, \text{ then } N_W(W_I) \cap N_W(W_J) \subset N_W(W_{I\cap J}). \tag{2.15}$$

$$\text{For } I \subset S, w \in N_W(W_I) \text{ if and only if } w \cdot \Phi_I = \Phi_I. \tag{2.16}$$

((2.15) follows from the well-known fact $W_I \cap W_J = W_{I\cap J}$. (2.16) follows immediately from (2.11).) Moreover, we have the following:

Lemma 2.22. *Let $I \subset J \subset S$ such that $J \smallsetminus I \subset I^{\perp}$. Then*

$$N_W(W_J) \cap N_W(W_I) \subset N_W(W_{J \smallsetminus I}).$$

Proof. Let $w \in N_W(W_J) \cap N_W(W_I)$ and $s \in J \smallsetminus I$. Then $w \cdot \Phi_J = \Phi_J$ and $w \cdot \Phi_I = \Phi_I$ by (2.16), so that we have $w \cdot \alpha_s \in \Phi_J$ and $w \cdot \alpha_s \notin \Phi_I$. Now by the hypothesis and (2.10), we have $\mathrm{supp}(w \cdot \alpha_s) \subset J \smallsetminus I$ and so $w \cdot \alpha_s \in \Phi_{J \smallsetminus I}$. Hence the claim follows from (2.16). $\square$

3 Main results

3.1 Direct indecomposability

In this subsection, we give the main result of this paper; all infinite irreducible Coxeter groups are in fact directly indecomposable, even if it has infinite rank (Theorem 3.3). As is mentioned in Introduction, this result was already shown in [9] for the case of finite rank, in which the finiteness of the ranks is essential and so cannot be removed immediately.

Our proof is based on the following complete description (proved in later sections) of the centralizers of normal subgroups, which are generated by involutions, in irreducible Coxeter groups (possibly of infinite rank):

Theorem 3.1. *(See Definition 2.8 for notations.) Let (W, S) be an irreducible Coxeter system of an arbitrary rank, and $H \lhd W$ a normal subgroup generated*

by involutions. Then:
(i) *If $H \subset Z(W)$, then $Z_W(H) = W$.*
(ii) *If $(W, S) = (W(B_n), S(B_n))$, $2 \leq n \leq \infty$, $\tau \in \mathrm{Aut}(\Gamma(B_n))$, $H \not\subset Z(W)$ and $H \subset \tau(G_{B_n})$, then $Z_W(H) = \tau(G_{B_n})$. (cf. Lemma 2.12 for definition of G_{B_n}.)*
(iii) *If $(W, S) = (W(D_n), S(D_n))$, $3 \leq n \leq \infty$, $\tau \in \mathrm{Aut}(\Gamma(D_n))$, $H \not\subset Z(W)$ and $H \subset \tau(G_{D_n})$, then $Z_W(H) = \tau(G_{D_n})$. (cf. Lemma 2.12 for definition of G_{D_n}.)*
(iv) *Otherwise, $Z_W(H) = Z(W)$.*

This theorem yields the following corollary. A group G is said to be a *central product* of two subgroups H_1, H_2 if $G = H_1H_2$ and $H_2 \subset Z_G(H_1)$ (or equivalently $H_1 \subset Z_G(H_2)$). Note that $H_1 \cap H_2 \subset Z(G)$ in this case.

Corollary 3.2. *Let (W, S) be an irreducible Coxeter system of an arbitrary rank, and suppose that W is a central product of two subgroups G_1, G_2 generated by involutions. Then either $G_1 \subset Z(W)$ or $G_2 \subset Z(W)$.*

Proof. By definition, we have $G_2 \subset Z_W(G_1)$, $W = G_1 Z_W(G_1)$ and $G_1 \lhd W$. Now if G_1 satisfies the condition of cases (ii) or (iii) of Theorem 3.1, then G_1 and $Z_W(G_1)$ are contained in the same proper subgroup of W. This is impossible, so that we have $G_1 \subset Z(W)$ (case (i)) or $G_2 \subset Z_W(G_1) = Z(W)$ (case (iv)). □

Now our main result follows immediately:

Theorem 3.3. *The only nontrivial direct product decompositions of an irreducible Coxeter group W (of an arbitrary rank) are the ones given in Theorem 2.16. In particular, W is directly indecomposable if and only if $W \not\simeq W(\mathcal{T})$ for $\mathcal{T} = B_{2k+1}$, $I_2(4k+2)$ $(k \geq 1)$, E_7, H_3.*

Proof. Assume that $W = G_1 \times G_2$ for nontrivial subgroups $G_1, G_2 \subset W$. Then both G_1 and G_2 are generated by involutions, since W is so. Thus by Corollary 3.2, we have either $G_1 = Z(W)$ or $G_2 = Z(W)$ (since $G_1, G_2 \neq 1$ and $|Z(W)| \leq 2$). Hence $Z(W) \neq 1$ and so the claim follows from Theorem 2.16. □

3.2 The Isomorphism Problem

By using these results, we give some results on the Isomorphism Problem of general Coxeter groups. Let (W, S) be a Coxeter system with canonical direct product decomposition $W = \prod_{\omega \in \Omega} W_\omega$ into irreducible components W_ω. Then we put

$$\Omega_{\mathrm{fin}} = \{\omega \in \Omega \mid |W_\omega| < \infty\},\ \Omega_{\mathrm{inf}} = \Omega \smallsetminus \Omega_{\mathrm{fin}}, W_{\mathrm{fin}} = \prod_{\omega \in \Omega_{\mathrm{fin}}} W_\omega,\ W_{\mathrm{inf}} = \prod_{\omega \in \Omega_{\mathrm{inf}}} W_\omega.$$

(Note that $W = W_{\mathrm{fin}} \times W_{\mathrm{inf}}$.) Moreover, we write $\Omega_{\mathcal{T}} = \{\omega \in \Omega \mid W_\omega \simeq W(\mathcal{T})\}$ for any type $\mathcal{T}$. Now our result (proved later) is stated as follows:

Theorem 3.4. *(See above for notations.) Let (W, S), (W', S') be two Coxeter systems with the decompositions $W = \prod_{\omega \in \Omega} W_\omega$, $W' = \prod_{\omega' \in \Omega'} W'_{\omega'}$ into irreducible components. Let $\pi_\omega : W \to W_\omega$, $\pi'_{\omega'} : W' \to W'_{\omega'}$ denote the projections.*
(i) *$W \simeq W'$ if and only if the following two conditions are satisfied:*
(I) *There is a bijection $\varphi : \Omega_{\mathrm{inf}} \to \Omega'_{\mathrm{inf}}$ such that $W_\omega \simeq W'_{\varphi(\omega)}$ for all*

$\omega \in \Omega_{\mathrm{inf}}$.

(II) *Each of the following subsets of Ω has the same cardinality as the corresponding subset of Ω':*

$$\Omega_{A_1} \cup \big(\bigcup_{k\geq 1} \Omega_{B_{2k+1}}\big) \cup \Omega_{E_7} \cup \Omega_{H_3} \cup \big(\bigcup_{k\geq 1} \Omega_{I_2(4k+2)}\big), \quad \Omega_{B_3} \cup \Omega_{A_3},$$
$$\Omega_{B_{2k+1}} \cup \Omega_{D_{2k+1}}, \quad \Omega_{I_2(6)} \cup \Omega_{A_2}, \quad \Omega_{I_2(4k+2)} \cup \Omega_{I_2(2k+1)} \quad (k \geq 2),$$
$$\Omega_{\mathcal{T}} \quad \text{for } \mathcal{T} = A_n \ (4 \leq n < \infty),\ B_n \ (n < \infty \text{ even}),\ D_n \ (4 \leq n < \infty \text{ even}),$$
$$E_6,\ E_7,\ E_8,\ F_4,\ H_3,\ H_4,\ I_2(4k)\ (2 \leq k < \infty).$$

(ii) *Suppose that $W \simeq W'$, and let $f \in \mathrm{Isom}(W, W')$. Then:*

(I) *$f(W_{\mathrm{fin}}) = W'_{\mathrm{fin}}$ (and so the map g_{fin} defined by $g_{\mathrm{fin}} = f|_{W_{\mathrm{fin}}}$ is an isomorphism $W_{\mathrm{fin}} \to W'_{\mathrm{fin}}$).*

(II) *There is a bijection $\varphi : \Omega_{\mathrm{inf}} \to \Omega'_{\mathrm{inf}}$ such that for all $\omega \in \Omega_{\mathrm{inf}}$, the map $g_\omega = \pi'_{\varphi(\omega)} \circ f|_{W_\omega}$ is an isomorphism $W_\omega \to W'_{\varphi(\omega)}$.*

(III) *Moreover, there is a map $g_Z \in \mathrm{Hom}(W_{\mathrm{inf}}, Z(W'))$ such that*

$$f(w) = \begin{cases} g_\omega(w) g_Z(w) & \text{if } \omega \in \Omega_{\mathrm{inf}},\ w \in W_\omega, \\ g_{\mathrm{fin}}(w) & \text{if } w \in W_{\mathrm{fin}}. \end{cases}$$

Note that this is an analogue of the Krull-Remak-Schmidt Theorem on direct product decompositions of groups, and follows from that (together with Theorem 3.3) if W has a composition series. (More precisely, the key property in the proof of the K-R-S Theorem, which follows from the existence of composition series, is that any surjective normal endomorphism of an indecomposable factor is either nilpotent or isomorphic. However, it is not clear whether or not an irreducible Coxeter group has this property.) Our result here is also a generalization of a result of [9].

In order to prove this theorem, we introduce the following "modified version" of irreducible components. Here a group G is said to be *admissible* if either G is a nontrivial directly indecomposable irreducible Coxeter group (cf. Theorem 3.3) or G is isomorphic to one of $W(E_7)^+$, $W(H_3)^+$.

Remark 3.5. *Let $W = \prod_{\omega\in\Omega} W_\omega$ be the usual decomposition of a Coxeter group W into irreducible components. Then, by subdividing every directly decomposable W_ω into the direct factors (cf. Theorem 3.3), we can obtain another decomposition $W = \prod_{\lambda\in\Lambda} G_\lambda$ into admissible subgroups G_λ. Moreover, since any infinite W_ω is directly indecomposable, we can take the index set Λ so that $\Omega_{\mathrm{inf}} \subset \Lambda$ and $G_\omega = W_\omega$ for all $\omega \in \Omega_{\mathrm{inf}}$.*

From now, we consider the following two conditions on a family $\mathcal{G}$ of groups:

If $\{G_\lambda\}_{\lambda\in\Lambda}$ is a subfamily of $\mathcal{G}$, $G' \in \mathcal{G}$ and $f : G = \prod_{\lambda\in\Lambda} G_\lambda \to G'$ is a surjective homomorphism, then f maps a G_λ onto G' (so that it maps all other G_μ into $Z(G')$). (3.1)

$$\text{If } G \in \mathcal{G}, \text{ then either } Z(G) = 1 \text{ or } Z(G) \text{ is cyclic of prime order.} \tag{3.2}$$

(Actually, the condition (3.2) can be slightly weakened to the form that $Z(G)$ is either trivial or a finite elementary abelian p-group with p prime. But we omit the detail here, since we do not need such a generalization in this paper.)

Remark 3.6. ***(i)*** *If $\mathcal{G}$ satisfies (3.1), then all groups $G \in \mathcal{G}$ are directly indecomposable. In fact, if G admits a nontrivial decomposition $G = G_1 \times G_2$ with projections $\pi_i : G \to G_i$ $(i = 1, 2)$, then the map $G \times G \to G$, $(w, u) \mapsto \pi_1(w)\pi_2(u)$ is surjective but does not satisfy the conclusion of (3.1).*
(ii) *If $\mathcal{G}$ satisfies (3.1) and (3.2), then any $G \in \mathcal{G}$ has the three properties (I)–(III) in Lemma 2.1 whenever $Z(G) \neq G$. This follows immediately from (i).*

Lemma 3.7. *Any family $\mathcal{G}$ of admissible groups satisfies (3.1) and (3.2).*

Proof. The condition (3.2) follows from Lemma 2.17. For (3.1), we may assume $G' \not\simeq W(A_1)$ (so that $Z(G') \neq G'$), since otherwise the conclusion is obvious. Then there is an index $\lambda \in \Lambda$ such that $f(G_\lambda) \not\subset Z(G')$. Put $G_1 = G_\lambda$ and $G_2 = \prod_{\mu \in \Lambda \smallsetminus \{\lambda\}} G_\mu$. Then the hypothesis of (3.1) implies that G' is a central product (cf. Section 3.1) of $f(G_1)$ and $f(G_2)$, so that $f(G_1) \cap f(G_2) \subset Z(G')$. Thus the conclusion follows from Lemma 2.17 if $G' \simeq W(E_7)^+$ or $W(H_3)^+$ (in fact, the central product is a direct product since $Z(G') = 1$, while G' is directly indecomposable).

On the other hand, suppose that G' is a directly indecomposable irreducible Coxeter group. Since both G_1 and G_2 are generated by involutions (cf. Lemma 2.17), $f(G_1)$ and $f(G_2)$ also have this property. Thus we have $f(G_2) \subset Z(G')$ by Corollary 3.2 (since $f(G_1) \not\subset Z(G')$). Now if $Z(G') \not\subset f(G_1)$ (so that $f(G_1) \cap Z(G') = 1$ since $|Z(G')| \leq 2$), then the central product becomes a (nontrivial) direct product, but this is impossible. This implies that $f(G_2) \subset Z(G') \subset f(G_1)$ and so $f(G_1) = G'$. Hence the claim holds. □

Remark 3.8. *By a similar argument, it is deduced that any family $\mathcal{G}$, consisting of cyclic groups of prime order and directly indecomposable groups with trivial center, also satisfies the conditions (3.1) and (3.2).*

We prepare some more notations. For a decomposition $G = \prod_{\lambda \in \Lambda} G_\lambda$ of G, put

$$\begin{aligned} &G_{\Lambda'} = \prod_{\lambda \in \Lambda'} G_\lambda \text{ (for } \Lambda' \subset \Lambda), \ \Lambda_Z = \{\lambda \mid Z(G_\lambda) = G_\lambda\}, \ \Lambda_{\neg Z} = \Lambda \smallsetminus \Lambda_Z, \\ &\Lambda_p = \{\lambda \mid |Z(G_\lambda)| = p\}, \Lambda_{Z,p} = \Lambda_Z \cap \Lambda_p, \Lambda_{\neg Z,p} = \Lambda_{\neg Z} \cap \Lambda_p \text{ (} p \text{ prime or 1).} \end{aligned} \tag{3.3}$$

Note that the proof of the following theorem is essentially the same as the proof of Theorem 2.1 of [9], but slightly more delicate by the lack of the assumption on finiteness of the index sets (not only by generality of the context). Note also that this is also an analogue of the Krull-Remak-Schmidt Theorem.

Theorem 3.9. *(See (3.3) for notations.) Let $G = \prod_{\lambda \in \Lambda} G_\lambda$, $G' = \prod_{\lambda' \in \Lambda'} G'_{\lambda'}$ be decompositions of two groups G, G' into nontrivial subgroups. Let $\pi_\lambda : G \to G_\lambda$ and $\pi'_{\lambda'} : G' \to G'_{\lambda'}$ be the projections. Suppose that $\mathcal{G} = \{G_\lambda \mid \lambda \in \Lambda\} \cup \{G'_{\lambda'} \mid \lambda' \in \Lambda'\}$ satisfies the conditions (3.1) and (3.2). Let $f \in \mathrm{Isom}(G, G')$. Then:*
(i) *There is a bijection $\varphi : \Lambda \to \Lambda'$ such that $G_\lambda \simeq G'_{\varphi(\lambda)}$ for all $\lambda \in \Lambda$. Moreover, for any $\lambda \in \Lambda_{\neg Z}$, the map $g_\lambda = \pi'_{\varphi(\lambda)} \circ f|_{G_\lambda}$ is an isomorphism*

$G_\lambda \to G'_{\varphi(\lambda)}$.

(ii) *Moreover, there is a map* $g_Z \in \mathrm{Hom}(G, Z(G'))$ *such that*

$$f(w) = \begin{cases} g_\lambda(w) g_Z(w) & \text{if } \lambda \in \Lambda_{\neg Z},\ w \in G_\lambda, \\ g_Z(w) & \text{if } w \in G_{\Lambda_Z} \end{cases}$$

and that $\pi'_{\varphi(\lambda)} \circ g_Z(G_\lambda) = 1$ *for all* $\lambda \in \Lambda_{\neg Z}$.

(iii) *If* $\bigcup_{p \neq 1} \Lambda_p \subset \Lambda^\natural \subset \Lambda$, *then* $\bigcup_{p \neq 1} \Lambda'_p \subset \varphi(\Lambda^\natural)$ *and* $f(G_{\Lambda^\natural}) = G'_{\varphi(\Lambda^\natural)}$.

Proof. Note that $\bigcup_{p \neq 1} \Lambda_p = \{\lambda \in \Lambda \mid Z(G_\lambda) \neq 1\}$. Then the claim (iii) is deduced from the other claims (since now $Z(G) \subset G_{\Lambda^\natural}$ and $Z(G') \subset G'_{\varphi(\Lambda^\natural)}$).

From now, we prove the claims (i) and (ii). First, we put (symmetrically)

$$f_{\lambda'} = \pi'_{\lambda'} \circ f \in \mathrm{Hom}(G, G'_{\lambda'})\ (\lambda' \in \Lambda'),\ f'_\lambda = \pi_\lambda \circ f^{-1} \in \mathrm{Hom}(G', G_\lambda)\ (\lambda \in \Lambda),$$

and define (symmetrically)

$$\mathcal{A}'_\lambda = \{\lambda' \in \Lambda' \mid f_{\lambda'}(G_\lambda) \not\subset Z(G'_{\lambda'})\} \subset \Lambda'_{\neg Z} \text{ for } \lambda \in \Lambda_{\neg Z},$$
$$\mathcal{A}_{\lambda'} = \{\lambda \in \Lambda \mid f'_\lambda(G'_{\lambda'}) \not\subset Z(G_\lambda)\} \subset \Lambda_{\neg Z} \text{ for } \lambda' \in \Lambda'_{\neg Z}.$$

Note that $\mathcal{A}'_\lambda \neq \emptyset$ since $f(G_\lambda) \not\subset Z(G')$ (and $\mathcal{A}_{\lambda'} \neq \emptyset$ by symmetry). Moreover, since $f_{\lambda'} : G \to G'_{\lambda'}$ is surjective, the condition (3.1) implies that

$$\text{if } \lambda' \in \mathcal{A}'_\lambda, \text{ then } f_{\lambda'}(G_\lambda) = G'_{\lambda'} \text{ and } f_{\lambda'}(G_\mu) \subset Z(G'_{\lambda'}) \text{ for all } \mu \in \Lambda \smallsetminus \{\lambda\}.$$

By symmetry, a similar property holds for $\lambda \in \mathcal{A}_{\lambda'}$ (with respect to the map f'_λ).

We prove the following claims:

Claim 1: If $\lambda, \mu \in \Lambda_{\neg Z}$ and $\lambda \neq \mu$, then $\mathcal{A}'_\lambda \cap \mathcal{A}'_\mu = \emptyset$.

Claim 2: If $\lambda' \in \mathcal{A}'_\lambda$, then $\lambda \in \mathcal{A}_{\lambda'}$. (Thus $|\mathcal{A}'_\lambda| = 1$ for all $\lambda \in \Lambda_{\neg Z}$, by Claim 1 and symmetry. Moreover, by symmetry, the map $\varphi : \Lambda_{\neg Z} \to \Lambda'_{\neg Z}$ defined by $\mathcal{A}'_\lambda = \{\varphi(\lambda)\}$ is a bijection with inverse map satisfying $\mathcal{A}_{\lambda'} = \{\varphi^{-1}(\lambda')\}$.)

Claim 3: The map g_λ ($\lambda \in \Lambda_{\neg Z}$) in (i) is an isomorphism $G_\lambda \to G'_{\varphi(\lambda)}$.

Claim 4: $f(Z(G_{\Lambda_{\neg Z,p}})) = Z(G'_{\Lambda'_{\neg Z,p}})$ for all primes p.

Claim 5: For each prime p, $\Lambda_{Z,p}$ and $\Lambda'_{Z,p}$ have the same cardinality.

Proof of Claim 1: Assume contrary that $\lambda' \in \mathcal{A}'_\lambda \cap \mathcal{A}'_\mu$. Then the relation $\lambda' \in \mathcal{A}'_\lambda$ means that $f_{\lambda'}(G_\lambda) \not\subset Z(G'_{\lambda'})$, while the relation $\lambda' \in \mathcal{A}'_\mu$ implies (by the above property) that $f_{\lambda'}(G_\lambda) \subset Z(G'_{\lambda'})$ (since $\lambda \neq \mu$). This is a contradiction.

Proof of Claim 2: Since $G'_{\lambda'} \neq Z(G'_{\lambda'})$, we can take an element $w \in G'_{\lambda'} \smallsetminus Z(G'_{\lambda'})$. Put $u_\mu = f'_\mu(w) \in G_\mu$ for $\mu \in \Lambda$, so that we have $w = f(\prod_{\mu \in \Lambda} u_\mu)$. It suffices to show $u_\lambda = f'_\lambda(w) \notin Z(G_\lambda)$. Now $f_{\lambda'}(u_\mu) \in Z(G'_{\lambda'})$ for all $\mu \in \Lambda \smallsetminus \{\lambda\}$, while $w = \pi'_{\lambda'}(w) \notin Z(G'_{\lambda'})$. Thus we have $f_{\lambda'}(u_\lambda) \notin Z(G'_{\lambda'})$ and so $u_\lambda \notin Z(G_\lambda)$ (since $f_{\lambda'}(G_\lambda) = G'_{\lambda'}$). Hence $\lambda \in \mathcal{A}_{\lambda'}$.

Proof of Claim 3: Note that $g_\lambda : G_\lambda \to G'_{\varphi(\lambda)}$ is surjective (as above). Now the following equivalence holds for all $w \in G_\lambda$:

$$f_{\varphi(\lambda)}(w) \in Z(G'_{\varphi(\lambda)}) \iff f(w) \in Z(G') \iff w \in Z(G) \iff w \in Z(G_\lambda)$$

(we use the fact $\mathcal{A}'_\lambda = \{\varphi(\lambda)\}$ for the first equivalence). This implies that $\ker g_\lambda$ is contained in the simple group $Z(G_\lambda)$ (cf. (3.2)), so that $\ker g_\lambda = 1$ or

$Z(G_\lambda)$. Thus g_λ is injective (and so an isomorphism) if $Z(G_\lambda) = 1$. Moreover, if $Z(G'_{\varphi(\lambda)}) = 1$, then $f'_\lambda|_{G'_{\varphi(\lambda)}}$ is an isomorphism $G'_{\varphi(\lambda)} \to G_\lambda$ by symmetry, so that we have $Z(G_\lambda) = 1$. Thus g_λ is injective (as above) also in this case.

On the other hand, suppose $Z(G'_{\varphi(\lambda)}) \neq 1$. Then by the above equivalence, there is an element $w \in Z(G_\lambda)$ such that $g_\lambda(w) \neq 1$ (since g_λ is surjective). Thus we have $\ker g_\lambda \neq Z(G_\lambda)$ and so $\ker g_\lambda = 1$. Hence g_λ is an isomorphism.

Proof of Claim 4: Note that $Z(G) = \prod_{p \neq 1} Z(G_{\Lambda_p})$ and each $Z(G_{\Lambda_p})$ is an elementary abelian p-group, by (3.2). $Z(G')$ also admits a similar decomposition. Thus the isomorphism $f|_{Z(G)} : Z(G) \to Z(G')$ maps each $Z(G_{\Lambda_p})$ onto $Z(G'_{\Lambda'_p})$. Moreover, for any $\lambda \in \Lambda_{\neg Z,p}$, the composite homomorphism $G_\lambda \xrightarrow{f} G' \to G'_{\Lambda'_{Z,p}}$ (where the latter map is the projection) maps $Z(G_\lambda)$ to 1, by Remark 3.6 (ii) (note that $Z(G'_{\Lambda'_{Z,p}}) = G'_{\Lambda'_{Z,p}}$). Thus we have $f(Z(G_\lambda)) \subset G'_{\Lambda'_{\neg Z,p}}$ for any $\lambda \in \Lambda_{\neg Z,p}$ and so $f(Z(G_{\Lambda_{\neg Z,p}})) \subset Z(G'_{\Lambda'_{\neg Z,p}})$. Now this claim holds by symmetry.

Proof of Claim 5: Note that $Z(G_{\Lambda_p}) = G_{\Lambda_{Z,p}} \times Z(G_{\Lambda_{\neg Z,p}})$ and $Z(G'_{\Lambda'_p})$ admits a similar decomposition. Moreover, we have $f(Z(G_{\Lambda_p})) = Z(G'_{\Lambda'_p})$ and $f(Z(G_{\Lambda_{\neg Z,p}})) = Z(G'_{\Lambda'_{\neg Z,p}})$ by Claim 4. Thus the complementary factors $G_{\Lambda_{Z,p}}$, $G'_{\Lambda'_{Z,p}}$, which are elementary abelian p-groups with basis having the same cardinality as $\Lambda_{Z,p}$, $\Lambda'_{Z,p}$ respectively, are also isomorphic. Now this claim follows from uniqueness of the dimension of a vector space.

Conclusion. Since Λ_Z, Λ'_Z are disjoint unions of $\Lambda_{Z,p}$, $\Lambda'_{Z,p}$ respectively (cf. (3.2)), Claim 5 implies that this φ extends (not uniquely) to a bijection $\varphi : \Lambda \to \Lambda'$ satisfying (i) (note that $\Lambda_{Z,1} = \Lambda'_{Z,1} = \emptyset$ by the hypothesis). Moreover, define a map $g_Z : G \to Z(G')$ componentwise by

$$g_Z(w) = \begin{cases} \prod_{\lambda' \in \Lambda' \smallsetminus \{\varphi(\lambda)\}} f_{\lambda'}(w) & \text{if } \lambda \in \Lambda_{\neg Z}, w \in G_\lambda, \\ f(w) & \text{if } w \in G_{\Lambda_Z}. \end{cases}$$

Note that $G_{\Lambda_Z} \subset Z(G)$, while in the above definition, we have $f_{\lambda'}(w) \in Z(G'_{\lambda'})$ by the fact $\mathcal{A}'_\lambda = \{\varphi(\lambda)\}$. Since $Z(G')$ is abelian, these facts imply that g_Z is a well-defined group homomorphism. Now the claim (ii) follows from definition. □

Proof of Theorem 3.4. Let $W = \prod_{\lambda \in \Lambda} G_\lambda$, $W' = \prod_{\lambda' \in \Lambda'} G'_{\lambda'}$ be the decompositions into admissible groups given in Remark 3.5.

(i) Each of the sets in the condition (II), except Ω_{E_7} and Ω_{H_3} in the last row, has the same cardinality as the set $\{\lambda \in \Lambda \mid G_\lambda \simeq W(\mathcal{T}')\}$ where $\mathcal{T}' = A_1$, A_3, D_{2k+1}, A_2, $I_2(2k+1)$ and $\mathcal{T}$, respectively (note that no two admissible finite groups of distinct types are isomorphic; cf. Lemma 2.17). Moreover, each of Ω_{E_7} and Ω_{H_3} has the same cardinality as $\{\lambda \in \Lambda \mid G_\lambda \simeq W(\mathcal{T}')^+\}$ for $\mathcal{T}' = E_7$ and H_3, respectively. Similar relations also hold for W'. Thus the two conditions (I), (II) are satisfied if and only if there is a bijection $\psi : \Lambda \to \Lambda'$ such that $G_\lambda \simeq G'_{\psi(\lambda)}$ for all $\lambda \in \Lambda$. Hence the claim follows from Theorem 3.9 (i) (which can be applied indeed to the case, by Lemma 3.7).

(ii) Take $\varphi : \Lambda \to \Lambda'$, $g_\lambda \in \mathrm{Isom}(G_\lambda, G'_{\varphi(\lambda)})$ ($\lambda \in \Lambda_{\neg Z}$) and $g'_Z \in \mathrm{Hom}(W, Z(W'))$ as in the conclusion of Theorem 3.9. By Remark 3.5, $g_\omega \in \mathrm{Isom}(W_\omega, W'_{\varphi(\omega)})$ for all $\omega \in \Omega_{\mathrm{inf}}$, so that the claim (II) holds. The claim (I) follows from Theorem

3.9 (iii) (by putting $\Lambda^\natural = \Lambda \setminus \Omega_{\mathrm{inf}}$). Moreover, the claim (III) also follows from Theorem 3.9, by putting $g_Z = g'_Z|_{W_{\mathrm{inf}}}$. Hence the proof is concluded. □

3.3 Automorphism groups

In this subsection, the *complete* direct product of groups is denoted by a symbol $\overline{\prod}$. Owing to Theorems 3.4 and 3.9, we can examine the automorphism groups of $W = \prod_{\omega\in\Omega} W_\omega$ and $G = \prod_{\lambda\in\Lambda} G_\lambda$ respectively (Theorem 3.10), under the notations and hypotheses in Section 3.2. Note that each $\mathrm{Aut}(G_\lambda)$, $\mathrm{Aut}(W_\omega)$ is embedded into $\mathrm{Aut}(G)$, $\mathrm{Aut}(W)$ respectively. The group $\mathrm{Aut}(W_{\mathrm{fin}})$ is also embedded into $\mathrm{Aut}(W)$.

On the other hand, the symmetric group on each isomorphism class of components of G or W is also embedded into the automorphism group, as follows. For the case of G, we partition the index set $\Lambda_{\neg Z}$ into subsets Λ_ξ ($\xi \in \Xi$) so that $\lambda, \lambda' \in \Lambda_{\neg Z}$ are in the same subset if and only if $G_\lambda \simeq G_{\lambda'}$. Moreover, for $\xi \in \Xi$, we choose an "identity map" $\mathrm{id}_{\mu,\lambda} \in \mathrm{Isom}(G_\lambda, G_\mu)$ for each $\lambda, \mu \in \Lambda_\xi$ so that $\mathrm{id}_{\lambda,\lambda} = \mathrm{id}_{G_\lambda}$, $\mathrm{id}_{\lambda,\mu} = {\mathrm{id}_{\mu,\lambda}}^{-1}$ and $\mathrm{id}_{\nu,\mu} \circ \mathrm{id}_{\mu,\lambda} = \mathrm{id}_{\nu,\lambda}$ for all $\lambda, \mu, \nu \in \Lambda_\xi$. (This can be done by taking a maximal tree in the category of groups G_λ ($\lambda \in \Lambda_\xi$) and group isomorphisms.) Then each element τ of the symmetric group $\mathrm{Sym}(\Lambda_\xi)$ on Λ_ξ induces an automorphism of the factor G_{Λ_ξ} of G; namely,

$$\tau(w) = \mathrm{id}_{\tau(\lambda),\lambda}(w) \in G_{\tau(\lambda)} \text{ for } \lambda \in \Lambda_\xi \text{ and } w \in G_\lambda.$$

In this manner, $\mathrm{Sym}(\Lambda_\xi)$ is embedded into $\mathrm{Aut}(G_{\Lambda_\xi})$, and so also into $\mathrm{Aut}(G)$. Similarly, we write $\Omega = \coprod_{\upsilon\in\Upsilon} \Omega_\upsilon$, choose "identity maps" $\mathrm{id}_{\omega',\omega} \in \mathrm{Isom}(W_\omega, W_{\omega'})$ and then embed every symmetric group $\mathrm{Sym}(\Omega_\upsilon)$ into $\mathrm{Aut}(W)$. Moreover, put

$$\Upsilon_{\mathrm{fin}} = \{\upsilon \in \Upsilon \mid |W_\omega| < \infty \text{ for } \omega \in \Omega_\upsilon\} \text{ and } \Upsilon_{\mathrm{inf}} = \Upsilon \setminus \Upsilon_{\mathrm{fin}}.$$

Recall (Section 2.1) the structure of the monoid $\mathrm{Hom}(G', Z(G'))$ (where G' is a group), the action of $\mathrm{Aut}(G')$ on it and the embedding $f \mapsto f^\flat$ into the monoid $\mathrm{End}(G')$ compatible with the action of $\mathrm{Aut}(G')$. For a subset $H \subset \mathrm{Hom}(G', Z(G'))$, the image of H by the embedding is denoted by $H^\flat$. In particular, the group $\mathrm{Hom}(G', Z(G'))^\times$ of invertible elements of $\mathrm{Hom}(G', Z(G'))$ is embedded into $\mathrm{Aut}(G')$ (as a subgroup $\mathrm{Hom}(G', Z(G'))^{\times\flat}$).

Now for the group G, let

$$\begin{aligned}\mathrm{Hom}(G, Z(G))_o = \{f \in \mathrm{Hom}(G, Z(G)) \mid & f(G_{\Lambda_Z}) = 1,\\ & f(G_\lambda) \subset Z(G_\lambda) \text{ for all } \lambda \in \Lambda_{\neg Z}\}\end{aligned}$$

(cf. (3.3) for notations). Since we assumed that each G_λ ($\lambda \in \Lambda_{\neg Z}$) satisfies the three conditions in Lemma 2.1 (cf. Remark 3.6 (ii)), we have $f(Z(G)) = 1$ for all $f \in \mathrm{Hom}(G, Z(G))_o$. Thus by Lemma 2.4 (i), $\mathrm{Hom}(G, Z(G))_o$ is an abelian subgroup of $\mathrm{Hom}(G, Z(G))^\times$ with multiplication $(f * g)(w) = f(w)g(w)$ ($f, g \in \mathrm{Hom}(G, Z(G))_o$, $w \in G$).

On the other hand, since $Z(W_{\mathrm{inf}}) = 1$, Lemma 2.4 (ii) implies that the set $\mathrm{Hom}(W_{\mathrm{inf}}, Z(W))$ forms an abelian normal subgroup of $\mathrm{Hom}(W, Z(W))^\times$ with multiplication $(f * g)(w) = f(w)g(w)$ ($f, g \in \mathrm{Hom}(W_{\mathrm{inf}}, Z(W))$, $w \in W_{\mathrm{inf}}$). Since now $Z(W)$ is an elementary abelian 2-group, $\mathrm{Hom}(W_{\mathrm{inf}}, Z(W))$ is also an elementary abelian 2-group.

Now our result is stated as follows. Here $G'_1 G'_2$ denotes (for two subgroups G'_1, G'_2 of a group G') the subgroup of G' generated by $G'_1 \cup G'_2$, as usual.

Theorem 3.10. *(See above for notations. See also Section 3.2.)*
(i) *Put* $H_1 = \mathrm{Hom}(G, Z(G))^{\times\flat}$, $H_2 = \overline{\prod}_{\lambda\in\Lambda_{\neg Z}}\mathrm{Aut}(G_\lambda)$, $H_3 = \overline{\prod}_{\xi\in\Xi}\mathrm{Sym}(\Lambda_\xi)$ *and* $H_4 = \mathrm{Hom}(G, Z(G))^\flat_o$. *Then*

$$\mathrm{Aut}(G) = (H_1H_2) \rtimes H_3,\ H_1 \lhd \mathrm{Aut}(G),\ H_2 \lhd H_2H_3,\ H_1 \cap H_2 = H_4.$$

(ii) *Put* $H_1' = \mathrm{Hom}(W_{\mathrm{inf}}, Z(W))^\flat$, $H_2' = \mathrm{Aut}(W_{\mathrm{fin}})$, $H_3' = \overline{\prod}_{\omega\in\Omega_{\mathrm{inf}}}\mathrm{Aut}(W_\omega)$ *and* $H_4' = \overline{\prod}_{\upsilon\in\Upsilon_{\mathrm{inf}}}\mathrm{Sym}(\Omega_\upsilon)$. *Then*

$$\mathrm{Aut}(W) = (H_1' \rtimes (H_2' \times H_3')) \rtimes H_4',\ H_2'H_4' = H_2' \times H_4',\ H_3'H_4' = H_3' \rtimes H_4'.$$

(iii) *The subgroup* $H = \left(\prod_{\omega\in\Omega}\mathrm{Aut}(W_\omega)\right)\left(\prod_{\upsilon\in\Upsilon}\mathrm{Sym}(\Omega_\upsilon)\right)$ *has finite index in* $\mathrm{Aut}(W)$ *if and only if, either* $Z(W) = 1$ *or the odd Coxeter graph (cf. Definition 2.10)* Γ^{odd} *of* W *consists of only finitely many connected components. (Hence the index is finite whenever* W *has finite rank.)*

From now, we prove this theorem. First, we prove (i) and (ii). Note that $H_2'H_3' = H_2' \times H_3'$ and $H_2'H_4' = H_2' \times H_4'$ by definition. Moreover, by definition,

$$\begin{aligned} H_2 &= \{f \in \mathrm{Aut}(G) \mid f(w) = w\ (w \in G_{\Lambda_Z}),\ f(G_\lambda) = G_\lambda\ (\lambda \in \Lambda_{\neg Z})\},\\ H_3' &= \{f \in \mathrm{Aut}(W) \mid f(w) = w\ (w \in W_{\mathrm{fin}}),\ f(W_\omega) = W_\omega\ (\omega \in \Omega_{\mathrm{inf}})\}. \end{aligned} \tag{3.4}$$

Claim 1. **(i)** $\mathrm{Aut}(G) = H_1H_2H_3$. **(ii)** $\mathrm{Aut}(W) = H_1'H_2'H_3'H_4'$.

Proof. **(i)** Let $f \in \mathrm{Aut}(G)$, and take φ, g_λ, g_Z as in Theorem 3.9. Note that $\varphi(\Lambda_\xi) = \Lambda_\xi$ for all $\xi \in \Xi$. Now define $f_1 \in \mathrm{Hom}(G, Z(G))$ by

$$f_1(w) = \begin{cases} g_Z \circ g_{\varphi^{-1}(\lambda)}^{-1}(w)^{-1} & \text{for } \lambda \in \Lambda_{\neg Z},\ w \in G_\lambda,\\ wf(w)^{-1} & \text{for } w \in G_{\Lambda_Z} \end{cases}$$

(this is well defined since $G_{\Lambda_Z} \subset Z(G)$). Then by definition and Theorem 3.9, we have $f = {f_1}^\flat \circ f_2 \circ f_3$, where

$$f_2 = (g_{\varphi^{-1}(\lambda)} \circ \mathrm{id}_{\varphi^{-1}(\lambda),\lambda})_{\lambda\in\Lambda_{\neg Z}} \in H_2,\ f_3 = (\varphi|_{\Lambda_\xi})_{\xi\in\Xi} \in H_3.$$

Moreover, we have ${f_1}^\flat = f \circ f_3^{-1} \circ f_2^{-1} \in \mathrm{Aut}(G)$ and so $f_1 \in \mathrm{Hom}(G, Z(G))^\times$ by Lemma 2.2 (ii). Hence ${f_1}^\flat \in H_1$ and so $f \in H_1H_2H_3$.
(ii) Let $f \in \mathrm{Aut}(W)$, and take φ, g_{fin}, g_λ, g_Z as in Theorem 3.4 (ii). Note that $\varphi(\Omega_\upsilon) = \Omega_\upsilon$ for all $\upsilon \in \Upsilon$. Now define $f_1 \in \mathrm{Hom}(W_{\mathrm{inf}}, Z(W))$ by

$$f_1(w) = g_Z \circ g_{\varphi^{-1}(\omega)}^{-1}(w)^{-1} \text{ for } \omega \in \Omega_{\mathrm{inf}},\ w \in W_\omega.$$

Then we have (by definition and Theorem 3.4 (ii))

$$f = {f_1}^\flat \circ g_{\mathrm{fin}} \circ (g_{\varphi^{-1}(\omega)} \circ \mathrm{id}_{\varphi^{-1}(\omega),\omega})_{\omega\in\Omega_{\mathrm{inf}}} \circ (\varphi|_{\Omega_\upsilon})_{\upsilon\in\Upsilon_{\mathrm{inf}}} \in H_1'H_2'H_3'H_4'.$$

Hence the proof is concluded. □

Claim 2. **(i)** If $f^\flat \in H_1$, $\lambda, \mu \in \Lambda_{\neg Z}$ and $f^\flat(G_\lambda) \subset G_\mu$, then $\lambda = \mu$ and $f(G_\lambda) \subset Z(G_\lambda)$.
(ii) If $f^\flat \in H_1'$, $\omega, \omega' \in \Omega_{\mathrm{inf}}$ and $f^\flat(W_\omega) \subset W_{\omega'}$, then $\omega = \omega'$ and $f(W_\omega) = 1$.

Proof. **(i)** By the choice of λ, we can take $w \in G_\lambda \setminus Z(G_\lambda)$. Now we have $\pi_\lambda(f(w)) \in Z(G_\lambda)$ (where π_λ is the projection $G \to G_\lambda$) and so $\pi_\lambda(f^\flat(w)) = w\pi_\lambda(f(w))^{-1} \neq 1$. Since $f^\flat(w) \in G_\mu$, this implies that $\mu = \lambda$. Now the latter part follows from definition of the map $f^\flat$.
(ii) By a similar argument to (i), we have $\omega = \omega'$ and $f(W_\omega) \subset Z(W_\omega)$. Hence the claim holds since $Z(W_\omega) = 1$. □

Claim 3. **(i)** $(H_1H_2) \cap H_3 = 1$. **(ii)** $(H_1'H_2'H_3') \cap H_4' = 1$.

Proof. **(i)** Let $f_1 \in H_1$, $f_2 \in H_2$ such that $f_1 \circ f_2 \in H_3$. By (3.4) and definition of H_3, both f_2^{-1} and $f_1 \circ f_2$ map each component G_λ ($\lambda \in \Lambda_{\neg Z}$) onto a component, so that f_1 also does so. By Claim 2 (i), f_1 maps each G_λ ($\lambda \in \Lambda_{\neg Z}$) onto itself, while f_2 also does so (cf. (3.4)). Thus $f_1 \circ f_2 \in H_3$ also has this property. By definition of H_3, this occurs only if $f_1 \circ f_2 = \mathrm{id}_G$. Hence the claim holds.
(ii) The proof is similar to (i); if $f_i \in H_i'$ ($i = 1, 2, 3$) and $f_4 = f_1 \circ f_2 \circ f_3 \in H_4'$, then $f_1 = f_4 \circ f_3^{-1} \circ f_2^{-1}$ must map each W_ω ($\omega \in \Omega_{\mathrm{inf}}$) onto some component, which is W_ω by Claim 2 (ii). This implies that f_4 maps each W_ω ($\omega \in \Omega_{\mathrm{inf}}$) onto itself, so that $f_4 = \mathrm{id}_W$ by definition of H_4'. Hence the claim holds. □

Claim 4. **(i)** $H_2 \lhd H_2H_3$. **(ii)** $H_3' \lhd H_3'H_4'$.

Proof. For (i), it is enough to show that $f_3 \circ f_2 \circ f_3^{-1} \in H_2$ for all $f_2 \in H_2$ and $f_3 \in H_3$. By definition, f_3 is identity on G_{Λ_Z} and maps each G_λ ($\lambda \in \Lambda_{\neg Z}$) onto a component. Now by (3.4), $f_3 \circ f_2 \circ f_3^{-1}$ also satisfies the condition in (3.4), so that it belongs to H_2. Hence the claim holds. The proof of (ii) is similar. □

Claim 5. **(i)** $H_1 \lhd \mathrm{Aut}(G)$. **(ii)** $H_1' \lhd \mathrm{Aut}(W)$.

Proof. **(i)** Note that $\mathrm{Aut}(G)$ acts on the monoid $\mathrm{Hom}(G, Z(G))$. Thus its subgroup $\mathrm{Hom}(G, Z(G))^\times$ of the invertible elements is invariant under the action. Now the claim follows from Lemma 2.2 (iii).
(ii) By Lemma 2.2 (iii), it is enough to show that the subgroup $\mathrm{Hom}(W_{\mathrm{inf}}, Z(W))$ of $\mathrm{Hom}(W, Z(W))$ is invariant under the action of $\mathrm{Aut}(W)$. Moreover, by Claim 1, it is enough to show that $h \circ f \circ h^{-1} \in \mathrm{Hom}(W_{\mathrm{inf}}, Z(W))$ for all $f \in \mathrm{Hom}(W_{\mathrm{inf}}, Z(W))$ and $h \in H_2'H_3'H_4'$. Now we have $h(W_{\mathrm{fin}}) = W_{\mathrm{fin}}$ by definition of H_2', H_3' and H_4', so that $h \circ f \circ h^{-1}(W_{\mathrm{fin}}) = h(f(W_{\mathrm{fin}})) = h(1) = 1$. Hence the claim holds. □

Claim 6. **(i)** $H_1 \cap H_2 = H_4$. **(ii)** $H_1' \cap (H_2'H_3') = 1$.

Proof. **(i)** Let $f^\flat \in H_1 \cap H_2$. Then by (3.4), we have $f^\flat(w) = w$ (or equivalently $f(w) = 1$) for all $w \in G_{\Lambda_Z}$ and $f^\flat(G_\lambda) = G_\lambda$ for all $\lambda \in \Lambda_{\neg Z}$. Thus we have $f \in \mathrm{Hom}(G, Z(G))_o$ by Claim 2 (i), so that $f^\flat \in H_4$. Conversely, $H_4 \subset H_1$ by definition, while $H_4 \subset H_2$ by (3.4) and definition of H_4. Hence the claim holds.
(ii) Let $f^\flat \in H_1' \cap (H_2'H_3')$. Then for any $\omega \in \Omega_{\mathrm{inf}}$, we have $f^\flat(W_\omega) = W_\omega$ by definition of H_2' and H_3'. Thus we have $f(W_\omega) = 1$ by Claim 2 (ii). Hence $f = 1$ and $f^\flat = \mathrm{id}_W$. □

Now the claims (i) and (ii) of Theorem 3.10 hold. Namely:
(i) We have $H_1 \cap H_2 = H_4$ (Claim 6) and $\mathrm{Aut}(G) = (H_1H_2)H_3$ (Claim 1). Now since $H_1 \lhd \mathrm{Aut}(G)$ (Claim 5) and $H_2 \lhd H_2H_3$ (Claim 4), the conjugation by an element of H_1, H_2 or H_3 maps H_1 and H_2 into the subgroup H_1H_2 generated by $H_1 \cup H_2$. Thus $H_1H_2 \lhd \mathrm{Aut}(G)$ and $\mathrm{Aut}(G) = (H_1H_2) \rtimes H_3$ (Claim 3).

(ii) We have $H'_2H'_3 = H'_2 \times H'_3$, $H'_2H'_4 = H'_2 \times H'_4$ (as the above remark), $H'_3H'_4 = H'_3 \rtimes H'_4$ (Claims 3, 4) and $H'_1 \lhd \mathrm{Aut}(W)$ (Claim 5), so $H'_1H'_2H'_3 = H'_1 \rtimes (H'_2 \times H'_3)$ (Claim 6). Now by a similar argument to (i), the conjugation by an element of H'_4 leaves $H'_1H'_2H'_3$ invariant, so $H'_1H'_2H'_3 \lhd \mathrm{Aut}(W)$ since $\mathrm{Aut}(W) = H'_1H'_2H'_3H'_4$ (Claim 1). Thus $\mathrm{Aut}(W) = (H'_1 \rtimes (H'_2 \times H'_3)) \rtimes H'_4$ (Claim 3).

Proof of Theorem 3.10 (iii). If $Z(W) = 1$, then all irreducible components of W are directly indecomposable (cf. Theorem 3.3), so that the decomposition $W = \prod_{\omega\in\Omega} W_\omega$ itself satisfies the conditions (3.1) and (3.2) in Section 3.2. Thus we can apply the result (i) to this decomposition. Now $H_1 = 1$ since $Z(W) = 1$. Moreover, $\Omega = \Omega_{\neg Z}$ in this case, so that we have $H = H_2H_3 = \mathrm{Aut}(W)$.

From now, we assume that $Z(W) \neq 1$. For $f \in \mathrm{Aut}(W)$, let $\mathrm{sep}(f)$ be the set of all $\omega \in \Omega$ such that $f(W_\omega) \not\subset W_{\omega'}$ for all $\omega' \in \Omega$. Since any element of H maps each component W_ω onto a component, the cardinality of the set $\mathrm{sep}(f)$ is invariant in each coset of $\mathrm{Aut}(W)/H$. Moreover, by definition, we have

$$H = \left(\overline{\prod}_{\omega\in\Omega_{\mathrm{fin}}} \mathrm{Aut}(W_\omega)\right)\left(\overline{\prod}_{\upsilon\in\Upsilon_{\mathrm{fin}}} \mathrm{Sym}(\Omega_\upsilon)\right) \times H'_3H'_4 \subset H'_2 \times (H'_3H'_4).$$

Case 1. Γ^{odd} consists of only finitely many connected components: This implies that $|\Omega| < \infty$ and $|\mathrm{Hom}(W_{\mathrm{inf}}, \{\pm 1\})| < \infty$ (cf. Lemma 2.11). Since $Z(W)$ is now a finite elementary abelian 2-group, (ii) implies that $H'_2H'_3H'_4$ has index $|H'_1| = |\mathrm{Hom}(W_{\mathrm{inf}}, Z(W))| < \infty$ in $\mathrm{Aut}(W)$. Moreover, since now $|W_{\mathrm{fin}}| < \infty$, the index of H in $H'_2H'_3H'_4$ is $\leq |H'_2| < \infty$. Thus H has finite index also in $\mathrm{Aut}(W)$.

Case 2. Γ^{odd} consists of infinitely many connected components: Now we have to show that H has infinite index in $\mathrm{Aut}(W)$.

Subcase 2-1. The odd Coxeter graph of some W_ω consists of infinitely many connected components: Note that $\omega \in \Omega_{\mathrm{inf}}$ in this case. Now by Lemma 2.11, we have $|\mathrm{Hom}(W_\omega, \{\pm 1\})| = \infty$ and so $|\mathrm{Hom}(W_{\mathrm{inf}}, Z(W))| = \infty$ (since we assumed that $Z(W) \neq 1$). Thus by (ii), the subgroup $H'_2H'_3H'_4$ ($\supset H$) has index $|H'_1| = \infty$, so that H also has infinite index in $\mathrm{Aut}(W)$.

Subcase 2-2. The odd Coxeter graph of every W_ω consists of only finitely many connected components: Then we have $|\Omega| = \infty$ by the hypothesis of Case 2. Since we assumed that $Z(W) \neq 1$, we can take an infinite sequence $\omega_0, \omega_1, \omega_2, \dots$ of distinct elements of Ω such that $Z(W_{\omega_0}) \neq 1$. Let u denote the unique element of $Z(W_{\omega_0}) \smallsetminus \{1\}$. Now for $k \geq 1$, we define $f_k \in \mathrm{Hom}(W, Z(W))$ componentwise by

$$f_k(w) = \begin{cases} u^{\ell(w)} & \text{if } \omega \in \{\omega_1, \dots, \omega_k\} \text{ and } w \in W_\omega, \\ 1 & \text{if } \omega \in \Omega \smallsetminus \{\omega_1, \dots, \omega_k\} \text{ and } w \in W_\omega. \end{cases}$$

Then we have $f_k \circ f_k = 1$ and so $f_k * f_k = 1$ since $Z(W)$ is an elementary abelian 2-group. This implies that $f_k \in \mathrm{Hom}(W, Z(W))^\times$ and so ${f_k}^\flat \in \mathrm{Aut}(W)$, while $\mathrm{sep}({f_k}^\flat) = \{\omega_1, \dots, \omega_k\}$ by definition. Thus by the above remark, all ${f_k}^\flat$ belong to distinct cosets in $\mathrm{Aut}(W)/H$ and so H has infinite index in $\mathrm{Aut}(W)$. Hence the proof is concluded. □

Example 3.11. *Let $m = (m_1, m_2, \dots)$ be an infinite sequence of nonnegative integers. Here we examine $\mathrm{Aut}(W_m)$ for the group $W_m = \prod_{n\geq 1}(\mathrm{Sym}_n)^{m_n}$ by* *using our result, where* $\mathrm{Sym}_n = \mathrm{Sym}(\{1,2,\dots,n\})$ *is the symmetric group of degree* n*. Note that* $\mathrm{Sym}_1 = 1$*.*

Since Sym_n $(n \geq 2)$ *is the Coxeter group* $W(A_{n-1})$*, which is directly indecomposable (cf. Theorem 3.3), we can apply Theorem 3.10 (i) to this decomposition of* W_m*. In this case, we have* $Z(\mathrm{Sym}_n) = 1$ *unless* $Z(\mathrm{Sym}_n) = \mathrm{Sym}_n$ *(namely* $n = 1, 2$*), so that* $\mathrm{Hom}(W_m, Z(W_m))_o = 1$*. Thus we have* $\mathrm{Aut}(W_m) = H_1 \rtimes H_2 \rtimes H_3$*.*

Note that $Z(W_m) = (\mathrm{Sym}_2)^{m_2} \simeq \{\pm 1\}^{m_2}$*, while* $|\mathrm{Hom}(\mathrm{Sym}_n, \{\pm 1\})| = 2$ *for all* $n \geq 2$ *by Lemma 2.11. Thus Lemmas 2.3 and 2.4 (ii) imply that*

$$\begin{aligned}
H_1 &= \mathrm{Hom}\left(\prod_{n\geq 3}(\mathrm{Sym}_n)^{m_n}, Z(W_m)\right)^{\flat} \rtimes \mathrm{Hom}(\mathrm{Sym}_2^{m_2}, Z(W_m))^{\times\flat} \\
&= \left(\overline{\prod}_{n\geq 3}\mathrm{Hom}\big((\mathrm{Sym}_n)^{m_n}, Z(W_m)\big)\right)^{\flat} \rtimes \mathrm{Aut}((\mathrm{Sym}_2)^{m_2}) \\
&\simeq \left(\overline{\prod}_{n\geq 3}\{\pm 1\}^{m_2 m_n}\right) \rtimes \mathrm{GL}_{m_2}(\mathbb{F}_2).
\end{aligned}$$

Secondly, recall the well-known fact that $\mathrm{Aut}(\mathrm{Sym}_n) = \mathrm{Inn}(\mathrm{Sym}_n)$ *(the group of inner automorphisms) if* $n \neq 6$ *and* $|\mathrm{Aut}(\mathrm{Sym}_6)/\mathrm{Inn}(\mathrm{Sym}_6)| = 2$*. This implies that* $\mathrm{Aut}(\mathrm{Sym}_2) = 1$*,* $|\mathrm{Aut}(\mathrm{Sym}_6)| = 2|\mathrm{Sym}_6|$ *and* $\mathrm{Aut}(\mathrm{Sym}_n) \simeq \mathrm{Sym}_n$ *if* $n \neq 2, 6$*. Thus we have*

$$H_2 \simeq \overline{\prod}_{n\geq 3}\mathrm{Aut}(\mathrm{Sym}_n)^{m_n} \simeq \left(\overline{\prod}_{3\leq n\neq 6}{\mathrm{Sym}_n}^{m_n}\right) \times \mathrm{Aut}(\mathrm{Sym}_6)^{m_6}.$$

Moreover, by definition, we have $H_3 \simeq \overline{\prod}_{n\geq 3}\mathrm{Sym}_{m_n}$*.*

As a special case, if all but finitely many terms in m *are* 0*, then (by putting* $|m| = \sum_n m_n < \infty$*) we have*

$$\begin{aligned}
|H_1| &= 2^{m_2(|m|-m_1-m_2)}\prod_{i=0}^{m_2-1}(2^{m_2} - 2^i) = 2^{m_2(|m|-m_1-m_2)+\binom{m_2}{2}}\prod_{i=1}^{m_2}(2^i - 1), \\
|H_2| &= 2^{m_6}\prod_{n\geq 3}(n!)^{m_n},\ |H_3| = \prod_{n\geq 3} m_n!.
\end{aligned}$$

Hence we have

$$\begin{aligned}
|\mathrm{Aut}(W_m)| &= |H_1| \cdot |H_2| \cdot |H_3| \\
&= 2^{m_2(|m|-m_1-m_2)+\binom{m_2}{2}+m_6}\prod_{i=1}^{m_2}(2^i - 1)\prod_{n\geq 3}((n!)^{m_n} m_n!) \\
&= \left(2^{m_2(|m|-m_1-m_2-1)+\binom{m_2}{2}+m_6}\prod_{i=1}^{m_2}(2^i - 1)\prod_{n\geq 3} m_n!\right)|W_m|.
\end{aligned}$$

4 Centralizers of normal subgroups generated by involutions

4.1 Proof of Theorem 3.1

In this section, we prove Theorem 3.1. From now, (W, S) always denotes a Coxeter system. In the proof, we use the notion of core subgroups (cf. Section

2.1). For a subgroup $G \leq W$, let X_G be the set of all elements in G of the form $w_0(I)$ ($I \subset S$) such that $1 \neq w_0(I) \in Z(W_I)$. Then we have the following relation (proved below):

Proposition 4.1. *Let $H \lhd W$ be a normal subgroup generated by involutions. Then H is the smallest normal subgroup of W containing X_H, and*

$$Z_W(H) = \bigcap_{w_0(I) \in X_H} \mathrm{Core}_W(N_W(W_I)).$$

On the other hand, the subgroups $\mathrm{Core}_W(N_W(W_I))$ are determined completely (for irreducible (W,S)) by the following theorem, which we prove in later subsections. Here we use the notation $(W(D_3), S(D_3))$ instead of $(W(A_3), S(A_3))$.

Theorem 4.2. *(See Definitions 2.5 and 2.8 for notations.) Let (W,S) be an irreducible Coxeter system of an arbitrary rank, and I nonempty proper subset of S. Then:*
(i) *If $(W,S) = (W(B_n), S(B_n))$, $1 \leq k < n \leq \infty$, $\tau \in \mathrm{Aut}(\Gamma(B_n))$ and $I = \tau(S(B_k))$, then $\mathrm{Core}_W(N_W(W_I)) = \tau(G_{B_n})$.*
(ii) *If $(W,S) = (W(D_n), S(D_n))$, $2 \leq k < n \leq \infty$, $\tau \in \mathrm{Aut}(\Gamma(D_n))$ and $I = \tau(S(D_k))$, then $\mathrm{Core}_W(N_W(W_I)) = \tau(G_{D_n})$.*
(iii) *Otherwise, $\mathrm{Core}_W(N_W(W_I)) = Z(W)$.*
(cf. Lemma 2.12 for definition of G_{B_n} and G_{D_n}.)

Note that, $\mathrm{Core}_W(N_W(W_I)) = N_W(W_I) = W$ if $I = \emptyset$ or S. Theorem 3.1 will be proved by combining Proposition 4.1 and Theorem 4.2.

In the proof of Proposition 4.1, we use the following two results:

Theorem 4.3 ([10], Theorem A). *Let w be an involution in W. Then w is conjugate in W to some element $w_0(I)$ ($I \subset S$) such that $w_0(I) \in Z(W_I)$.*

Lemma 4.4. *Let W_I be a finite parabolic subgroup of W such that $w_0(I) \in Z(W_I)$. Then $Z_W(w_0(I)) = N_W(W_I)$.*

Proof. First, assume $u \in Z_W(w_0(I))$. Then $u^{-1} w_0(I) u = w_0(I) \in Z(W_I)$ and so $w_0(I) \cdot (u \cdot \alpha_s) = u w_0(I) \cdot \alpha_s = -u \cdot \alpha_s$ for all $s \in I$. This implies that $u \cdot \alpha_s \in \Phi_I$ for all $s \in I$, so that $u \in N_W(W_I)$ by (2.16).

Conversely, assume $u \in N_W(W_I)$. Put $u' = u w_0(I) u^{-1} \in W_I$. Then we have $u' \cdot \alpha_s = -\alpha_s$ for all $s \in I$ (since $w_0(I)$ maps $u^{-1} \cdot \alpha_s \in \Phi_I$ (cf. (2.16)) to $-u^{-1} \cdot \alpha_s$). Hence we have $u' = w_0(I)$ and so $u \in Z_W(w_0(I))$. □

Proof of Proposition 4.1. By Theorem 4.3, every involution in H is conjugate to some element of X_H (since $H \lhd W$). This implies that any normal subgroup of W containing X_H also contains all the generators of H. Thus the first claim follows. For the second one, apply Lemmas 2.7 and 4.4. □

Proof of Theorem 3.1. The claim (i) is obvious. From now, we assume $H \not\subset Z(W)$. Note that $Z(W) \subset Z_W(H)$. Note also that, by Proposition 4.1,

$$Z_W(H) \subset \mathrm{Core}_W(N_W(W_I)) \text{ for all } w_0(I) \in X_H. \tag{4.1}$$

Case 1. $(W,S) = (W(B_n), S(B_n))$, $n \geq 2$ or $(W(D_n), S(D_n))$, $n \geq 3$: Let $\mathcal{T} = B$, $L = 1$ for the former case, $\mathcal{T} = D$, $L = 2$ for the latter case.

Subcase 1-1. $\mathcal{T} = B$**,** $n \neq 2$ **or** $\mathcal{T} = D$**,** $n \neq 4$**:** Note that in this case, any automorphism of $\Gamma(\mathcal{T}_n)$ preserves the sets $S(\mathcal{T}_k)$, elements $w_0(S(\mathcal{T}_k))$ $(k \geq L)$ and so the subgroup $G_{\mathcal{T}_n}$.

Subsubcase 1-1-1. $H \subset G_{\mathcal{T}_n}$**:** This is a case (ii) or (iii) (for τ identity), and so we have to show $Z_W(H) = G_{\mathcal{T}_n}$. The inclusion $\supset$ holds since $G_{\mathcal{T}_n}$ is abelian. Conversely, since $H \not\subset Z(W)$, X_H contains an element other than $w_0(S)$, so that we have $Z_W(H) \subset G_{\mathcal{T}_n}$ by (4.1) and Theorem 4.2.

Subsubcase 1-1-2. $H \not\subset G_{\mathcal{T}_n}$**:** By the above remark, this is actually not a case (ii) or (iii), so that we have to show $Z_W(H) \subset Z(W)$. Now X_H contains an element $w_0(I)$ such that $I \neq S(\mathcal{T}_k)$ for any $L \leq k \leq n$, since otherwise $H \subset G_{\mathcal{T}_n}$ by Lemma 2.12. For this I, we have $\mathrm{Core}_W(N_W(W_I)) = Z(W)$ by Theorem 4.2, so that the claim follows from (4.1).

Subcase 1-2. $\mathcal{T} = B$**,** $n = 2$**:** Note that $X_H \subset \{s_1, s_2, w_0(S)\}$ in this case. Moreover, $X_H \not\subset \{w_0(S)\}$ since $H \not\subset Z(W)$.

Subsubcase 1-2-1. $s_1 \in X_H$ **and** $s_2 \notin X_H$**:** In this case, we have $X_H \subset \{s_1, w_0(S)\}$ and so $H \subset G_{B_2}$ by Lemma 2.12. This is a case (ii) (for τ identity). Now we have $G_{B_2} \subset Z_W(H)$ since G_{B_2} is abelian, while $Z_W(H) \subset G_{B_2}$ by (4.1) and Theorem 4.2 (applying to $\{s_1\} \subset S$). Thus the claim holds.

Subsubcase 1-2-2. $s_1 \notin X_H$ **and** $s_2 \in X_H$**:** By symmetry, this is also a case (ii) (for the unique $\tau \neq \mathrm{id}_S$) and the claim holds similarly.

Subsubcase 1-2-3. $s_1 \in X_H$ **and** $s_2 \in X_H$**:** Note that $H = W$. This is not a case (ii) or (iii), and actually $Z_W(H) = Z(W)$.

Subcase 1-3. $\mathcal{T} = D$**,** $n = 4$**:** Note that (by definition)

$$X_H \subset \{s_1,\ s_2,\ s_3,\ s_4,\ s_1s_2s_4,\ s_1s_2,\ s_2s_4,\ s_4s_1,\ w_0(S)\}.$$

Subsubcase 1-3-1. X_H **contains one of the first five elements:** Now we have $H \not\subset \tau(G_{D_4})$ for any τ, so that this is not a case (iii) and we have to show $Z_W(H) \subset Z(W)$. This claim follows from (4.1) (applying to the element of X_H given in the hypothesis here) and Theorem 4.2.

Subsubcase 1-3-2. X_H **contains at least two of the elements** s_1s_2**,** s_2s_4**,** s_4s_1**:** Now we have $H \not\subset \tau(G_{D_4})$ for any τ, so that this is not a case (iii) and we have to show $Z_W(H) \subset Z(W)$. Let X_H contain two such elements s_is_j, s_js_k, and put $I = \{s_i, s_j\}$, $J = \{s_j, s_k\}$. Then we have

$$\mathrm{Core}_W(N_W(W_I)) \cap \mathrm{Core}_W(N_W(W_J)) \subset \mathrm{Core}_W(N_W(W_{\{s_j\}}))$$

by (2.4), (2.15) and (2.2). Thus we have $Z_W(H) \subset \mathrm{Core}_W(N_W(W_{\{s_j\}})) = Z(W)$ by (4.1) and Theorem 4.2.

Subsubcase 1-3-3. X_H **contains none of the first five elements and at most one of** s_1s_2**,** s_2s_4**,** s_4s_1**:** Note that $X_H \not\subset \{w_0(S)\}$ since $H \not\subset Z(W)$. Thus we have $s_is_j \in X_H \subset \{s_is_j, w_0(S)\}$ for one of $(i,j) = (1,2)$, $(2,4)$, $(4,1)$. Lemma 2.12 implies that this is a case (iii) (namely $H \subset \tau(G_{D_4})$), by taking $\tau \in \mathrm{Aut}(\Gamma)$ mapping s_1, s_2 to s_i, s_j respectively. Now $\tau(G_{D_4}) \subset Z_W(H)$ since $\tau(G_{D_4})$ is abelian. Conversely, we have $\mathrm{Core}_W(N_W(W_{\{s_i,s_j\}})) = \tau(G_{D_4})$ by Theorem 4.2, so that $Z_W(H) \subset \tau(G_{D_4})$ by (4.1). Thus the claim holds.

Case 2. $(W,S) \not\simeq (W(B_n), S(B_n))$ $(n \geq 2)$**,** $(W(D_n), S(D_n))$ $(n \geq 3)$**:** This is not a case (ii) or (iii), so that we have to show $Z_W(H) \subset Z(W)$. Since $H \not\subset Z(W)$, X_H contains an element other than $w_0(S)$, so that we have $Z_W(H) \subset Z(W)$ by (4.1) and Theorem 4.2. Hence the proof is concluded. $\square$

4.2 Some lemmas

In the rest of this paper, we prove Theorem 4.2. In this subsection, we prepare some lemmas used in our proof. From now, we abbreviate the notation $\mathrm{Core}_W(N_W(W_I))$ to C_I.

First, by combining Lemma 2.22, (2.4) and (2.2), we have:

$$\text{If } I \subset J \subset S \text{ and } J \smallsetminus I \subset I^{\perp}, \text{ then } C_J \cap C_I \subset C_{J \smallsetminus I}. \tag{4.2}$$

Lemma 4.5 (Expanding Lemma). *If $I \subset S$ and $s \in S \smallsetminus (I \cup I^{\perp})$, then $C_I \subset C_{I \cup \{s\}}$.*

Proof. It is enough (by (2.3)) to show that $C_I \subset N_W(W_{I \cup \{s\}})$. Let $w \in C_I$. By the hypothesis, we have $c = \langle \alpha_s, \alpha_t \rangle < 0$ for some $t \in I$. Now since $sws \in C_I \subset N_W(W_I)$, we have $sws \cdot \alpha_t \in \Phi_I$ (by (2.16)) and so $ws \cdot \alpha_t \in \Phi_{I \cup \{s\}}$. On the other hand, we have $ws \cdot \alpha_t = w \cdot \alpha_t - 2cw \cdot \alpha_s$. Thus $w \cdot \alpha_s \in \Phi_{I \cup \{s\}}$ since $w \cdot \alpha_t \in \Phi_I$ (by (2.16)). Hence we have $w \in N_W(W_{I \cup \{s\}})$ by (2.16). $\square$

For $s \in S$ and $I \subset S$, let $d_\Gamma(s, I) = \min\{d_\Gamma(s,t) \mid t \in I\}$ denote the distance from s to the set I in the Coxeter graph Γ of (W, S).

Lemma 4.6 (Cutting Lemma). *Let (W, S) be irreducible, $I \subset S$ and $s \in S \smallsetminus I$. Then for $d_\Gamma(s, I) < k < \infty$, we have $C_I \subset C_J$, where $J = \{t \in I \mid d_\Gamma(s,t) \geq k\}$.*

Proof. It is enough (by (2.3) and (2.16)) to show that $w \cdot \Phi_J \subset \Phi_J$ (or equivalently, $w \cdot \Pi_J \subset \Phi_J$) for all $w \in C_I$. Assume contrary that $t \in J$ and $w \cdot \alpha_t \notin \Phi_J$. Note that $w \cdot \alpha_t \in \Phi_I$ (by (2.16)) and so $s \notin \mathrm{supp}(w \cdot \alpha_t)$. Then by definition of J, we have

$$(d =)\ d_\Gamma(s, \mathrm{supp}(w \cdot \alpha_t)) < k \leq d_\Gamma(s,t).$$

Take a shortest path $s_0 = s, s_1, \dots, s_{d-1}, s_d \in \mathrm{supp}(w \cdot \alpha_t)$ in Γ from s to the set $\mathrm{supp}(w \cdot \alpha_t)$. Then by the above inequality, we have $s_i \in \{t\}^{\perp}$ for all $0 \leq i \leq d-1$. Put $u = ss_1 \cdots s_{d-1} \in W$. Then we have $uwu^{-1} \cdot \alpha_t = uw \cdot \alpha_t$ and so (by (2.13))

$$\mathrm{supp}(uwu^{-1} \cdot \alpha_t) = \mathrm{supp}(w \cdot \alpha_t) \cup \{s, s_1, \dots, s_{d-1}\} \not\subset I$$

(note that $s \notin I$). On the other hand, we have $uwu^{-1} \in C_I$ and so $uwu^{-1} \cdot \alpha_t \in \Phi_I$ (by (2.16)). This is a contradiction. Hence the claim holds. $\square$

Lemma 4.7 (Shifting Lemma). *Suppose that $s, t \in S$ are in the same connected component of the odd Coxeter graph Γ^{odd} of (W, S). Then $C_{\{s\}} = C_{\{t\}}$.*

Proof. By definition of Γ^{odd}, and by symmetry, it is enough to show that $C_{\{s\}} \subset C_{\{t\}}$ for any s, t such that $m(s,t) = 2k+1$ is odd. Now by putting $u = (st)^k \in W$, we have $t = usu^{-1}$. Thus for $w \in C_{\{s\}}$, we have

$$wtw^{-1} = wusu^{-1}w^{-1} = u(u^{-1}wu)s(u^{-1}wu)^{-1}u^{-1} = usu^{-1} = t$$

since $u^{-1}wu \in C_{\{s\}}$. Thus $w \in N_W(W_{\{t\}})$. Hence the claim follows from (2.3). $\square$

Moreover, we have:

Lemma 4.8. *Let (W,S) be irreducible and I a nontrivial proper subset of S. Then* $\mathrm{Core}_W(W_I) = 1$.

Proof. Assume contrary that $1 \neq w \in \mathrm{Core}_W(W_I)$ (so that $w \cdot \Phi_I = \Phi_I$ by (2.16)). Fix $s \in S \smallsetminus I$ and take $\gamma \in \Phi_I^+$ such that $w \cdot \gamma \in \Phi_I^-$.

Case 1. $(d =)$ $d_\Gamma(s, \mathrm{supp}(\gamma)) \leq d_\Gamma(s, \mathrm{supp}(w \cdot \gamma))$**:** Take a shortest path $s_0 = s, s_1, \dots, s_{d-1}, s_d \in \mathrm{supp}(\gamma)$ in Γ from s to the set $\mathrm{supp}(\gamma)$. Then by the above inequality, we have $s_i \notin \mathrm{supp}(w \cdot \gamma)$ for all $0 \leq i \leq d-1$. Put $u = ss_1 \cdots s_{d-1} \in W$. Then we have $u \cdot \gamma \in \Phi^+$ (by (2.12)), $\mathrm{supp}(u \cdot \gamma) = \mathrm{supp}(\gamma) \cup \{s, s_1, \dots, s_{d-1}\} \not\subset I$ (by (2.13)) and so $u \cdot \gamma \in \Phi^+ \smallsetminus \Phi_I$. On the other hand, we have $uwu^{-1} \cdot (u \cdot \gamma) = u \cdot (w \cdot \gamma) \in \Phi^-$ (by (2.12)). This is a contradiction, since $uwu^{-1} \in \mathrm{Core}_W(W_I) \subset W_I$.

Case 2. $d_\Gamma(s, \mathrm{supp}(\gamma)) > d_\Gamma(s, \mathrm{supp}(w \cdot \gamma))$**:** Now by applying Case 1 to the elements $w^{-1} \in \mathrm{Core}_W(W_I)$ and $-w \cdot \gamma \in \Phi_I[w^{-1}]$, we have a contradiction again. Hence the claim holds in any case. □

Owing to Lemma 4.8, we have the following results:

$$\text{If } (W,S) \text{ is irreducible, } |W| = \infty \text{ and } s \in S, \text{ then } C_{S \smallsetminus \{s\}} = 1. \tag{4.3}$$

$$\begin{aligned} &\text{If } (W,S) \text{ is irreducible, } J \subsetneq S \text{ and } I \text{ is an irreducible component of } J \\ &\text{such that } |W_I| = \infty, \text{ then } C_J = 1. \end{aligned} \tag{4.4}$$

Indeed, for (4.3), Corollary 2.20 (iii) implies $N_W(W_{S \smallsetminus \{s\}}) = W_{S \smallsetminus \{s\}}$ and so Lemma 4.8 proves the claim. For (4.4), we have $N_W(W_J) \subset W_{I \cup I^\perp}$ by Proposition 2.18 (ii), while $\mathrm{Core}_W(W_{I \cup I^\perp}) = 1$ by Lemma 4.8 since $I \cup I^\perp \subsetneq S$, hence the claim follows from (2.2).

4.3 Proof for finite case

In this subsection, we prove Theorem 4.2 for the case $|W| < \infty$. From now, we abbreviate often the terms "Expanding Lemma", "Cutting Lemma", "Shifting Lemma" to 'EL', 'CL', 'SL', respectively.

Lemma 4.9. *Let (W,S) be irreducible, $|W| < \infty$ and $s \in S$. Suppose that no condition below is satisfied:* ***(I)*** *$W = W(B_n)$, $n \geq 2$, $s = s_1$,* ***(II)*** *$W = W(B_2)$, $s = s_2$,* ***(III)*** *$W = W(I_2(m))$, m even. Then $C_{\{s\}} = Z(W)$.*

Proof. Since $Z(W) \subset C_{\{s\}}$ and $\bigcap_{t \in S} N_W(W_{\{t\}}) = Z(W)$, it is enough to show that $C_{\{s\}} \subset C_{\{t\}}$ for all $t \in S$.

Case 1. The odd Coxeter graph Γ^{odd} of (W,S) is connected: Then the claim follows from the Shifting Lemma.

Case 2. $W = W(B_n)$**,** $n \geq 3$ **and** $s \neq s_1$**:** We have $C_{\{s\}} \overset{\mathrm{SL}}{=} C_{\{s_i\}}$ for all $2 \leq i \leq n$, while $C_{\{s_2\}} \overset{\mathrm{EL}}{\subset} C_{\{s_1, s_2\}} \overset{\mathrm{CL}}{\subset} C_{\{s_1\}}$ (since $n \geq 3$). Thus the claim holds.

Case 3. $W = W(F_4)$**:** By symmetry, we may assume $s = s_1$ or s_2. Now we have $C_{\{s_1\}} \overset{\mathrm{SL}}{=} C_{\{s_2\}} \overset{\mathrm{EL}}{\subset} C_{\{s_2, s_3\}} \overset{\mathrm{CL}}{\subset} C_{\{s_3\}} \overset{\mathrm{SL}}{=} C_{\{s_4\}}$. Hence the claim holds. □

Corollary 4.10. *Let (W,S) be irreducible, $|W| < \infty$, $s \in S$ and suppose that there is a unique vertex t of Γ farthest from s. Suppose further that W and t do not satisfy any of the three conditions (I)–(III) in Lemma 4.9. Then $C_{S \smallsetminus \{s\}} = Z(W)$.*

Proof. Now we have $C_{S \smallsetminus \{s\}} \overset{\mathrm{CL}}{\subset} C_{\{t\}}$ by the choice of t. Then apply Lemma 4.9. $\square$

Lemma 4.11. *Suppose that one of the following conditions is satisfied:* ***(I)*** $W = W(B_3)$, $s = s_2$, ***(II)*** $W = W(D_4)$, $s = s_3$, ***(III)*** $W = W(H_3)$, $s = s_2$, ***(IV)*** $W = W(I_2(m))$ *($m \geq 6$ even), $s \in S$. Then $C_I = Z(W)$, where $I = S \smallsetminus \{s\}$.*

Proof. By the hypothesis and Corollary 2.20 (ii), we have $N_W(W_I) = W_I \times Z(W)$. Now a direct computation shows that $sW_Is \cap N_W(W_I) = 1$, so that $W_I \cap C_I = 1$ by (2.5). Since $Z(W) \subset C_I$, we have $C_I = Z(W)$. $\square$

Lemma 4.12. ***(i)*** *If $W = W(B_n)$, $1 \leq n < \infty$, then* $\mathrm{Core}_W(G_{B_n}) = G_{B_n}$.
(ii) *If $W = W(D_n)$, $3 \leq n < \infty$, then* $\mathrm{Core}_W(G_{D_n} \rtimes \langle s_1 \rangle) = G_{D_n}$.

Proof. The claim (i) is obvious, since $G_{B_n} \lhd W$ (cf. Lemma 2.12). For (ii), we have $G_{D_n} \subset \mathrm{Core}_W(G_{D_n} \rtimes \langle s_1 \rangle)$ since $G_{D_n} \lhd W$, while $s_1 \notin \mathrm{Core}_W(G_{D_n} \rtimes \langle s_1 \rangle)$ since $(s_1s_3)s_1(s_1s_3)^{-1} = s_3 \notin G_{D_n} \rtimes \langle s_1 \rangle$. Thus the claim holds. $\square$

Proof of Theorem 4.2 (for finite W). Note that $Z(W) \subset C_I$ by definition.

Case 1. $(W,S) = (W(\mathcal{T}_n), S(\mathcal{T}_n))$ **for** $\mathcal{T} = B$**,** $n \geq 3$ **or** $\mathcal{T} = D$**,** $3 \leq n \neq 4$**:** Put $L = 1$ in the former case, $L = 2$ in the latter case. Note that in this case, any automorphism of $\Gamma(\mathcal{T}_n)$ preserves the sets $S(\mathcal{T}_k)$, elements $w_0(S(\mathcal{T}_k))$ $(k \geq L)$ and so the subgroup $G_{\mathcal{T}_n}$.

Subcase 1-1. $I = S(\mathcal{T}_k)$ **for some** $L \leq k < n$**:** This is a case (i) or (ii) of Theorem 4.2 (for τ identity), so that we have to show $C_I = G_{\mathcal{T}_n}$. Note that

$$C_{S(\mathcal{T}_i)} \overset{\mathrm{EL}}{\subset} C_{S(\mathcal{T}_j)} \overset{\mathrm{CL}}{\subset} C_{S(\mathcal{T}_i)} \text{ and so } C_{S(\mathcal{T}_i)} = C_{S(\mathcal{T}_j)} \text{ for all } L \leq i < j < n.$$

Thus we may assume $I = S(\mathcal{T}_L)$, and we have $C_I \subset \bigcap_{i=L}^{n-1} N_W(W_{S(\mathcal{T}_i)})$. By Corollary 2.21, (2.3) and Lemma 4.12, we have $C_I \subset G_{\mathcal{T}_n}$. Conversely, since $G_{\mathcal{T}_n}$ is abelian and contains $w_0(I)$, we have $G_{\mathcal{T}_n} \subset Z_W(w_0(I)) = N_W(W_I)$ by Lemma 4.4. Thus $G_{\mathcal{T}_n} \subset C_I$ since $G_{\mathcal{T}_n} \lhd W$. Hence $C_I = G_{\mathcal{T}_n}$.

Subcase 1-2. $I \neq S(\mathcal{T}_k)$ **for all** $L \leq k < n$**:** By the above remark, this is not a case (i) or (ii), and so we have to show $C_I \subset Z(W)$. Note that $I \neq S$. Let M be the first index ≥ 1 such that $s_M \notin I$, so that $S(\mathcal{T}_{M-1}) \subset I$ (where we put $S(\mathcal{T}_0) = \emptyset$). If $\mathcal{T} = D$ and $M = 2$, then we have $C_I \overset{\mathrm{EL}}{\subset} C_{S \smallsetminus \{s_M\}}$ since $I \neq \emptyset$. Otherwise, there is some $M < i \leq n$ such that $s_i \in I$ (since otherwise we have $I = S(\mathcal{T}_{M-1})$; a contradiction), and so $M < n$ and $C_I \overset{\mathrm{EL}}{\subset} C_{S \smallsetminus \{s_M\}}$. In any case, we may assume that $I = S \smallsetminus \{s_M\}$. Now there are the following three cases:

Subsubcase 1-2-1. $M \leq L + 1$**:** Note that $M < n$, and so $(\mathcal{T}_n, M) \neq (D_3, 3)$. If $\mathcal{T}_n = B_3$ and $M = 2$, then $C_I = Z(W)$ by Lemma 4.11. Otherwise, we have a unique vertex of Γ farthest from s; that is s_{3-M} if $\mathcal{T}_n = D_3$ and $M \leq 2$, and s_n otherwise (note that $\mathcal{T}_n \neq D_4$). Thus $C_I = Z(W)$ by Corollary 4.10.

Subsubcase 1-2-2. $L + 2 \leq M \leq n - 2$**:** This hypothesis implies that

$$C_I \overset{\mathrm{CL}}{\subset} C_{I \smallsetminus \{s_{M-1}, s_{M+1}\}} \overset{\mathrm{EL}}{\subset} C_{S \smallsetminus \{s_{M-1}\}},$$

so that the claim follows inductively from the case of smaller M.

Subsubcase 1-2-3. $L+2 \leq M = n-1$**:** Note that $n \geq L+3$ and $I = S(\mathcal{T}_{n-2}) \cup \{s_n\}$. Now we have $C_I \overset{\mathrm{CL}}{\subset} C_{S(\mathcal{T}_{n-3})} \overset{\mathrm{EL}}{\subset} C_{S(\mathcal{T}_{n-2})}$ and so $C_I \subset C_{\{s_n\}}$ by (4.2). Thus $C_I \subset C_{\{s_n\}} = Z(W)$ by Lemma 4.9.

Case 2. $(W,S) = (W(B_2), S(B_2))$**:** Since I is proper and nonempty, we have $I = \{s_i\}$ ($i = 1$ or 2). This is a case (i), by taking $\tau = \mathrm{id}_S$ (if $i = 1$), $\tau \neq \mathrm{id}_S$ (if $i = 2$). Now we have to show $C_I = \tau(G_{B_2})$. We have $C_I \subset N_W(W_{\tau(\{s_1\})}) = \tau(G_{B_2})$ by Corollary 2.21 (i). Conversely, we have $\tau(G_{B_2}) \subset C_I$ by a similar argument to Subcase 1-1. Thus $C_I = \tau(G_{B_2})$.

Case 3. $(W,S) = (W(D_4), S(D_4))$**:** Note that I is proper and nonempty.

Subcase 3-1. $|I| = 1$**:** This is not a case (i) or (ii), so that we have to show $C_I \subset Z(W)$. This follows from Lemma 4.9.

Subcase 3-2. $|I| = 2$ **and** $s_3 \in I$**:** This is also not a case (i) or (ii), so that we have to show $C_I \subset Z(W)$. Let $I = \{s_3, s_i\}$. Then we have $C_I \overset{\mathrm{CL}}{\subset} C_{\{s_i\}}$, while $C_{\{s_i\}} = Z(W)$ by the previous case. Thus $C_I \subset Z(W)$.

Subcase 3-3. $|I| = 2$ **and** $s_3 \notin I$**:** Note that there is $\tau \in \mathrm{Aut}(\mathit{\Gamma})$ such that $\tau(S(D_2)) = I$. This is a case (ii), so that we have to show $C_I = \tau(G_{D_4})$. By symmetry, we may assume $\tau = \mathrm{id}_S$. First, we have $C_I \overset{\mathrm{EL}}{\subset} C_{S(D_3)}$ and so $C_I \subset \bigcap_{i=2}^{3} N_W(W_{S(D_i)}) = G_{D_4} \rtimes \langle s_1 \rangle$ by Corollary 2.21 (ii). Thus we have $C_I \subset G_{D_4}$ by (2.3) and Lemma 4.12. Conversely, we have $G_{D_4} \subset C_I$ by a similar argument to Subcase 1-1. Hence we have $C_I = G_{D_4}$.

Subcase 3-4. $|I| = 3$ **and** $s_3 \in I$**:** Note that there is $\tau \in \mathrm{Aut}(\mathit{\Gamma})$ such that $\tau(S(D_3)) = I$. This is a case (ii), so that we have to show $C_I = \tau(G_{D_4})$. By symmetry, we may assume $\tau = \mathrm{id}_S$. Now we have $C_I \overset{\mathrm{CL}}{\subset} C_{S(D_2)} \overset{\mathrm{EL}}{\subset} C_I$, while $C_{S(D_2)} = G_{D_4}$ by the previous subcase. Thus $C_I = G_{D_4}$.

Subcase 3-5. $I = S \smallsetminus \{s_3\}$**:** This is not a case (i) or (ii), so that we have to show $C_I \subset Z(W)$. This follows from Lemma 4.11.

Case 4. $(W,S) \not\simeq (W(B_n), S(B_n))$ **(**$n \geq 2$**),** $(W(D_n), S(D_n))$ **(**$n \geq 3$**):** This is not a case (i) or (ii), so that we have to show $C_I \subset Z(W)$. Note that $|S| \geq 2$ since I is proper and nonempty.

Subcase 4-1. $|S| = 2$**:** Namely, $(W,S) = (W(\mathcal{T}), S(\mathcal{T}))$, $\mathcal{T} = A_2$ or $I_2(m)$ ($5 \leq m < \infty$), and $|I| = 1$. Then we have $C_I = Z(W)$ by Lemma 4.11 (for the latter case, with m even) or Lemma 4.9 (the other cases).

Subcase 4-2. $|S| = 3$**:** Namely, $(W,S) = (W(H_3), S(H_3))$ (note that $W(A_3) \simeq W(D_3)$). Now we have $C_I \overset{\mathrm{EL}}{\subset} C_{S \smallsetminus \{s_i\}}$ for some i, while $C_{S \smallsetminus \{s_i\}} = Z(W)$ by Lemma 4.11 (if $i = 2$) or Corollary 4.10 (if $i \neq 2$). Thus $C_I \subset Z(W)$.

Subcase 4-3. $|S| \geq 4$**:** Namely, $(W,S) = (W(\mathcal{T}), S(\mathcal{T}))$ for $\mathcal{T} = A_n$ ($n \geq 4$), E_n ($n = 6,7,8$), F_4 or H_4. Now we have $C_I \overset{\mathrm{EL}}{\subset} C_{S \smallsetminus \{s_i\}}$ for some i. Thus we may assume $I = S \smallsetminus \{s_i\}$.

Subsubcase 4-3-1. There is a unique vertex of $\mathit{\Gamma}$ **farthest from** s_i**:** Now we have $C_I = Z(W)$ by Corollary 4.10.

Subsubcase 4-3-2. There are at least two vertices of $\mathit{\Gamma}$ **farthest from** s_i**:** Namely, we have $(\mathcal{T}, i) = (A_{2k+1}, k+1)$ ($k \geq 2$), $(E_6, 2)$, $(E_6, 4)$ or $(E_8, 5)$. Now there are exactly two vertices s, t of $\mathit{\Gamma}$ farthest from s_i, and there is a vertex $\neq s, t$ adjacent to s and not adjacent to t. This implies that $C_I \overset{\mathrm{CL}}{\subset} C_{\{s,t\}} \overset{\mathrm{CL}}{\subset} C_{\{t\}}$, while $C_{\{t\}} = Z(W)$ by Lemma 4.9. Thus $C_I \subset Z(W)$. Hence the proof is concluded. $\square$

4.4 Proof for infinite case

In this subsection, we prove Theorem 4.2 in the case $|W| = \infty$. The key facts are (4.3) and (4.4).

In the proof, we use a characterization (Proposition 4.14) of certain infinite Coxeter systems, which is based on the characterization of connected Coxeter graphs of finite type. Before stating this, we prepare the following graph-theoretic lemma.

Lemma 4.13. *Let $\mathcal{G}$ be a connected acyclic graph (i.e. a tree) on nonempty vertex set $V(\mathcal{G})$ of an arbitrary cardinality (with no edge labels here).*
(i) *If all vertices of $\mathcal{G}$ have degree ≤ 2 and $\mathcal{G}$ has a terminal vertex (i.e. vertex of degree 1) s_0, then $\mathcal{G} \simeq \mathit{\Gamma}(A_n)$ (as unlabelled graphs) for some $1 \leq n \leq \infty$.*
(ii) *If $s_0 \in V(\mathcal{G})$ and all vertices of $\mathcal{G}$ except s_0 have degree ≤ 2, then each connected component $\mathcal{G}'$ of $\mathcal{G} \smallsetminus \{s_0\}$ contains exactly one vertex s adjacent to s_0, $\mathcal{G}' \simeq \mathit{\Gamma}(A_n)$ (as unlabelled graphs) for some $1 \leq n \leq \infty$ and s is a terminal vertex of $\mathcal{G}'$.*
(iii) *If all vertices of $\mathcal{G}$ have degree 2, then $\mathcal{G} \simeq \mathit{\Gamma}(A_{\infty,\infty})$ (as unlabelled graphs).*

Proof. **(i)** By the hypothesis, for any $s \in V(\mathcal{G})$, $\mathcal{G}$ contains a unique simple path $P_s = (t_s^{(0)} = s_0, t_s^{(1)}, \dots, t_s^{(\ell-1)}, t_s^{(\ell)} = s)$ from s_0 to s. Let $\ell(s) = \ell$, the length of P_s. Then for all $s_1, s_2 \in V(\mathcal{G})$, we have either $P_{s_1} \subset P_{s_2}$ or $P_{s_2} \subset P_{s_1}$: Otherwise, for the first index k such that $t_{s_1}^{(k)} \neq t_{s_2}^{(k)}$, the vertex $t_{s_1}^{(k-1)} = t_{s_2}^{(k-1)}$ is adjacent to distinct vertices $t_{s_1}^{(k)}$, $t_{s_2}^{(k)}$ (and $t_{s_1}^{(k-2)}$ if $k \geq 2$) but this is impossible by the hypothesis on the degree of $t_{s_1}^{(k-1)}$.

This observation shows that the map $\ell : V(\mathcal{G}) \to \{0, 1, 2, \dots\}$ is injective and satisfies that $i \in \ell(V(\mathcal{G}))$ whenever $0 \leq i < j$ and $j \in \ell(V(\mathcal{G}))$. Thus the set $V(\mathcal{G})$ is finite or countable. Moreover, it also implies that two vertices s_1, s_2 are adjacent if $\ell(s_1) = \ell(s_2) \pm 1$, while by definition of ℓ, these are not adjacent if $\ell(s_1) \neq \ell(s_2) \pm 1$. Thus the claim holds.
(ii) First, take a vertex t of $\mathcal{G}'$ and a simple path P in $\mathcal{G}$ from s_0 to t. Then the vertex s of P next to s_0 is adjacent to s_0 and contained in $\mathcal{G}'$. On the other hand, if $\mathcal{G}'$ contains two vertices adjacent to s_0, then s_0 and a path in $\mathcal{G}'$ between these two vertices form a closed path in $\mathcal{G}$. This is a contradiction, so that the first claim follows. Since s has degree ≤ 2 in $\mathcal{G}$ and adjacent to $s_0 \notin V(\mathcal{G}')$, s is a terminal vertex of $\mathcal{G}'$. Now the second claim is deduced by applying (i) to $\mathcal{G}'$ and s.
(iii) This follows from (ii), since $\mathcal{G}$ is nonempty and has no terminal vertices. □

Proposition 4.14. *Let (W, S) be an irreducible Coxeter system of an arbitrary rank, with Coxeter graph $\mathit{\Gamma}$. Suppose that $|W| = \infty$ and $|W_I| < \infty$ for all finite subsets $I \subset S$. Then $\mathit{\Gamma} \simeq \mathit{\Gamma}(A_\infty)$, $\mathit{\Gamma}(B_\infty)$, $\mathit{\Gamma}(D_\infty)$ or $\mathit{\Gamma}(A_{\infty,\infty})$.*

Proof. In this proof, a full subgraph $\mathit{\Gamma}_I$ of $\mathit{\Gamma}$ is said to be *forbidden* if $|I| < \infty$ and $|W_I| = \infty$. The hypothesis means that $|W| = \infty$ and $\mathit{\Gamma}$ is connected and contains no forbidden subgraphs. This implies $|S| = \infty$ immediately.

Step 1. $\mathit{\Gamma}$ is acyclic: This follows immediately from the fact that any nontrivial cycle in $\mathit{\Gamma}$ forms a forbidden subgraph.

Step 2. No $s \in S$ has degree ≥ 4 in $\mathit{\Gamma}$: Otherwise, this s and the four adjacent vertices form a forbidden subgraph of $\mathit{\Gamma}$. This is a contradiction.

Step 3. At most one $s \in S$ has degree 3 in $\mathit{\Gamma}$: Assume contrary that two distinct vertices $s, t \in S$ have degree 3. Since Γ is connected, there is a path P in Γ between s and t. Then s, t, P and all the vertices adjacent to s or t form a forbidden subgraph. This is a contradiction.

Step 4. If some $s \in S$ has degree 3 in Γ, then $\Gamma \simeq \Gamma(D_\infty)$: By Steps 1–3, we can apply Lemma 4.13 (ii) to this case. This lemma shows that $\Gamma_{S \smallsetminus \{s\}}$ consists of three connected components $\simeq \Gamma(A_{n_1})$, $\Gamma(A_{n_2})$, $\Gamma(A_{n_3})$ (as unlabelled graphs) respectively, of which a terminal vertex is adjacent to s in Γ. By symmetry, we may assume $n_1 \geq n_2 \geq n_3 \geq 1$.

Now we have $n_1 = \infty$ since $|S| = \infty$. If $n_2 \geq 2$, then Γ must contain a forbidden subgraph ($\simeq \Gamma(\widetilde{E_8})$ as unlabelled graphs), but this is a contradiction. Thus we have $n_2 = n_3 = 1$ and so $\Gamma \simeq \Gamma(D_\infty)$ as unlabelled graphs. Moreover, every edge of Γ must have no label (or label '3'), since otherwise Γ must contain a forbidden subgraph again. Hence $\Gamma \simeq \Gamma(D_\infty)$ (as Coxeter graphs) in this case.

Step 5. If all vertices of Γ have degree ≤ 2, then $\Gamma \simeq \Gamma(A_\infty)$, $\Gamma(B_\infty)$ or $\Gamma(A_{\infty,\infty})$: First, we consider the case that Γ has a terminal vertex. Then Lemma 4.13 (i) implies that $\Gamma \simeq \Gamma(A_\infty)$ as unlabelled graphs (note that $|S| = \infty$). Moreover, by a similar argument to Step 4, the hypothesis (Γ contains no forbidden subgraphs) detects the edge-labels of Γ, so that we have $\Gamma \simeq \Gamma(A_\infty)$ or $\Gamma(B_\infty)$ (as Coxeter graphs). The other case is similar; we have $\Gamma \simeq \Gamma(A_{\infty,\infty})$ as Coxeter graphs by Lemma 4.13 (iii) and the hypothesis. Hence the proof is concluded. □

Proof of Theorem 4.2 (for infinite W). Note that $Z(W) = 1$ in this case.

Case 1. $(W, S) = (W(\mathcal{T}_n), S(\mathcal{T}_n))$ for $\mathcal{T}_n = A_\infty$, B_∞, D_∞ or $A_{\infty,\infty}$: Put $L = 1$ if $\mathcal{T}_n = B_\infty$, $L = 2$ if $\mathcal{T}_n = D_\infty$. Moreover, for $k \geq 1$, put

$$J_k = \{s_1, s_2, \dots, s_k\} \text{ if } \mathcal{T}_n \neq A_{\infty,\infty}, \ J_k = \{s_{-k}, s_{-k+1}, \dots, s_k\} \text{ if } \mathcal{T}_n = A_{\infty,\infty}.$$

Subcase 1-1. $\mathcal{T}_n = B_\infty$ or D_∞, and $I = S(\mathcal{T}_k)$ for some $L \leq k < \infty$: This is a case (i) or (ii) (for τ identity), so that we have to show $C_I = G_{\mathcal{T}_\infty}$. Put $G_i = W_{J_{k+i}}$ and $H_i = N_{G_i}(W_I)$ for $i \geq 1$. Then we have $\bigcup_{i=1}^\infty G_i = W$ and $\bigcup_{i=1}^\infty H_i = N_W(W_I)$, so that $C_I \subset \bigcup_{i=1}^\infty \mathrm{Core}_{G_i}(H_i)$ by Lemma 2.6. Moreover, by the result of finite case (Section 4.3), we have $\mathrm{Core}_{G_i}(H_i) = G_{\mathcal{T}_{k+i}}$ for all $i \geq 1$. Since $\bigcup_{i=1}^\infty G_{\mathcal{T}_{k+i}} = G_{\mathcal{T}_\infty}$ (cf. Lemma 2.12), we have $C_I \subset G_{\mathcal{T}_\infty}$.

On the other hand, we have $C_{S(\mathcal{T}_L)} \overset{\mathrm{EL}}{\subset} C_I$, while $G_{\mathcal{T}_\infty} \subset Z_W(w_0(S(\mathcal{T}_L)))$ since $w_0(S(\mathcal{T}_L)) \in G_{\mathcal{T}_\infty}$ and $G_{\mathcal{T}_\infty}$ is abelian. Thus $G_{\mathcal{T}_\infty} \subset N_W(W_{S(\mathcal{T}_L)})$ by Lemma 4.4, $G_{\mathcal{T}_\infty} \subset C_{S(\mathcal{T}_L)}$ by (2.3) and so $G_{\mathcal{T}_\infty} \subset C_I$. Hence $C_I = G_{\mathcal{T}_\infty}$.

Subcase 1-2. The hypothesis of Subcase 1-1 is not satisfied: This is not a case (i) or (ii), so that we have to show $C_I = 1$.

Subsubcase 1-2-1. $|I| < \infty$: Let $w \in C_I$. Now take a sufficiently large $4 \leq k < \infty$ so that $I \subset J_k$ and $w \in W_{J_k}$. Put $G_i = W_{J_{k+i}}$ and $H_i = N_{G_i}(W_I)$ for $i \geq 1$, so that $\bigcup_{i=1}^\infty G_i = W$ and $\bigcup_{i=1}^\infty H_i = N_W(W_I)$. Now by the hypothesis of Subcase 1-2, and by the result for finite case (Section 4.3), we have $\mathrm{Core}_{G_i}(H_i) \subset Z(G_i) \subset \{1, w_0(J_{k+i})\}$ for all i. Moreover, by Lemma 2.6, we have $C_I \subset \bigcup_{i=1}^\infty \mathrm{Core}_{G_i}(H_i)$. Since $w_0(J_{k+i}) \notin W_{J_k}$ for any $i \geq 1$, this implies that $w = 1$ by the choice of k. Hence we have $C_I = 1$.

Subsubcase 1-2-2. $|I| = \infty$: If I has an irreducible component J of infinite cardinality, then $C_I = 1$ by (4.4). Thus we may assume that I is a union of infinitely many irreducible components of finite cardinality. Now we can choose indices $4 \le i \le j < \infty$ so that $s_k \notin I$ for all $i \le k \le j$, $s_{i-1} \in I$ and $s_{j+1} \in I$. Let K_1, K_2 be the (distinct) irreducible components of I containing s_{i-1}, s_{j+1} respectively. Then we have $C_I \overset{\mathrm{CL}}{\subset} C_{I \smallsetminus (K_1 \cup K_2)}$ and so $C_I \subset C_{K_1 \cup K_2}$ by (4.2). Moreover, we have $C_{K_1 \cup K_2} = 1$ by Subsubcase 1-2-1. Thus $C_I = 1$.

Case 2. $(W,S) \not\simeq (W(\mathcal{T}), S(\mathcal{T}))$ **for** $\mathcal{T} = A_\infty$**,** B_∞**,** D_∞**,** $A_{\infty,\infty}$**:** This is not a case (i) or (ii), so that we have to show $C_I = 1$. By Proposition 4.14, there is a finite subset $J_0 \subset S$ such that $|W_{J_0}| = \infty$. This J_0 consists of only finitely many irreducible components, and so we have $|W_J| = \infty$ for some irreducible component of J_0. Since Γ is connected and $|J| < \infty$, there is a (finite) sequence $s_1, s_2, \dots, s_r$ of elements of S such that $s_i \notin I_{i-1} \cup {I_{i-1}}^{\perp}$ for all $1 \le i \le r$ and $J \subset I_r$, where we put $I_0 = I$ and $I_i = I_{i-1} \cup \{s_i\}$ $(1 \le i \le r)$ inductively. Now we have $C_{I_{i-1}} \overset{\mathrm{EL}}{\subset} C_{I_i}$ for all $1 \le i \le r$, so that $C_I \subset C_{I_{r-1}}$ and $C_I \subset C_{I_r}$.

Subcase 2-1. $I_r \neq S$**:** Now an irreducible component of I_r (namely, the one containing J) generates an infinite group. Thus $C_I \subset C_{I_r} = 1$ by (4.4).

Subcase 2-2. $I_r = S$**:** Note that $r \ge 1$ since I is proper. Since (W,S) is irreducible, we have $C_I \subset C_{I_{r-1}} = 1$ by (4.3). Hence the proof is concluded. $\square$

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Part II

Almost central involutions in split extensions of Coxeter groups by graph automorphisms

ALMOST CENTRAL INVOLUTIONS IN SPLIT EXTENSIONS OF COXETER GROUPS BY GRAPH AUTOMORPHISMS

KOJI NUIDA

In this paper, given a split extension of an arbitrary Coxeter group by automorphisms of the Coxeter graph, we determine the involutions in that extension whose centralizer has finite index. Our result has applications to many problems such as the isomorphism problem of general Coxeter groups. In the argument, some properties of certain special elements and of the fixed-point subgroups by graph automorphisms in Coxeter groups, which are of independent interest, are also given.

1 Introduction

Let (W, S) be an arbitrary Coxeter system, possibly of infinite rank, and G a group acting on W. We assume that the action of G preserves the set S; namely, each element of G gives rise to an automorphism of the Coxeter graph of (W, S). The subject of this paper is the almost central involutions in the semidirect product $W \rtimes G$ corresponding the action of G; that is, involutions which is central in some subgroup of $W \rtimes G$ of finite index. We determine those involutions in $W \rtimes G$, hence the subgroup generated by those involutions, in terms of the structure of the Coxeter system (W, S) and the action of G on W (Theorem 3.1). Actually, this subgroup is the product of some finite irreducible components of W, specified in terms of the action of G, and a subgroup of G. Note that this subgroup is determined by the group structure of $W \rtimes G$ only, so our result can extract some information on the Coxeter group W from the group structure of $W \rtimes G$. Moreover, if $W \rtimes G$ admits another expression $W' \rtimes G'$ of this type, our result exhibits some relation between the Coxeter groups W and W' through the subgroup in problem (Theorem 3.2).

The main motive of this research is an application to the isomorphism problem of general Coxeter groups; that is, the problem of deciding which Coxeter groups are isomorphic as abstract groups. An important phase of the problem is to determine whether a given group isomorphism f between two Coxeter groups W and W' maps the reflections in W onto those of W'. As summarized in Section 3.3, it is shown by a result of the author's preceding paper [14] that both the centralizer of a reflection t in W and that of $f(t)$ in W' are semidirect products satisfying the hypothesis of our main theorem. Since those centralizers are isomorphic via f, our main theorem can derive some properties of $f(t)$ from those of W and of t. In particular, $f(t)$ is a reflection in W' whenever W and t satisfy a certain condition which is independent on the choice of W' and f (Theorem 3.7). When the condition is actually satisfied will be investigated in a forthcoming paper [13] of the author. Note that this argument works without any assumption on finiteness of ranks of W or of W', in contrast with most of the preceding results on the isomorphism problem which covers the case of finite ranks only.

For other applications, our result implies that the product of all finite irreducible components of a Coxeter group W is independent on the choice of the generating set S of W (Example 3.3). On the other hand, regarding certain semidirect product decompositions of W into two Coxeter groups which arise from the partition of S into conjugacy classes, our result shows that, under a certain condition, the normal factor possesses no finite irreducible component (Example 3.6). See Section 3.2 for further examples.

This paper is organized as follows. Section 2 is a preliminary for basics and further remarks on abstract groups and Coxeter groups. Section 3 summarizes the main result and its applications mentioned above. In Section 4, we recall the notion of essential elements in Coxeter groups introduced by Daan Krammer [10], and summarize some properties studied by Krammer and by Luis Paris [16]. In Section 5, we give some results on the fixed-point subgroup of a Coxeter group by an automorphism of the Coxeter graph, together with preceding results given by Robert Steinberg [18], by Bernhard Mühlherr [11] and by Masayuki Nanba [12]. Finally, Section 6 is devoted to the proof of the main theorem.

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2 Preliminaries

2.1 On abstract groups

In this subsection, we fix notations for abstract groups, and give some definitions and facts. Let G be an arbitrary group. We denote $H \leq G$ if H is a subgroup of G, and $H \trianglelefteq G$ if H is a normal subgroup of G. For a subset $X \subseteq G$, let $\langle X \rangle$ and $\langle X \rangle_{\lhd G}$ denote the subgroup and the normal subgroup, respectively, of G generated by X. Put

$$Z_H(X) = \{g \in H \mid gx = xg \text{ for all } x \in X\} \text{ for } H \leq G,$$

so $Z_G(X)$ is the centralizer of X in G. Write

$$x^g = g^{-1}xg \text{ and } X^g = \{x^g \mid x \in X\} \text{ for } g, x \in G \text{ and } X \subseteq G.$$

For $H \leq G$, put

$$\mathrm{Core}_G H = \bigcap_{g \in G} H^g,$$

the *core* of H in G. It is easily verified that $\mathrm{Core}_G H$ is the unique largest normal subgroup of G contained in H.

Lemma 2.1. *Let G be a group.*

1. *If $H \trianglelefteq G$, then $Z_G(H) \trianglelefteq G$.*
2. *If $X \subseteq G$, then $Z_G(\langle X \rangle_{\lhd G}) = \mathrm{Core}_G Z_G(X)$.*

Proof. The proof of (1) is straightforward. For (2), the inclusion $\subseteq$ follows from (1) since $\langle X\rangle \subseteq \langle X\rangle_{\lhd G}$, so it suffices to show that $H \subseteq Z_G(\langle X\rangle_{\lhd G})$ whenever $H \trianglelefteq G$ and $H \subseteq Z_G(X)$. Now we have $X \subseteq Z_G(H) \trianglelefteq G$ by (1), so $\langle X\rangle_{\lhd G} \subseteq Z_G(H)$, proving the claim. $\square$

Let $[G : H]$ denote the index of a subgroup $H \leq G$ in G. Recall the following well-known properties:

$$\text{if } G \geq H_1 \geq H_2, \text{ then } [G : H_2] = [G : H_1]\,[H_1 : H_2]\,; \tag{2.1}$$

$$\text{if } H_1, H_2 \leq G, \text{ then } [G : H_1] \geq [H_2 : H_1 \cap H_2]\,. \tag{2.2}$$

From these properties it is easy to deduce that

$$\begin{gathered}\text{if } H_1, H_2 \leq G \text{ and } [G : H_2] < \infty, \text{ then the followings are equivalent:}\\ [G : H_1] < \infty; \quad [G : H_1 \cap H_2] < \infty; \quad [H_2 : H_1 \cap H_2] < \infty.\end{gathered} \tag{2.3}$$

Lemma 2.2. *Let $H \leq G$. Then $[G : H] < \infty$ if and only if $[G : \mathrm{Core}_G H] < \infty$.*

Proof. The only nontrivial part is the "only if" part. Let $G = \bigsqcup_{i=1}^n Hg_i$ (where $n = [G : H] < \infty$) be a decomposition into cosets. Then $\mathrm{Core}_G H = \bigcap_{i=1}^n H^{g_i}$. Now for $1 \leq k \leq n$, two subgroups H^{g_k} and H have the same (finite) index in G, so the subgroup $\bigcap_{i=1}^k H^{g_i}$ has finite index in $\bigcap_{i=1}^{k-1} H^{g_i}$ by (2.2). Now iterative use of (2.1) yields the desired conclusion. $\square$

We say that an element $g \in G$ is *almost central* in G if $[G : Z_G(g)] < \infty$.

Corollary 2.3. *Let G be a group and $g \in G$.*

1. *We have $[G : Z_G(\langle g\rangle_{\lhd G})] < \infty$ if and only if g is almost central in G.*
2. *If g is almost central in G, then all $h \in \langle g\rangle_{\lhd G}$ are almost central in G.*

Proof. The claim (1) follows immediately from Lemmas 2.1 (2) and 2.2, and (2) is a consequence of (1) and the observation $Z_G(h) \geq Z_G(\langle g\rangle_{\lhd G})$. $\square$

Lemma 2.4. *Let $G_1 \rtimes G_2$ be a semidirect product of two groups, and suppose that $H_i \leq G_i$ has finite index in G_i for $i = 1, 2$. Then $[G_1 \rtimes G_2 : H_1H_2] < \infty$.*

Proof. Decompose G_i as $\bigsqcup_{j=1}^{r_i} g_{i,j}H_i$, where $r_i < \infty$. Then

$$G_1 \rtimes G_2 = \bigcup_{1\leq j\leq r_1,\, 1\leq k\leq r_2} g_{1,j}H_1g_{2,k}H_2 = \bigcup_{j,k} g_{1,j}g_{2,k}H_1^{g_{2,k}}H_2.$$

Since $[G_1 : H_1] < \infty$, we have $\left[H_1^{g_{2,k}} : H_1^{g_{2,k}} \cap H_1\right] < \infty$ by (2.2). Let $H_1^{g_{2,k}} = \bigsqcup_{\ell=1}^{n_k} h_{k,\ell}(H_1^{g_{2,k}} \cap H_1)$ (where $n_k < \infty$) be the corresponding coset decomposition. Then we have

$$G_1 \rtimes G_2 = \bigcup_{j,k}\bigcup_{\ell=1}^{n_k} g_{1,j}g_{2,k}h_{k,\ell}(H_1^{g_{2,k}} \cap H_1)H_2 \subseteq \bigcup_{j,k,\ell} g_{1,j}g_{2,k}h_{k,\ell}H_1H_2,$$

where the last union is taken over the finite set of the (j, k, ℓ), as desired. $\square$

2.2 Coxeter groups

This subsection summarizes some basic definitions and facts for Coxeter groups, which are found in the book [9] unless otherwise noticed, and give further results and remarks. Some more preliminaries focusing into the two topics, essential elements and fixed-point subgroups by Coxeter graph automorphisms, will be given in Sections 4 and 5.

Definitions

A pair (W,S) of a group W and its generating set S is called a *Coxeter system* if W admits the following presentation

$$W = \langle S \mid (st)^{m_{s,t}} = 1 \text{ for all } s,t \in S \text{ such that } m_{s,t} < \infty \rangle,$$

where the $m_{s,t} \in \{1,2,\dots\} \cup \{\infty\}$ are symmetric in $s,t \in S$, and $m_{s,t} = 1$ if and only if $s = t$. A group W is called a *Coxeter group* if some $S \subseteq W$ makes (W,S) a Coxeter system. The cardinality $|S|$ of S is called the *rank* of (W,S) or of W, which is *not* assumed to be finite unless otherwise noticed. Now $m_{s,t}$ coincides with the order of $st \in W$, so the system (W,S) determines uniquely (up to isomorphism) the *Coxeter graph* denoted by Γ, that is a simple unoriented graph with vertex set S in which every two vertices $s,t \in S$ is joined by an edge with label $m_{s,t}$ if and only if $m_{s,t} \geq 3$. (By convention, the label '3' is usually omitted when drawing a picture.)

An automorphism of the Coxeter graph Γ is briefly called a *graph automorphism of* (W,S) or *of* W. Let $\operatorname{Aut}\Gamma$ denote the set of the graph automorphisms of W. Then $m_{\tau(s),\tau(t)} = m_{s,t}$ for $\tau \in \operatorname{Aut}\Gamma$ and $s,t \in S$, so this τ extends uniquely to an automorphism of the group W denoted also by τ.

For $I \subseteq S$, let W_I denote the *standard parabolic subgroup* $\langle I \rangle$ of W generated by I. A subgroup conjugate to some W_I is called a *parabolic subgroup*. (In some context, the term "parabolic subgroups" signifies the subgroups W_I themselves only.) Now (W_I, I) is also a Coxeter system, of which the Coxeter graph Γ_I is the full subgraph of Γ with vertex set I. If I is (the vertex set of) a connected component of Γ, then W_I is called an *irreducible component* of (W,S) (or of W, if the set S is obvious from the context). If Γ is connected, then (W,S) and W are called *irreducible*. Now W is the (restricted) direct product of the irreducible components; however, each irreducible component is *not* necessarily directly indecomposable as an abstract group.

Regarding the standard parabolic subgroups, it is well known that

$$\text{if } I,J \subseteq S, \text{ then } W_I \cap W_J = W_{I \cap J}. \tag{2.4}$$

Then, since each $w \in W$ is a product of a finite number of elements of S, it follows that W possesses a unique minimal standard parabolic subgroup containing w, called the *standard parabolic closure* of w and denoted here by $\mathrm{SP}(w)$. Now the *support* $\mathrm{supp}(w) \subseteq S$ of $w \in W$ is defined by

$$W_{\mathrm{supp}(w)} = \mathrm{SP}(w).$$

On the other hand, we have the following fact for parabolic subgroups:

Proposition 2.5 (See e.g. [7, Corollary 7]). *Let $I,J \subseteq S$ and $w \in W$. Then $W_I \cap (W_J)^w = (W_K)^u$ for some $K \subseteq I$ and $u \in W_I$. Moreover, we have $K \neq I$ whenever $W_I \neq (W_J)^w$.*

This proposition denies the existence of an infinite, properly descending sequence $(W_{I_1})^{w_1} \supset (W_{I_2})^{w_2} \supset \cdots$ of parabolic subgroups with I_1 finite, since it enables us to choose the I_i inductively as descending properly. Thus W also possesses a unique minimal parabolic subgroup containing a given $w \in W$, called the *parabolic closure* of w and denoted here by $\mathrm{P}(w)$.

Let ℓ denote the length function of (W, S), namely $\ell(w)$ (where $w \in W$) is the minimal length n of an expression $w = s_1 \cdots s_n$ with $s_i \in S$ (so $\ell(w^{-1}) = \ell(w)$). Such an expression of w with $n = \ell(w)$ is called a *reduced expression*. The following three well-known properties will be used in the arguments below, without references:

if $w \in W$ and $s \in S$, then $\ell(ws) = \ell(w) \pm 1$;
for $I \subseteq S$, the length function ℓ_I of (W_I, I) agrees with ℓ on W_I;
$\mathrm{supp}(w) = \{s_1, \dots, s_n\}$ for any reduced expression $w = s_1 \cdots s_n$.

Theorem 2.6 (Exchange Condition). *Let $w = s_1 \cdots s_n \in W$, $s_i \in S$ and $t \in S$ with $\ell(wt) < \ell(w)$. Then there exists an index i such that $wt = s_1 \cdots \widehat{s_i} \cdots s_n$ (s_i omitted).*

Geometric representation and root systems

Let V denote the geometric representation space of W, that is an $\mathbb{R}$-vector space equipped with the basis $\Pi = \{\alpha_s \mid s \in S\}$ and the symmetric bilinear form $\langle\, ,\, \rangle$ determined by

$$\langle \alpha_s, \alpha_t \rangle = -\cos\frac{\pi}{m_{s,t}} \text{ if } m_{s,t} < \infty \text{ and } \langle \alpha_s, \alpha_t \rangle = -1 \text{ if } m_{s,t} = \infty.$$

W acts faithfully on V by $s \cdot v = v - 2\langle \alpha_s, v \rangle \alpha_s$ for $s \in S$ and $v \in V$, making $\langle\, ,\, \rangle$ W-invariant. Let $\Phi = W \cdot \Pi$, $\Phi^+ = \Phi \cap \mathbb{R}_{\geq 0}\Pi$ and $\Phi^- = -\Phi^+$ denote, respectively, the *root system*, the set of *positive roots* and the set of *negative roots*. We have $\Phi = \Phi^+ \sqcup \Phi^-$, and Φ consists of unit vectors with respect to $\langle\, ,\, \rangle$. For any subset $\Psi \subseteq \Phi$ and $w \in W$, write

$$\Psi^+ = \Psi \cap \Phi^+,\ \Psi^- = \Psi \cap \Phi^- \text{ and } \Psi[w] = \{\gamma \in \Psi^+ \mid w \cdot \gamma \in \Phi^-\}.$$

Then $\ell(w)$ coincides with the cardinality $|\Phi[w]|$ of $\Phi[w]$, so $w = 1$ if and only if $\Phi[w] = \emptyset$. This implies a further property that

$$\text{for } w, u \in W, \text{ we have } w = u \text{ if and only if } \Phi[w] = \Phi[u]. \tag{2.5}$$

For any $v = \sum_{s \in S} c_s \alpha_s \in V$, the *support* $\mathrm{supp}(v) \subseteq S$ of v is defined by

$$\mathrm{supp}(v) = \{s \in S \mid c_s \neq 0\}.$$

For $I \subseteq S$, put

$$\Pi_I = \{\alpha_s \mid s \in I\} \subset \Pi,\ V_I = \mathrm{span}_{\mathbb{R}}\Pi_I \subset V \text{ and } \Phi_I = \Phi \cap V_I.$$

Then it is well known (see e.g. [8, Lemma 4]) that

$$\Phi_I = W_I \cdot \Pi_I, \tag{2.6}$$

the root system of a Coxeter system (W_I, I). Note that $\Phi[w] \subseteq \Phi_{\mathrm{supp}(w)}$ for $w \in W$. Moreover, it is well known that for $I \subseteq S$, any $w \in W$ admits a unique decomposition $w = w^I w_I$ with $w_I \in W_I$ and $\Phi_I[w^I] = \emptyset$. Note that $\Phi[w_I] = \Phi_I[w]$. This implies that

$$\text{if } w \in W \text{ and } s \in \operatorname{supp}(w), \text{ then } s \in \operatorname{supp}(\gamma) \text{ for some } \gamma \in \Phi[w] \tag{2.7}$$

(if this fails, then $\Phi[w] = \Phi_I[w] = \Phi[w_I]$ where $I = \operatorname{supp}(w) \setminus \{s\}$, so $w = w_I \in W_I$ by (2.5), contradicting the definition of $\operatorname{supp}(w)$). Now we prepare a technical lemma which will be used in later sections.

Lemma 2.7. *Let $1 \neq w \in W$ and $I = \operatorname{supp}(w) \subseteq S$.*

1. *Let $\gamma \in \Phi^+$, $J = \operatorname{supp}(\gamma)$ and suppose that $I \cap J = \emptyset$ and J is adjacent to I in the Coxeter graph Γ. Then $w \cdot \gamma \in \Phi^+_{I \cup J} \setminus \Phi_J$.*
2. *Suppose that $s \in S \setminus I$ is adjacent to I in Γ. Then $\operatorname{supp}(w^s) = I \cup \{s\}$.*
3. *For $i = 1, 2$, let $1 \neq u_i \in W$, $J_i = \operatorname{supp}(u_i)$ and suppose that $J_i \cap I = \emptyset$ and J_2 is adjacent to I in Γ. Then $u_1 w u_2 \neq w$.*

Proof. **(1)** Since the action of $w \in W_I$ leaves the coefficient in γ of any $\alpha_s \in \Pi_J$ unchanged, it suffices to show that $w \cdot \gamma \neq \gamma$. Take $s \in I$ adjacent to J, and $\beta \in \Phi_I^+$ such that $s \in \operatorname{supp}(\beta)$ and $w \cdot \beta \in \Phi_I^-$ (see (2.7)). This choice yields that $\langle \beta, \gamma \rangle < 0$ and $\langle w \cdot \beta, \gamma \rangle \geq 0$ since $I \cap J = \emptyset$, showing that $w \cdot \gamma \neq \gamma$ since $\langle\, , \,\rangle$ is W-invariant.
(2) Put $J = \operatorname{supp}(w^s)$. Then we have $w = (w^s)^s \in W_{J \cup \{s\}}$ and so $I \subseteq J \cup \{s\}$, therefore $I \subseteq J$ since $s \notin I$. On the other hand, $ws \cdot \alpha_s = -w \cdot \alpha_s \in \Phi^- \setminus \{-\alpha_s\}$ by (1), so we have $w^s \cdot \alpha_s \in \Phi^-$ and $s \in J$. Thus we have $I \cup \{s\} \subseteq J$, while $w^s \in W_{I \cup \{s\}}$, proving the claim.
(3) Take $s \in J_2$ adjacent to I, and $\gamma \in \Phi[u_2{}^{-1}] \subseteq \Phi^+_{J_2}$ with $s \in \operatorname{supp}(\gamma)$ (see (2.7)), so $\beta = u_2{}^{-1} \cdot \gamma$ lies in $\Phi^-_{J_2}$. Then $w \cdot \beta \in \Phi^-$ since $I \cap J_2 = \emptyset$, while $wu_2 \cdot \beta = w \cdot \gamma \in \Phi^+ \setminus \Phi_{S \setminus I}$ by (1) and so $u_1 w u_2 \cdot \beta \in \Phi^+$ since $J_1 \cap I = \emptyset$. Thus we have $u_1 w u_2 \neq w$ as desired. □

For $\gamma = w \cdot \alpha_s \in \Phi$, let $s_\gamma = wsw^{-1}$ denote the *reflection* along the root γ acting on V by $s_\gamma \cdot v = v - 2\langle \gamma, v \rangle \gamma$ for $v \in V$. Let

$$S^W = \bigcup_{w \in W} wSw^{-1}$$

denote the set of the reflections in W, which depends on the set S in general.

Lemma 2.8. *Let W be an infinite irreducible Coxeter group. Then the orbit $W \cdot \gamma \subseteq \Phi$ of any root $\gamma \in \Phi$ is an infinite set.*

The proof of this lemma requires the following two results:

Proposition 2.9 ([4, proof of Proposition 4.2]). *Let W be an infinite irreducible Coxeter group of finite rank, and $I \subset S$ a proper subset. Then $|\Phi \setminus \Phi_I| = \infty$.*

Proposition 2.10 ([15, Lemma 2.9]). *Let $w \in W$ and suppose that $I, J \subseteq S$ are disjoint subsets such that $w \cdot \Pi_I = \Pi_I$ and $w \cdot \Pi_J \subseteq \Phi^-$. Then we have $\Phi_{I \cup J}[w] = \Phi^+_{I \cup J} \setminus \Phi_I$.*

Proof of Lemma 2.8. First we show that, for any $\beta \in \Phi^+$, we have $\langle \beta, \alpha_s \rangle < 0$ for some $s \in S$. This is obvious if $|S| = \infty$ (choose $s \in S \smallsetminus \mathrm{supp}(\beta)$ adjacent in the infinite connected graph Γ to the finite set $\mathrm{supp}(\beta)$), so suppose that $|S| < \infty$. Assume contrary that $\langle \beta, \alpha_s \rangle \geq 0$ for all $s \in S$. Put $I = \{s \in S \mid \langle \beta, \alpha_s \rangle = 0\} \neq S$ (note that $\langle \beta, \beta \rangle = 1$), so s_β fixes Φ_I pointwise. Then for any $s \in S \smallsetminus I$, we have $\langle \beta, \alpha_s \rangle > 0$ and $s_\beta \cdot \alpha_s = \alpha_s - 2\langle \beta, \alpha_s \rangle \beta \in \Phi^-$. Thus Proposition 2.10 implies that $\Phi[s_\beta] = \Phi^+ \smallsetminus \Phi_I$, which has cardinality $\ell(s_\beta) < \infty$, contradicting Proposition 2.9. Hence the claim of this paragraph holds.

For the lemma, we may assume that $\gamma \in \Phi^+$. Then by taking $s \in S$ with $\langle \gamma, \alpha_s \rangle < 0$ and putting $\gamma_1 = s \cdot \gamma$, we have $\gamma_1 \neq \gamma$ and $\gamma_1 - \gamma \in \mathbb{R}_{\geq 0}\Pi$. Iterating, we obtain an infinite sequence $\gamma_0 = \gamma$, γ_1, $\gamma_2, \dots$ of distinct positive roots in $W \cdot \gamma$ inductively, proving the claim. $\square$

We also prepare a technical lemma.

Lemma 2.11. *Let $\beta, \gamma \in \Phi^+$, $I \subseteq S$ and suppose that* $\mathrm{supp}(\gamma) \not\subseteq \mathrm{supp}(\beta)$ *and* $\mathrm{supp}(\gamma) \not\subseteq I$. *Then $s_\gamma \notin s_\beta W_I$.*

Proof. Assume contrary that $s_\gamma = s_\beta w$ for some $w \in W_I$. Then we have $w \cdot \gamma = s_\beta s_\gamma \cdot \gamma = -s_\beta \cdot \gamma$, while $w \cdot \gamma \in \Phi^+$ and $s_\beta \cdot \gamma \in \Phi^+$ by the hypothesis. This is a contradiction. $\square$

Finite, affine and hyperbolic Coxeter groups

The finite irreducible Coxeter groups are completely classified, as summarized in [9, Chapter 2]. If $I \subseteq S$ and W_I is finite, let $w_0(I)$ denote the unique *longest element* of W_I, which is an involution and maps Π_I onto $-\Pi_I$. If W_I is irreducible (but not necessarily finite) and $1 \neq w \in W_I$, then we have $I^w = I$ if and only if W_I is finite and $w = w_0(I)$. This implies the well-known fact that the center $Z(W_I)$ of an arbitrary W_I is an elementary abelian 2-group. Moreover, if W_I is finite but not irreducible, then $w_0(I) = w_0(I_1) \cdots w_0(I_k)$ where $W_{I_1}, \dots, W_{I_k}$ are the irreducible components of W_I. It is well known that, if $w \in W_I$ and $\ell(ws) < \ell(w)$ for all $s \in I$, then W_I is finite and $w = w_0(I)$.

Theorem 2.12 ([17, Theorem A]). *For any involution $w \in W$, there is a finite W_I (where $I \subseteq S$) such that w is conjugate to $w_0(I)$ and $w_0(I) \in Z(W_I)$.*

The cases where $|W_I| < \infty$ and $w_0(I) \in Z(W_I)$ are determined as well.

Let W be an irreducible Coxeter group of finite rank. Then W is called *affine* or *compact hyperbolic*, respectively, if the bilinear form $\langle\,,\,\rangle$ satisfies that (1) it is positive semidefinite or nondegenerate, respectively; (2) it is not positive definite; and (3) its restriction to any proper subspace $V_I \subset V$ (where $I \subset S$) is positive definite. (See [9, Section 6.8] for another definition of compact hyperbolicness and its equivalence to ours.) The next proposition says that these are the minimal non-finite irreducible Coxeter groups.

Proposition 2.13. *Let W be a Coxeter group of finite rank.*

1. *([9, Theorem 6.4]) We have $|W| < \infty$ if and only if $\langle\,,\,\rangle$ is positive definite.*
2. *If $|W| = \infty$ and every proper standard parabolic subgroup $W_I \subset W$ is finite, then W is irreducible, and is either affine or compact hyperbolic.*

Proof. For (2), it is easy to show that this W is irreducible. Thus by (1) and the definition of compact hyperbolicness, it now suffices to show that this $\langle\, , \,\rangle$ is positive semidefinite if it is degenerate. This follows from the observation that now V is the sum of a positive definite subspace $V_{S \smallsetminus \{s\}}$ (where $s \in S$; see (1)) of codimension 1 and the nonzero radical $V^{\perp}$ of V (note that $V^{\perp} \not\subseteq V_{S \smallsetminus \{s\}}$). □

The affine and the compact hyperbolic Coxeter groups are completely determined in [9, Chapter 2 and Section 6.9]. See the lists in Figures 1 and 2, where we abbreviate s_i to i. Note that the names of the compact hyperbolic Coxeter groups given here are not standard and are very temporary.

On the other hand, it is shown in [15, Proposition 4.14] that the infinite irreducible Coxeter groups of infinite ranks, in which every proper standard parabolic subgroup of finite rank is finite, are exhausted by Figure 3.

On centralizers and normalizers in Coxeter groups

The centralizers and the normalizers in Coxeter groups play important roles in our arguments. Here we summarize some properties which we require.

Lemma 2.14 (e.g. [15, Lemma 4.4]). *Let W_I be a finite standard parabolic subgroup of W such that $w_0(I) \in Z(W_I)$. Then the centralizer $Z_W(w_0(I))$ coincides with the normalizer $N_W(W_I)$ of W_I in W.*

Proposition 2.15. *Let W be an infinite irreducible Coxeter group. Then no involution in W is almost central in W (see Section 2.1 for terminology).*

Proof. First, if $s \in S$, then W acts transitively on the conjugacy class of s in W, which is an infinite set (Lemma 2.8), so the kernel of this action is $Z_W(s)$ and has infinite index in W. Thus s is not almost central.

By Theorem 2.12, it suffices to prove that the longest element $w_0(I)$ of any finite $W_I \neq 1$ with $w_0(I) \in Z(W_I)$ is not almost central. Note that $Z_W(w_0(I)) = N_W(W_I)$ (Lemma 2.14), while $[N_W(W_I) : Z_W(W_I)] < \infty$ since $|W_I| < \infty$. By the first paragraph, $Z_W(s)$ has infinite index in W for any $s \in I$, so do $Z_W(W_I)$ (see (2.1)) and $Z_W(w_0(I))$, as desired. □

Finally, in [15, Theorem 3.1], the centralizer of a normal subgroup generated by involutions in an irreducible W is completely determined. The following observation is an easy consequence of the result.

Proposition 2.16 (See [15, Theorem 3.1]). *Suppose that W is an arbitrary Coxeter group, and $H \leq W$ is a subgroup generated by involutions which is normal in W. Then $Z_W(H)$ is also generated by involutions.*

3 The main theorem and its applications

The first subsection of this section summarizes the main theorem of this paper (Theorem 3.1) and its corollary (Theorem 3.2) together with some notational remarks. The second subsection consists of some examples, and explains what our theorem yields in these cases. Finally, the third subsection is devoted to an application of our theorem to the analysis of the isomorphism problem of Coxeter groups (the problem of deciding which Coxeter groups are isomorphic as abstract groups), which is the original motive of this research.

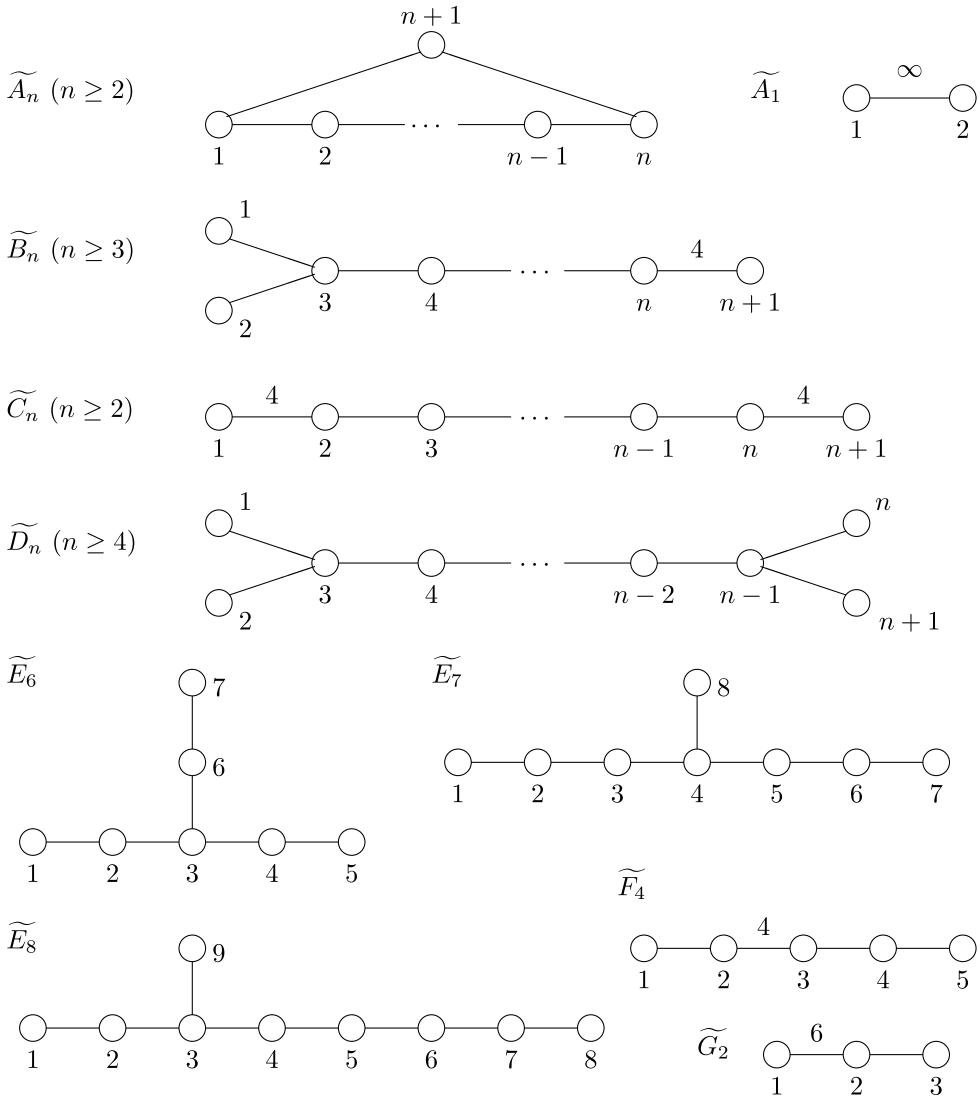

Figure 1: List of the affine Coxeter groups

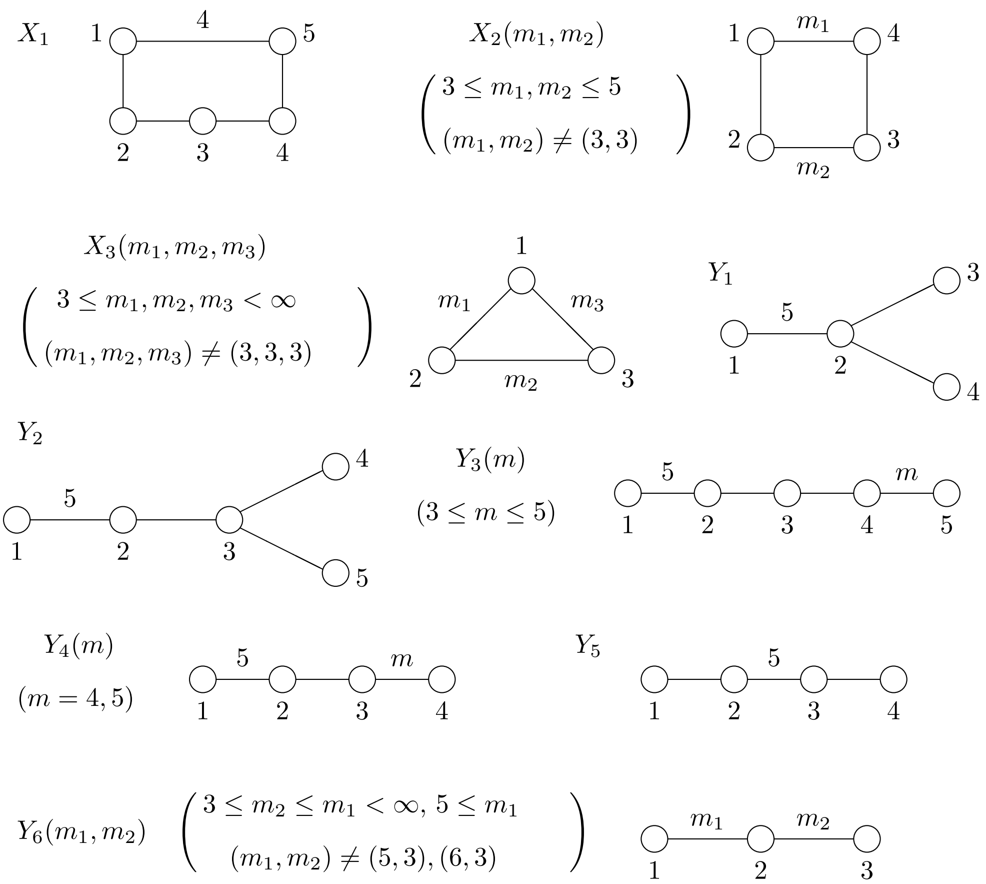

Figure 2: List of the compact hyperbolic Coxeter groups

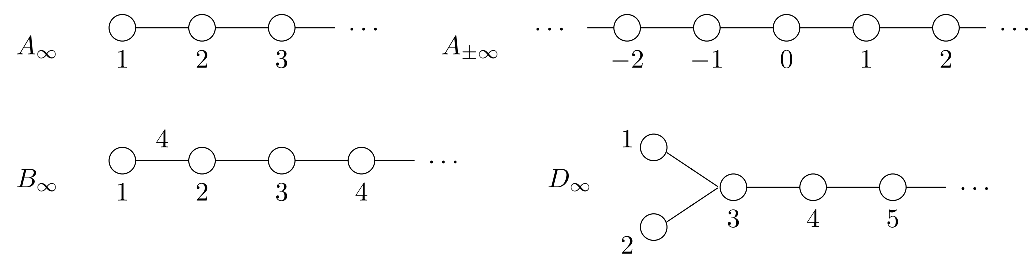

Figure 3: Some Coxeter groups of infinite ranks

3.1 Main theorem

First we prepare some notations. Let W be an arbitrary Coxeter group, and G a group acting on W via a map $\rho : G \to \mathrm{Aut}\,\Gamma$, $g \mapsto \rho_g$, yielding the semidirect product $W \rtimes G$ with respect to ρ. Let $\mathcal{C}_W^{\mathrm{fin}}$ and $\mathcal{C}_W^{\mathrm{inf}}$ be the set of the finite and the infinite irreducible components of W, respectively, and $\mathcal{C}_W = \mathcal{C}_W^{\mathrm{fin}} \cup \mathcal{C}_W^{\mathrm{inf}}$. Then the G-action permutes the elements of each of $\mathcal{C}_W$, $\mathcal{C}_W^{\mathrm{fin}}$ and $\mathcal{C}_W^{\mathrm{inf}}$. Let $\rho^\dagger : G \to \mathrm{Sym}(\mathcal{C}_W^{\mathrm{fin}})$, $g \mapsto \rho_g^\dagger$, denote the induced permutation representation of G on $\mathcal{C}_W^{\mathrm{fin}}$. For $\mathcal{C} \subseteq \mathcal{C}_W$, let $W(\mathcal{C})$ be the product of the irreducible components in $\mathcal{C}$, and put $W_{\mathrm{fin}} = W(\mathcal{C}_W^{\mathrm{fin}})$ and $W_{\mathrm{inf}} = W(\mathcal{C}_W^{\mathrm{inf}})$. Moreover, for an arbitrary group H, let H_{ACI} be the set of the almost central involutions in H (see Section 2.1 for the terminology).

Now our main theorem is as follows:

Theorem 3.1. *Here we adopt the above notations.*

1. *Let wg be an involution in $W \rtimes G$ with $w \in W$ and $g \in G$. Then wg is almost central in $W \rtimes G$ if and only if $w \in W(\mathcal{O}_\rho)$ and $g \in G_\rho \cup \{1\}$, where G_ρ is the set of all $h \in G_{\mathrm{ACI}}$ satisfying the following condition:*

$$\begin{gathered}\rho_h \textit{ is identity on all irreducible components of } W \\ \textit{except a finite number of finite irreducible components}, \end{gathered} \tag{3.1}$$

   and $\mathcal{O}_\rho \subseteq \mathcal{C}_W^{\mathrm{fin}}$ is the union of the $\rho^\dagger(G)$-orbits with finite cardinalities.

2. *We have*
$$\langle (W \rtimes G)_{\mathrm{ACI}} \rangle = W(\mathcal{O}_\rho) \rtimes \langle G_\rho \rangle.$$

Note that, assuming Theorem 5.1 below, the condition (3.1) is equivalent to the finiteness of the index $[\,W : W^{\rho_g}\,]$ of the fixed-point subgroup W^{ρ_g} by ρ_g. The proof of Theorem 3.1 is postponed until Section 6.

Since the subgroup $\langle H_{\mathrm{ACI}} \rangle$ of a group H is determined by the isomorphism type of H only, we obtain the following consequence.

Theorem 3.2. *For $i = 1, 2$, let $W_i \rtimes G_i$ be a semidirect product (via $\rho_i : G_i \to \mathrm{Aut}\,\Gamma_i$) as in Theorem 3.1, and $f : W_1 \rtimes G_1 \xrightarrow{\sim} W_2 \rtimes G_2$ a group isomorphism. Then f maps $W_1(\mathcal{O}_{\rho_1}) \rtimes (G_1)_{\rho_1}$ onto $W_2(\mathcal{O}_{\rho_2}) \rtimes (G_2)_{\rho_2}$.*

3.2 Examples

First we observe that, if $|G| < \infty$, then every $\rho^\dagger(G)$-orbit in $\mathcal{C}_W^{\mathrm{fin}}$ is finite, so $\mathcal{O}_\rho = \mathcal{C}_W^{\mathrm{fin}}$ in Theorem 3.1, therefore $\langle (W \rtimes G)_{\mathrm{ACI}} \rangle = W_{\mathrm{fin}} \rtimes G_\rho$ and G_ρ is generated by all involutions $h \in G$ satisfying (3.1).

Example 3.3. *Let W be an arbitrary Coxeter group. Then, by putting $G = 1$, Theorem 3.1 shows that $\langle W_{\mathrm{ACI}} \rangle = W_{\mathrm{fin}}$. Thus if $f : W \xrightarrow{\sim} W'$ is a group isomorphism between two Coxeter groups, then $f(W_{\mathrm{fin}}) = W'{}_{\mathrm{fin}}$; hence, by taking $W' = W$ and $f = \mathrm{id}_W$, it follows that the factor W_{fin} is independent on the choice of the generating set $S \subseteq W$.*

Example 3.3 is slightly generalized as follows:

Example 3.4. *Let $W \operatorname{wr} S_n = W^n \rtimes S_n$ denote the wreath product of W with the symmetric group S_n on n letters, so $\sigma \in S_n$ acts on $(w_1, \dots, w_n) \in W^n$ by $\rho_\sigma(w_1, \dots, w_n) = (w_{\sigma^{-1}(1)}, \dots, w_{\sigma^{-1}(n)})$. Then Theorem 3.1 implies that*

$$\langle (W \operatorname{wr} S_n)_{\mathrm{ACI}} \rangle = \begin{cases} W \operatorname{wr} S_n & \text{if } |W| < \infty; \\ {W_{\mathrm{fin}}}^n & \text{if } |W| = \infty. \end{cases}$$

Indeed, if $|W| = \infty$, then W possesses either an infinite irreducible component or infinitely many finite irreducible components, so no non-identity $\sigma \in S_n$ satisfies the condition (3.1) in any case.

We say that an irreducible component W_I of W has *finite multiplicity in W* if W possesses only finitely many irreducible components with Coxeter graph isomorphic to Γ_I. Note that, even if $|G| = \infty$, the factor $W(\mathcal{O}_\rho)$ in the theorem contains all $W_I \in \mathcal{C}_W^{\mathrm{fin}}$ with finite multiplicities.

Example 3.5. *Let $G = \operatorname{Aut}\Gamma * \operatorname{Aut}\Gamma$ be the free product of two copies of $\operatorname{Aut}\Gamma$, and $\rho : G \to \operatorname{Aut}\Gamma$ the map obtained by forgetting the distinction of the two factors $\operatorname{Aut}\Gamma$ of G. Then $\mathcal{O}_\rho$ is the set of all $W_I \in \mathcal{C}_W^{\mathrm{fin}}$ with finite multiplicities, and $G_\rho = 1$ since we have $G_{\mathrm{ACI}} = \emptyset$ by properties of free products. Roughly speaking, Theorem 3.1 extracts the finite irreducible components of W with finite multiplicities in this manner.*

For the final example, we prepare some further facts and notations. Let Γ^{odd} denote the *odd Coxeter graph* of a Coxeter group W, which is the subgraph of Γ obtained by removing all the edges with non-odd labels. It is well known (see [9, Exercise 5.3]) that two orbits $W \cdot \alpha_s$ and $W \cdot \alpha_t$ (where $s, t \in S$) intersects nontrivially if and only if s and t lie in the same connected component of Γ^{odd}. Let $S = S_1 \sqcup S_2$ be a partition where both factors are unions of connected components of Γ^{odd}, and $\Phi_{\sim S_1} = \bigcup_{s \in S_1} W \cdot \alpha_s \subseteq \Phi$. Now a general theorem of Vinay V. Deodhar [5] or of Matthew Dyer [6] shows that the subgroup $W(\Phi_{\sim S_1})$ generated by the reflections s_γ along $\gamma \in \Phi_{\sim S_1}$, which is normal in W since $\Phi_{\sim S_1}$ is W-invariant, is a Coxeter group. Moreover, the set $\Phi_{\sim S_1}$ plays the role of a root system of $W(\Phi_{\sim S_1})$; for example, any non-identity $w \in W(\Phi_{\sim S_1})$ sends some $\gamma \in \Phi_{\sim S_1}^+$ to a negative root.

Now we show that W decomposes as $W(\Phi_{\sim S_1}) \rtimes W_{S_2}$. First, if $1 \neq w \in W(\Phi_{\sim S_1}) \cap W_{S_2}$, then $w \cdot \gamma \in \Phi^-$ for some $\gamma \in \Phi_{\sim S_1}^+$ as mentioned above, and $\gamma \in \Phi_{S_2}$ since $w \in W_{S_2}$. Now by (2.6), we have $\gamma \in W \cdot \alpha_s \cap W \cdot \alpha_t$ for some $s \in S_1$ and $t \in S_2$, contradicting the choice of the partition $S = S_1 \sqcup S_2$. Thus we have $W(\Phi_{\sim S_1}) \cap W_{S_2} = 1$, while $S \subseteq W(\Phi_{\sim S_1}) W_{S_2}$ generates W, yielding the desired decomposition.

Moreover, this argument also shows that each $s \in S_2$ preserves the set $\Phi_{\sim S_1}^+$ of positive roots of $W(\Phi_{\sim S_1})$ (since $\alpha_s \not\in \Phi_{\sim S_1}$), so also the set of simple roots of $W(\Phi_{\sim S_1})$, therefore W_{S_2} acts on $W(\Phi_{\sim S_1})$ as graph automorphisms. Thus Theorem 3.1 yields the following observation:

Example 3.6. *In the situation, suppose further that W is infinite and irreducible. Then $\langle W_{\mathrm{ACI}} \rangle = 1$ (Example 3.3), while $W(\Phi_{\sim S_1})(\mathcal{O}_\rho)$ contains all the finite irreducible components of $W(\Phi_{\sim S_1})$ with finite multiplicities as mentioned above. Since $1 = W(\Phi_{\sim S_1})(\mathcal{O}_\rho) \rtimes (W_{S_2})_\rho$ (Theorem 3.2), it follows that no finite irreducible component of $W(\Phi_{\sim S_1})$ has finite multiplicity in $W(\Phi_{\sim S_1})$.*

In addition, if W_{S_2} is finite, then we have $W(\Phi_{\sim S_1})(\mathcal{O}_\rho) = W(\Phi_{\sim S_1})_{\mathrm{fin}}$. Now it follows that $W(\Phi_{\sim S_1})$ possesses no finite irreducible component.

3.3 Application to the isomorphism problem of Coxeter groups

An important phase of the isomorphism problem of Coxeter groups is of deciding whether a given group isomorphism $f : W \xrightarrow{\sim} W'$ between two Coxeter groups W and W' (with generating sets S and S', respectively) maps the set S^W of reflections in W onto that ${S'}^{W'}$ in W'; or, whether the subset S^W of W is independent on the choice of S. Note that, as is shown in [2, Lemma 3.7], we have $f(S^W) = {S'}^{W'}$ if and only if $f(S) \subseteq {S'}^{W'}$. Roughly speaking, our result below measures how $f(s)$ differs from reflections for each $s \in S$, within a certain compass. In most successful cases, the result is able to show that all $f(s)$ are reflections in W' (see Theorem 3.7).

Note that our results cover the case $|S| = \infty$ as well, in contrast with almost all of the preceding results on the isomorphism problem which cover the case of finite ranks only.

Preliminaries on centralizers and normalizers

The central tools of our argument are the centralizers $Z_W(W_I)$ and the normalizers $N_W(W_I)$ of standard parabolic subgroups W_I, which are described by the author [14] in a general setting (note that the normalizers had already been described by Brigitte Brink and Robert B. Howlett [3]). Here we summarize some of the author's results which we use.

Here we require the result only for the case that $|W_I| < \infty$ and $w_0(I) \in Z(W_I)$. Now $Z_W(W_I)$ and $N_W(W_I)$ admit the following decompositions:

$$Z_W(W_I) = (Z(W_I) \times W^{\perp I}) \rtimes Y_I \text{ and } N_W(W_I) = (W_I \times W^{\perp I}) \rtimes \widetilde{Y}_I. \quad (3.2)$$

Here $W^{\perp I}$ denotes the subgroup of W generated by the reflections in the set $Z_W(W_I) \smallsetminus W_I$, which is a Coxeter group by a theorem of Deodhar [5] or of Dyer [6]. Since $Z(W_I)$ is an elementary abelian 2-group, both $Z(W_I) \times W^{\perp I}$ and $W_I \times W^{\perp I}$ are also Coxeter groups. The factor $\widetilde{Y}_I$ of $N_W(W_I)$ acts on $W_I \times W^{\perp I}$ as graph automorphisms, preserving the factor W_I. The factor Y_I of $Z_W(W_I)$ is torsion-free and is the kernel of the induced action of $\widetilde{Y}_I$ on W_I, so Y_I is normal and has finite index in $\widetilde{Y}_I$ since $|W_I| < \infty$.

The results

Let $f : W \xrightarrow{\sim} W'$ be a group isomorphism between two Coxeter groups W and W' as above, and $I \subseteq S$ a subset with $|W_I| < \infty$ and $w_0(I) \in Z(W_I)$. Our temporal subject is the element $f(w_0(I)) \in W'$. Since $f(w_0(I))$ is an involution in W' as well as $w_0(I)$, Theorem 2.12 allows us to assume for simplicity that $f(w_0(I)) = w_0(J)$ for some $J \subseteq S'$ with $|W'_J| < \infty$ and $w_0(J) \in Z(W'_J)$. Let Y'_J and $\widetilde{Y}'_J$ denote the last factors of $Z_{W'}(W'_J)$ and of $N_{W'}(W'_J)$, respectively (see (3.2)).

We start with a very simple observation: since the isomorphism f maps $w_0(I)$ to $w_0(J)$, it also maps $Z_W(w_0(I))$ onto $Z_{W'}(w_0(J))$, so the combination of Lemma 2.14 and (3.2) yields the following isomorphism

$$f : (W_I \times W^{\perp I}) \rtimes \widetilde{Y}_I \xrightarrow{\sim} (W'_J \times {W'}^{\perp J}) \rtimes \widetilde{Y}'_J. \quad (3.3)$$

Let ρ and ρ' denote the maps representing the actions of $\widetilde{Y}_I$ and $\widetilde{Y}'_J$ in (3.3), respectively. Then by (3.3) and the results in Section 3.3, Theorem 3.2 yields the following isomorphism

$$f : (W_I \times W^{\perp I})(\mathcal{O}_\rho) \rtimes (\widetilde{Y}_I)_\rho \xrightarrow{\sim} (W'_J \times W'^{\perp J})(\mathcal{O}'_{\rho'}) \rtimes (\widetilde{Y}'_J)_{\rho'}. \tag{3.4}$$

Now the left and the right sides of (3.4) contain, as normal subgroups, W_I and W'_J which are $\rho(\widetilde{Y}_I)$-invariant and $\rho'(\widetilde{Y}'_J)$-invariant, respectively. Thus if we know much enough of the structure of the left side of (3.4), then we would be able to say something about the variation of the set J, so about the property of $f(w_0(I))$. This is hopeful at least for individual cases, since [14] also gives a method for computing the explicit structure of the decompositions (3.2).

From now, we assume further that $\widetilde{Y}_I = Y_I$ (this is satisfied if W_I admits no nontrivial graph automorphism). For an arbitrary group G, let G_{INV} be the set of the involutions in G, so $\langle G_{\mathrm{INV}} \rangle \trianglelefteq G$ and $\langle G_{\mathrm{INV}} \rangle$ is determined by the isomorphism type of G only as well as $\langle G_{\mathrm{ACI}} \rangle$. Then, since both $W_I \times W^{\perp I}$ and $W'_J \times W'^{\perp J}$ are generated by involutions and the torsion-free group Y_I possesses no involution, we can derive from (3.3) the following isomorphism

$$f : W_I \times W^{\perp I} \xrightarrow{\sim} (W'_J \times W'^{\perp J}) \rtimes G, \quad \text{where } G = \langle (\widetilde{Y}'_J)_{\mathrm{INV}} \rangle, \tag{3.5}$$

by taking the $\langle (*)_{\mathrm{INV}} \rangle$ of both sides. Now consider the centralizers of the normal subgroups $f^{-1}(W'_J)$ and W'_J in the left and the right sides of (3.5), respectively, which are also isomorphic via f. Since $f^{-1}(W'_J)$ is generated by involutions, Proposition 2.16 implies that the centralizer in the left side is also generated by involutions, so is the centralizer in the right side. The latter is the intersection of the right side of (3.5) and $Z_{W'}(W'_J) = (Z(W'_J) \times W'^{\perp J}) \rtimes Y'_J$, that is

$$(Z(W'_J) \times W'^{\perp J}) \rtimes (Y'_J \cap G),$$

and all of its involutions are contained in the former factor since $Y'_J \cap G$ is torsion-free as well as Y'_J. Thus it follows that $Y'_J \cap G = 1$, so the G-action on the finite group W'_J is faithful, therefore G is also finite. Hence, as mentioned in the first paragraph of Section 3.2, (3.5) and Theorem 3.2 yield the following isomorphism

$$f : W_I \times W^{\perp I}{}_{\mathrm{fin}} \xrightarrow{\sim} (W'_J \times W'^{\perp J}{}_{\mathrm{fin}}) \rtimes G_{\rho'}. \tag{3.6}$$

This reduces our problem to the study of semidirect product decompositions of Coxeter groups whose irreducible components are finite.

Finally, specializing to the case $I = \{s\}$, we obtain the following result.

Theorem 3.7. *Let (W, S) be an arbitrary Coxeter system.*

1. *Suppose that $s \in S$, and $W^{\perp s}{}_{\mathrm{fin}}$ is either trivial or generated by a single reflection conjugate to s. Then $f(s) \in S'^{W'}$ for any Coxeter system (W', S') and any group isomorphism $f : W \xrightarrow{\sim} W'$.*

2. *Suppose that every $s \in S$ satisfies the hypothesis of (1). Then $f(S) \subseteq S'^{W'}$ for any Coxeter system (W', S') and any group isomorphism $f : W \xrightarrow{\sim} W'$, so f preserves the set of reflections. Hence the set S^W is determined by W only and independent on the choice of $S \subseteq W$.*

Proof. We only prove (1), since (2) follows immediately from (1) and the first remark of Section 3.3. Now the above argument works for $I = \{s\}$, so it suffices to deduce that $|J| = 1$, implying that $f(s) = w_0(J) \in S'$ as desired. This is immediately done if $W^{\perp s}{}_{\mathrm{fin}} = 1$, since $J \neq \emptyset$ and now both sides of (3.6) have cardinality 2.

Suppose that $W^{\perp s}{}_{\mathrm{fin}} = \langle t \rangle$ with $t \in W$ conjugate to s. Then both sides of (3.6) have cardinality 4. Thus if $|J| \neq 1$, then it follows that $J = \{s', t'\}$ for two commuting generators $s', t' \in S'$ and the right side of (3.6) is W'_J itself, so we have an isomorphism $f : \langle s \rangle \times \langle t \rangle \xrightarrow{\sim} \langle s' \rangle \times \langle t' \rangle$. Since we assumed that $f(s) = w_0(J) = s't'$, it follows that $f(t)$ is either s' or t', which cannot be conjugate to $f(s) = s't'$ in W', contradicting the choice of t. Hence $|J| = 1$. □

Moreover, a forcecoming paper [13] of the author will describe for which $s \in S$ the hypothesis is indeed satisfied, and show that this case occurs very frequently.

4 Essential elements and Coxeter elements

Krammer introduced in his Ph.D. thesis [10] the notion of essential elements of Coxeter groups. An element w of a Coxeter group W is called *essential* in W if the parabolic closure $\mathrm{P}(w)$ of w is W itself (see Section 2.2 for terminology). Note that any W of infinite rank cannot possess an essential element, while a Coxeter element $s_1 s_2 \cdots s_n$ of an infinite irreducible W of finite rank (where $S = \{s_1, s_2, \dots, s_n\}$) is always essential in W (see Theorem 4.1). Here we summarize some properties of essential elements required in later sections, as follows:

Theorem 4.1. *Let W be an infinite irreducible Coxeter group of finite rank.*

1. *Any essential element of W has infinite order.*
2. *Let $0 \neq k \in \mathbb{Z}$. Then $w \in W$ is essential in W if and only if w^k is essential in W.*
3. *If $n = |S|$ and $\gamma_1, \dots, \gamma_n \in \Phi$ are linearly independent, then $s_{\gamma_1} \cdots s_{\gamma_n}$ is essential in W. Hence any Coxeter element of W is essential in W.*

The claim (1) is an immediate consequence of a well-known theorem of Jacques Tits, which says that any finite subgroup of a Coxeter group is contained in a finite parabolic subgroup (see e.g. [1, Lemma 1.2] for a proof). On the other hand, (2) and (3) are shown by Paris in his recent preprint [16]; however, he proved (3) only for Coxeter elements though his idea is adaptable applicable to the generalized version. Here we include proofs of (2) and (3) along Paris' idea for the sake of completeness.

For (2), we fix W and w as in the statement. For $\gamma \in \Phi$, let $\sigma_\gamma^w = ((\sigma_\gamma^w)_n)_{n \in \mathbb{Z}}$ be the infinite sequence of $+$ and $-$ such that $(\sigma_\gamma^w)_n = \varepsilon$ if and only if $w^n \cdot \gamma \in \Phi^\varepsilon$. We define $(\sigma_\gamma^w)_\infty$ (or $(\sigma_\gamma^w)_{-\infty}$, respectively) to be $\varepsilon \in \{+, -\}$ if $(\sigma_\gamma^w)_n = \varepsilon$ (or $(\sigma_\gamma^w)_{-n} = \varepsilon$, respectively) for all sufficiently large n. Following [10], we say that γ is *w-periodic* if $w^n \cdot \gamma = \gamma$ for some $n \neq 0$. Now we include the proofs of the following two lemmas for the sake of completeness.

Lemma 4.2 (See [10, Proposition 5.2.2]). *If $\gamma \in \Phi$ is not w-periodic, then only finitely many sign-changes occur in the sequence σ_γ^w.*

Proof. By the hypothesis, all roots $w^n \cdot \gamma$ such that $(\sigma_\gamma^w)_n \neq (\sigma_\gamma^w)_{n+1}$ are distinct and contained in the finite set $\Phi\,[w] \cup -\Phi\,[w]$. □

A root $\gamma \in \Phi$ is called *w-odd* (see [10]) if it is not w-periodic (so both $(\sigma_\gamma^w)_{\pm\infty}$ are defined; see Lemma 4.2) and $(\sigma_\gamma^w)_\infty \neq (\sigma_\gamma^w)_{-\infty}$. A reflection s_γ is called *w-odd* if γ is w-odd.

Lemma 4.3 (See [10, Lemma 5.2.7]). *For $k \in \mathbb{Z} \smallsetminus \{0\}$, a root of W is w-odd if and only if it is w^k-odd.*

Proof. Note that $\gamma \in \Phi$ is w-periodic if and only if it is w^k-periodic. Thus for a non-w-periodic γ, all of $(\sigma_\gamma^w)_{\pm\infty}$ and $(\sigma_\gamma^{w^k})_{\pm\infty}$ are defined (Lemma 4.2) and we have

$$(\sigma_\gamma^{w^k})_{\pm\infty} = \begin{cases} (\sigma_\gamma^w)_{\pm\infty} & \text{if } k > 0; \\ (\sigma_\gamma^w)_{\mp\infty} & \text{if } k < 0, \end{cases}$$

respectively. Thus the claim follows. □

Let $\mathrm{P}^\infty(w)$ denote the subgroup of W generated by the w-odd reflections. The following result of [10] is crucial in our argument.

Proposition 4.4 (See [10, Corollary 5.8.7]). *The parabolic closure $\mathrm{P}(w)$ is a direct product of $\mathrm{P}^\infty(w)$ and a finite number of finite groups.*

Moreover, the following result of the author [15] is also required. See also [16, Theorem 4.1] for the case of finite ranks.

Proposition 4.5 ([15, Theorem 3.3]). *If W is an infinite irreducible Coxeter group, then W is directly indecomposable as an abstract group.*

Corollary 4.6. *Suppose that W is infinite and irreducible. Then $w \in W$ is essential in W if and only if $\mathrm{P}^\infty(w) = W$.*

Proof. The 'if' part is a consequence of Proposition 4.4. For the "only if" part, assume that $\mathrm{P}(w) = W$. Then Proposition 4.4 implies that W is the direct product of $\mathrm{P}^\infty(w)$ and certain finite groups, while W is directly indecomposable (Proposition 4.5). Thus W must coincide with one of the direct factors, which cannot be finite since $|W| = \infty$, so $W = \mathrm{P}^\infty(w)$ as desired. □

Now the claim (2) of Theorem 4.1 follows easily from Lemma 4.3 and Corollary 4.6, since the w^k-odd reflections are precisely the w-odd reflections.

For the proof of (3), we prepare two lemmas. Here we say that (W, S) *is (non)degenerate* to signify the (non)degenerateness of the bilinear form $\langle\,,\,\rangle$, respectively.

Lemma 4.7 (See [16, Lemma 3.2]). *Let W be a Coxeter group of finite rank. Then there is a nondegenerate Coxeter system $(\widetilde{W}, \widetilde{S})$ of finite rank such that $S \subseteq \widetilde{S}$ and $\widetilde{W}_S = W_S$.*

Proof. We put $n = |S|$ and $S = \{s_1, s_3, \dots, s_{2n-1}\}$, and apply the following algorithm inductively for $1 \leq k \leq n$, beginning with $\widetilde{S} = S$:

if the Coxeter system $(\langle I_k\rangle, I_k)$ (where $I_k = \{s_i \in \widetilde{S} \mid i \le 2k\}$) is degenerate, add a new generator s_{2k} to $\widetilde{S}$ so that $s_{2k-1}s_{2k}$ has infinite order and s_{2k} commutes with the other elements of $\widetilde{S}$.

By computing the determinant of the matrix of the bilinear form with respect to the basis $\{\alpha_{s_i}\}_i$, it is checked inductively that the Coxeter system $(\langle I_k\rangle, I_k)$ will be nondegenerate when the k-th step is done. Hence the Coxeter system $(\widetilde{W}, \widetilde{S}) = (\langle I_n\rangle, I_n)$ obtained finally is the desired one. □

Lemma 4.8. *Any element of a proper standard parabolic subgroup W_I of W has a nonzero 1-eigenvector in V.*

Proof. It suffices to consider the case that $S = \{s_1, s_2, \dots, s_n\}$ is finite and $I = S \smallsetminus \{s_n\}$. Then, by definition of the W-action, the n-th row of the representation matrix A_w of $w \in W_I$ relative to the basis Π of V is $(0\ 0\ \cdots\ 0\ 1)$. Thus the matrix $I_n - A_w$ is singular as desired. □

The following property is the essence of the claim (3) of Theorem 4.1.

Proposition 4.9. *Let W be a Coxeter group with $|S| = n < \infty$, and suppose that $\gamma_1, \dots, \gamma_n \in \Phi$ are linearly independent. Then the standard parabolic closure of $s_{\gamma_1} \cdots s_{\gamma_n} \in W$ is W itself.*

Proof. Assume contrary that $w = s_{\gamma_1} \cdots s_{\gamma_n} \in W_I$ for a proper $W_I \subset W$. We may assume without loss of generality that (W, S) is nondegenerate, since we can extend S to $\widetilde{S} = S \sqcup \{t_1, \dots, t_m\}$ as in Lemma 4.7 and consider $t_1 \cdots t_m w \in \widetilde{W}_J$ instead of w, where $J = \widetilde{S} \smallsetminus (S \smallsetminus I)$. Choose a nonzero $v \in V$ such that $w \cdot v = v$ (Lemma 4.8). Then, since (W, S) is nondegenerate, there is an index i such that $\langle v, \gamma_i\rangle \ne 0$ and $\langle v, \gamma_j\rangle = 0$ for all $j > i$. This implies that $w \cdot v = s_{\gamma_1} \cdots s_{\gamma_i} \cdot v$, which is the sum of $s_{\gamma_i} \cdot v = v - 2\langle v, \gamma_i\rangle\gamma_i$ and a linear combination of $\gamma_1, \dots, \gamma_{i-1}$. Now the property $w \cdot v = v$ yields an expression of $2\langle v, \gamma_i\rangle\gamma_i$ as a linear combination of the other γ_j, contradicting the linear independence of $\gamma_1, \dots, \gamma_n$. Hence the claim follows. □

Now the claim (3) of Theorem 4.1 is easily proved, since the hypothesis of Proposition 4.9 is invariant under the action of W. Hence the proof of Theorem 4.1 is concluded.

5 On the fixed-point subgroups by Coxeter graph automorphisms

The subject of this section is the fixed-point subgroup

$$W^\tau = \{w \in W \mid \tau(w) = w\}$$

of a Coxeter group W by a graph automorphism $\tau \in \operatorname{Aut}\Gamma$ (as mentioned in Section 2.2, the automorphism of W induced by τ is also denoted by τ). Let $\tau\backslash S$ denote the set of the $\langle\tau\rangle$-orbits in S. Then it was shown by Steinberg [18, Theorem 1.32] that W^τ is a Coxeter group with respect to the following generating set

$$S(W^\tau) = \{w_0(I) \in W \mid I \in \tau\backslash S \text{ and } |W_I| < \infty\}$$

(see also [11] and [12]). Here we show the following properties of the subgroup W^τ, which will be used in the proof of the main theorem.

Theorem 5.1. *Let W be an arbitrary Coxeter group and $\tau \in \operatorname{Aut}\Gamma$. Then W^τ has finite index in W if and only if τ is identity on all irreducible components of W except a finite number of finite irreducible components.*

Theorem 5.2. *Let W be an infinite irreducible Coxeter group and $\tau \in \operatorname{Aut}\Gamma$.*

1. *If $|W_I| < \infty$ for all $I \in \tau\backslash S$, then the Coxeter group W^τ is also infinite and irreducible with respect to the generating set $S(W^\tau)$.*
2. *Suppose that the hypothesis of (1) fails and every orbit $I \in \tau\backslash S$ is finite. Then for any $1 \neq w \in W^\tau$, there is an element $u \in W$ of infinite order such that $u^k w \tau(u)^{-k} \neq w$ for all $0 \neq k \in \mathbb{Z}$.*

Note that the result on infiniteness of W^τ in Theorem 5.2 (1) is mentioned in [11, Section 5] without proof in a generalized setting.

5.1 Proof of Theorem 5.1

Our first step is to prove the following lemma:

Lemma 5.3. *Let W be an (irreducible) affine or compact hyperbolic Coxeter group with* type $W \neq \widetilde{A_1}$ *(see Section 2.2 for terminology). Suppose further that* $\operatorname{Aut}\Gamma \neq \{\mathrm{id}_S\}$*. Then for any $\mathrm{id}_S \neq \tau \in \operatorname{Aut}\Gamma$, there is an element $w \in W$ of infinite order such that $\langle w \rangle \cap \langle \tau(w) \rangle = 1$.*

From now until the end of the proof of Lemma 5.3, we assume that S is finite and the base field of the (finite-dimensional) geometric representation space V is extended from $\mathbb{R}$ to $\mathbb{C}$. Then the bilinear form $\langle\ ,\ \rangle$ and the faithful W-action also extend naturally so that W is embedded injectively in the group of orthogonal linear transformations of V relative to $\langle\ ,\ \rangle$. For $\lambda \in \mathbb{C}$, let $V_\lambda(w)$ denote the λ-eigenspace of $w \in W$, and let $V_{\sqrt{1}}(w)$, $V_{\neq\sqrt{1}}(w)$ be the sum of $V_\lambda(w)$ where λ runs over the roots of unity, over $\mathbb{C} \smallsetminus \{0\}$ except the roots of unity, respectively. Then some elementary linear algebra shows that, if $w \in W$, $0 \neq \lambda \in \mathbb{C}$ and $0 \neq k \in \mathbb{Z}$, then $V_\lambda(w^k)$ is the sum of $V_\mu(w)$ where $\mu \in \mathbb{C}$ varies subject to $\mu^k = \lambda$. Hence we have $V_{\sqrt{1}}(w^k) = V_{\sqrt{1}}(w)$ and $V_{\neq\sqrt{1}}(w^k) = V_{\neq\sqrt{1}}(w)$ whenever $k \neq 0$.

Now we have the following:

Lemma 5.4. *Let $w_1, w_2 \in W$ and suppose that either $V_{\sqrt{1}}(w_1) \neq V_{\sqrt{1}}(w_2)$ or $V_{\neq\sqrt{1}}(w_1) \neq V_{\neq\sqrt{1}}(w_2)$. Then $\langle w_1 \rangle \cap \langle w_2 \rangle = 1$.*

Proof. Assume contrary that $k, \ell \in \mathbb{Z} \smallsetminus \{0\}$ and ${w_1}^k = {w_2}^\ell$. Then, in the first case $V_{\sqrt{1}}(w_1) \neq V_{\sqrt{1}}(w_2)$, the above observation implies that

$$V_{\sqrt{1}}({w_1}^k) = V_{\sqrt{1}}(w_1) \neq V_{\sqrt{1}}(w_2) = V_{\sqrt{1}}({w_2}^\ell),$$

contradicting the assumption ${w_1}^k = {w_2}^\ell$. The other case is similar. □

Define actions of $\tau \in \operatorname{Aut}\Gamma$ on V and the dual space V^* with dual basis $\{\alpha_s^* \mid s \in S\}$ (as linear transformations) by

$$\tau(\alpha_s) = \alpha_{\tau(s)} \text{ and } \tau(\alpha_s^*) = \alpha_{\tau(s)}^* \text{ for } s \in S.$$

Then τ preserves the bilinear form $\langle\, ,\, \rangle$, and we have $\tau(w) \cdot \tau(v) = \tau(w \cdot v)$ for $w \in W$ and $v \in V$. Thus for $0 \neq \lambda \in \mathbb{C}$ and $w \in W$, it follows that $V_\lambda(\tau(w)) = \tau(V_\lambda(w))$, $V_{\sqrt{1}}(\tau(w)) = \tau(V_{\sqrt{1}}(w))$ and $V_{\neq\sqrt{1}}(\tau(w)) = \tau(V_{\neq\sqrt{1}}(w))$. Moreover, we have $\tau(\eta)(\tau(v)) = \eta(v)$ for $\eta \in V^*$ and $v \in V$. Note also that $\mathrm{Ann}(\tau(V')) = \tau(\mathrm{Ann}(V'))$ for any subspace $V' \subseteq V$, where $\mathrm{Ann}(V') = \{\eta \in V^* \mid \eta(V') = 0\}$ denotes the annihilator of V'.

By these observations, we have the following lemmas. In these lemmas, write $v^\perp = \{v' \in V \mid \langle v, v' \rangle = 0\}$ for $v \in V$.

Lemma 5.5. *Let* $\mathrm{id}_S \neq \tau \in \mathrm{Aut}\,\Gamma$*,* $\beta, \gamma \in \Phi^+$ *and* $V' \subset V$ *a subspace of codimension* 1*. Suppose that* $\langle \beta, \gamma \rangle = -1$*,* $V' \subseteq \beta^\perp \cap \gamma^\perp$ *and* $\mathrm{Ann}(V')$ *is not* τ*-invariant. Then* $w = s_\beta s_\gamma \in W$ *has infinite order and* $\langle w \rangle \cap \langle \tau(w) \rangle = 1$*.*

Proof. Since $\langle \beta, \gamma \rangle = -1$, we have $w^k \cdot \beta = (2k+1)\beta + 2k\gamma \neq \beta$ for all $k \geq 1$, showing that w has infinite order. Thus $V_{\sqrt{1}}(w) \neq V$, since otherwise we have $V_1(w^k) = V$ and $w^k = 1$ for a sufficiently large k, a contradiction. Now we have

$$V' \subseteq \beta^\perp \cap \gamma^\perp \subseteq V_1(w) \subseteq V_{\sqrt{1}}(w) \subset V \text{ and } \dim V - \dim V' = 1,$$

implying that $V' = V_{\sqrt{1}}(w)$. Since $\mathrm{Ann}(V')$ is not τ-invariant, we have

$$\mathrm{Ann}(V_{\sqrt{1}}(w)) \neq \tau(\mathrm{Ann}(V_{\sqrt{1}}(w))) = \mathrm{Ann}(V_{\sqrt{1}}(\tau(w))),$$

so $V_{\sqrt{1}}(w) \neq V_{\sqrt{1}}(\tau(w))$. Hence Lemma 5.4 completes the proof. □

Lemma 5.6. *For* $i = 1, 2$*, let* $\beta_i, \gamma_i \in \Phi^+$ *and* $V^{(i)} \subset V$ *a subspace of codimension* 3*, and suppose that* $\langle \beta_i, \gamma_i \rangle < -1$*,* $V^{(i)} \subseteq {\beta_i}^\perp \cap {\gamma_i}^\perp$ *and* $\mathbb{C}\beta_1 + \mathbb{C}\gamma_1 \neq \mathbb{C}\beta_2 + \mathbb{C}\gamma_2$*. Then each* $w_i = s_{\beta_i} s_{\gamma_i} \in W$ *has infinite order and* $\langle w_1 \rangle \cap \langle w_2 \rangle = 1$*.*

Proof. Put ${v_i}^\pm = (-c_i \pm \sqrt{c_i^2 - 1})\beta_i + \gamma_i$ and ${\lambda_i}^\pm = 2c_i^2 - 1 \mp 2c_i\sqrt{c_i^2 - 1}$, respectively, where $c_i = \langle \beta_i, \gamma_i \rangle$. Then a direct computation shows that $w_i \cdot {v_i}^\pm = {\lambda_i}^\pm {v_i}^\pm$ and $|{\lambda_i}^\pm| \neq 1$, respectively, and ${\lambda_i}^+ {\lambda_i}^- = 1$, so w_i has infinite order. Moreover, since ${\beta_i}^\perp \cap {\gamma_i}^\perp \subseteq V_1(w_i)$, the hypothesis implies that $\dim V - \dim V_1(w_i) \leq 3$, so the characteristic polynomial $\chi_{w_i}(x) = \det(x \cdot \mathrm{id}_V - w_i)$ of w_i decomposes as

$$\chi_{w_i}(x) = (x-1)^{|S|-3}(x - {\lambda_i}^+)(x - {\lambda_i}^-)(x - \mu_i) \text{ where } \mu_i \in \mathbb{C}.$$

Now we have $\pm 1 = \det w_i = \pm \chi_{w_i}(0) = \pm {\lambda_i}^+ {\lambda_i}^- \mu_i$ since w_i is a product of involutions, so $\mu_i = \pm 1$. Thus $V_{\neq\sqrt{1}}(w_i) = \mathbb{C}{v_i}^+ + \mathbb{C}{v_i}^- = \mathbb{C}\beta_i + \mathbb{C}\gamma_i$, so $V_{\neq\sqrt{1}}(w_1) \neq V_{\neq\sqrt{1}}(w_2)$ by the hypothesis. Hence Lemma 5.4 completes the proof. □

Corollary 5.7. *Let* $\mathrm{id}_S \neq \tau \in \mathrm{Aut}\,\Gamma$*,* $\beta, \gamma \in \Phi^+$ *and* $V' \subset V$ *a subspace of codimension* 3*. Suppose that* $\langle \beta, \gamma \rangle < -1$*,* $V' \subseteq \beta^\perp \cap \gamma^\perp$ *and* $\mathbb{C}\beta + \mathbb{C}\gamma$ *is not* τ*-invariant. Then* $w = s_\beta s_\gamma \in W$ *has infinite order and* $\langle w \rangle \cap \langle \tau(w) \rangle = 1$*.*

Proof. Note that $\tau(\mathbb{C}\beta + \mathbb{C}\gamma) = \mathbb{C}\tau(\beta) + \mathbb{C}\tau(\gamma)$ and $\tau(s_\beta s_\gamma) = s_{\tau(\beta)} s_{\tau(\gamma)}$. Then the claim follows from Lemma 5.6, where $\beta_1 = \beta$, $\gamma_1 = \gamma$, $\beta_2 = \tau(\beta)$, $\gamma_2 = \tau(\gamma)$, $V^{(1)} = V'$ and $V^{(2)} = \tau(V')$. □

Table 1: List for the proof of Lemma 5.3, affine case

W	τ	β	γ	$\mathrm{Ann}(V')$
$\widetilde{A_n}$ $(n \geq 2)$	$\tau(1) \neq 1$	α_1	$011\cdots11$	$2\alpha_1^* - \alpha_2^* - \alpha_{n+1}^*$
$\widetilde{B_n}$ $(n \geq 3)$		α_1	$0122\cdots22\sqrt{2}$	$2\alpha_1^* - \alpha_3^*$
$\widetilde{C_n}$ $(n \geq 2)$		α_1	$0\sqrt{2}\sqrt{2}\cdots\sqrt{2}\sqrt{2}1$	$\sqrt{2}\alpha_1^* - \alpha_2^*$
$\widetilde{D_n}$ $(n \geq 4)$	$\tau(1) \neq 1$	α_1	$0122\cdots2211$	$2\alpha_1^* - \alpha_3^*$
$\widetilde{E_6}$	$\tau(1) \neq 1$	α_1	0232121	$2\alpha_1^* - \alpha_2^*$
$\widetilde{E_7}$		α_1	02343212	$2\alpha_1^* - \alpha_2^*$

Table 2: List for the proof of Lemma 5.3, compact hyperbolic case

W	τ	β	γ	V'
X_1		α_1	01111	α_3, α_4
$X_2(m_1, m_2)$		α_4	$s_1 s_2 \cdot \alpha_3$	α_2
$X_3(m_1, m_2, m_3)$	$\tau(3) \neq 3$	α_3	$s_2 \cdot \alpha_1$	$\emptyset$
Y_1		α_4	$\widetilde{\alpha}_4$	α_3
Y_2		α_5	$\widetilde{\alpha}_5$	α_1, α_2
$Y_3(5)$		α_5	$\widetilde{\alpha}_5$	α_1, α_2
$Y_4(5)$		α_4	$\widetilde{\alpha}_4$	α_1
Y_5		α_4	$\widetilde{\alpha}_4$	α_1
$Y_6(m, m)$ $(m \geq 5)$		α_3	$s_2 \cdot \alpha_1$	$\emptyset$

Proof of Lemma 5.3. This lemma is deduced from Lemma 5.5 for affine case and Corollary 5.7 for compact hyperbolic case, by constructing the β, γ and V' as in Tables 1 and 2 (see also Figures 1 and 2). Note that $\beta + \gamma$ is the null root of W in an affine case. If $|\mathrm{Aut}\,\Gamma| \geq 3$, we assume by symmetry that τ satisfies the condition in the second column of the lists, where we abbreviate s_i to i. In the next two columns, a word $c_1 c_2 \cdots c_r$ (where $r = |S|$) signifies $\sum_{i=1}^{r} c_i \alpha_i \in V$ and $\widetilde{\alpha}_i$ denotes the unique highest root of the finite Coxeter group $W_{S \smallsetminus \{s_i\}}$. Finally, the last column gives a basis of V' or of $\mathrm{Ann}(V')$. $\square$

Now we cancel the assumption $|S| < \infty$ placed above. To prove Theorem 5.1, note that if $\tau \in \mathrm{Aut}\,\Gamma$ leaves $W_I \subseteq W$ invariant, then W_I possesses its own fixed-point subgroup ${W_I}^{\tau}$ which coincides with $W^{\tau} \cap W_I$.

Proof of Theorem 5.1. The only nontrivial part is the "only if" part, so we prove it. Note that, by (2.2), the hypothesis implies that

$$[G : W^{\tau} \cap G] < \infty \text{ for any subgroup } G \leq W, \tag{5.1}$$

so $W^{\tau} \cap G \neq 1$ for every infinite subgroup G of W.

Step 1: if $I \subseteq S$ is finite, and W_I is infinite and irreducible, then $\tau(W_I) = W_I$.

Assume contrary that $\tau(I) \neq I$, or equivalently $I \not\subseteq \tau(I)$. Then we have $I \cap \tau(I) \neq I$, while $W^{\tau} \cap W_I \subseteq W_I \cap W_{\tau(I)} = W_{I \cap \tau(I)}$ (see (2.4)), therefore no essential element in W_I lies in W^{τ}. Hence by Theorem 4.1, any power w^k (with $k \neq 0$) of a Coxeter element w of W_I has infinite order and is not in W^{τ}, so we have $W^\tau \cap \langle w \rangle = 1$, contradicting (5.1).

Step 2: the claim holds if W has finite rank.

Now it suffices to show that τ is identity on every infinite irreducible component W_I. Moreover, since (by Step 1) $\tau(W_I) = W_I$ and (by (5.1)) $[W_I : {W_I}^\tau] < \infty$, it actually suffices to consider the case $W_I = W$, namely W itself is infinite and irreducible. In this case, our aim is to show that τ is identity.

First, we consider the case that W is not of type $\widetilde{A_1}$ and every proper $W_J \subset W$ is infinite. Then by combining Proposition 2.13 (2) and Lemma 5.3, we have $\langle w \rangle \cap \langle \tau(w) \rangle = 1$ for some $w \in W$ of infinite order whenever $\tau \neq \mathrm{id}_S$. This implies that $W^\tau \cap \langle w \rangle = 1$, contradicting (5.1). Thus τ must be identity now, as desired. On the other hand, the claim also holds if type $W = \widetilde{A_1}$, since now we have $W^\tau = 1$ whenever $\tau \neq \mathrm{id}_S$.

Finally, we consider the remaining case that a proper $W_J \subset W$ is infinite. We may assume that $J = S \smallsetminus \{s\}$ for some $s \in S$, so it suffices to show that $\tau|_J = \mathrm{id}_J$. Since $|S| < \infty$, we may assume further that W_J is irreducible: indeed, if W_J is not irreducible and W_K is an infinite irreducible component of W_J (which exists since $|J| < \infty$), then the set $S \smallsetminus \{s'\}$, where s' is an element of $J \smallsetminus K$ farthest from s in Γ, possesses the desired properties. Now Step 1 implies that $\tau(J) = J$, so $[W_J : {W_J}^\tau] < \infty$ (by (5.1)), therefore the induction on $|S|$ shows that $\tau|_J = \mathrm{id}_J$, as desired.

Step 3: if $I \in \tau \backslash S$, then $|I| < \infty$.

Assume contrary that $|I| = \infty$. Then for any $w \in W_I$ with $J = \mathrm{supp}(w)$ (finite and) nonempty, we have $J \neq I$ and so $J \neq \tau(J)$ (since I is a $\langle \tau \rangle$-orbit), therefore $J \not\subseteq \tau(J)$ and $w \notin W_{\tau(J)}$. This means that $\tau(w) \neq w$. Thus we have $W^\tau \cap W_I = 1$, contradicting (5.1).

Step 4: τ is identity on every infinite irreducible component W_I.

First, we consider the case that a (not necessarily proper) $W_J \subseteq W_I$ of finite rank is infinite. We can take an irreducible W_J. Now assume contrary that τ is not identity on W_I, so $\tau(s) \neq s$ for some $s \in I$. Then, since W_I is irreducible and $|J| < \infty$, an irreducible $W_K \subseteq W_I$ of finite rank contains both W_J and s. This W_K is also infinite, so $\tau(K) = K$ (Step 1), therefore $[W_K : {W_K}^\tau] < \infty$ by (5.1). Now Step 2 implies that τ is identity on W_K, contradicting the choice of s. Hence the claim holds in this case.

In the remaining case, W_I is of type A_∞, $A_{\pm\infty}$, B_∞ or D_∞ (see Figure 3) as mentioned in Section 2.2. Note that $\tau(W_I) = W_I$, since otherwise we have $W^\tau \cap W_I = 1$, contradicting (5.1). Now the claim is trivial in the first and the third cases where $\mathrm{Aut}\,\Gamma = \{\mathrm{id}_S\}$.

In the case type $W_I = A_{\pm\infty}$, if τ is not identity on W_I, then Step 3 implies that τ is a turning of the infinite path Γ_I, so there is an infinite $J \subset I$ with $J \cap \tau(J) = \emptyset$. Now we have $W^\tau \cap W_J = 1$, contradicting (5.1). This verifies the claim.

Finally, in the case type $W_I = D_\infty$, if τ is not identity on W_I, then $\tau(s_1) = s_2$, $\tau(s_2) = s_1$ and τ fixes $J = I \smallsetminus \{s_1, s_2\}$ pointwise. Put $K = J \cup \{s_2\}$. Since any $w \in W_K$ satisfies that $\tau(w) \in W_{J \cup \{s_1\}}$, we have $W^\tau \cap W_K = W_J$ (see (2.4)), so $[W_K : W_J] < \infty$ by (5.1). However, putting $\gamma_k = \sum_{i=2}^{k} \alpha_{s_i} \in \Phi_K^+$ for $k \geq 3$, Lemma 2.11 implies that all of the infinitely many reflections s_{γ_k} belong to distinct cosets in W_K / W_J. This contradiction yields the claim.

Step 5: conclusion.

Assume that the "only if" part fails. Then by Step 4, W possesses infinitely many finite irreducible components $W_{I_1}, W_{I_2}, \dots$ on which τ is not identity. Since every $\langle\tau\rangle$-orbit is finite (Step 3), there is an infinite sequence $s_1, s_2, \dots$ of distinct elements of S such that $J = \{s_i \mid i \geq 1\}$ satisfies that $\tau(J) \cap J = \emptyset$; take s_1 as any element of I_1 with $\tau(s_1) \neq s_1$, and if $s_1, \dots, s_k$ are already chosen, then take $s_{k+1} \in I_i$ where I_i does not intersect with the $\langle\tau\rangle$-orbits of the preceding s_j and $\tau(s_{k+1}) \neq s_{k+1}$. Now we have $W^\tau \cap W_J = 1$ and $|W_J| = \infty$, contradicting (5.1). Hence the proof is concluded. $\square$

5.2 Proof of Theorem 5.2

We start with some preliminaries. Let $\tau \in \operatorname{Aut}\Gamma$ and $w \in W^\tau$, and denote the support of w as an element of $(W^\tau, S(W^\tau))$ by $\operatorname{supp}^\tau(w)$. The following (part of a) result of [12] shows a relation between $\operatorname{supp}(w)$ and $\operatorname{supp}^\tau(w)$.

Proposition 5.8 (See [12, Proposition 3.3]). *Let $w = w_0(I_1) \cdots w_0(I_r)$ (where $w_0(I_i) \in S(W^\tau)$) be a reduced expression of $w \in W^\tau$ with respect to $S(W^\tau)$. Then any expression of w obtained by replacing each $w_0(I_i)$ with its reduced expression, with respect to S, is also reduced with respect to S. Hence* $\operatorname{supp}(w) = \bigcup_{i=1}^{r} I_i$.

Secondly, we give a remark on the Coxeter graph of the Coxeter system $(W^\tau, S(W^\tau))$, denoted here by Γ^τ. Let $\tau\backslash\Gamma$ be the graph with vertex set $\tau\backslash S$, in which two orbits $I, J \in \tau\backslash S$ are joined if and only if these sets are adjacent in Γ. Then the vertex set $S(W^\tau)$ of Γ^τ is regarded as a subset of the vertex set $\tau\backslash S$ of $\tau\backslash\Gamma$ via an embedding $w_0(I) \mapsto I$. Now we have the following result on a relation between Γ^τ and $\tau\backslash\Gamma$.

Lemma 5.9. *Under the embedding $S(W^\tau) \hookrightarrow \tau\backslash S$ of the vertex set, the underlying graph of Γ^τ is a full subgraph of $\tau\backslash\Gamma$.*

Proof. Let $I, J \in \tau\backslash S$ be two distinct orbits with both W_I and W_J finite. It is obvious that I and J are not adjacent in Γ^τ (i.e. $w_0(I)$ and $w_0(J)$ commute) if these are not adjacent in $\tau\backslash S$. Thus our remaining task is to show that $w_0(I)$ and $w_0(J)$ do not commute if I and J are adjacent in $\tau\backslash S$, namely some $s \in I$ is adjacent to J in Γ. Now Lemma 2.7 (1) implies that $w_0(J) \cdot \alpha_s \in \Phi^+_{J \cup \{s\}} \smallsetminus \Phi_{\{s\}}$, so $w_0(I)w_0(J) \cdot \alpha_s \in \Phi^+$. On the other hand, we have $w_0(I) \cdot \alpha_s \in \Phi^-_I$, so $w_0(J)w_0(I) \cdot \alpha_s \in \Phi^-$. Thus we have $w_0(I)w_0(J) \neq w_0(J)w_0(I)$ as desired. $\square$

Moreover, note that $\tau\backslash\Gamma$ is connected whenever Γ is. Indeed, for any $I, J \in \tau\backslash S$, a path in the connected graph Γ between any $s \in I$ and any $t \in J$ gives rise to a path in $\tau\backslash\Gamma$ between I and J.

Proof of Theorem 5.2 (1). As is remarked above, the irreducibility of W yields the connectedness of $\tau\backslash\Gamma$, while the hypothesis implies that the embedding $\Gamma^\tau \hookrightarrow \tau\backslash\Gamma$ in Lemma 5.9 is now an isomorphism. Thus Γ^τ is connected as desired.

For the infiniteness of W^τ, assume the contrary. Then W^τ possesses the longest element w_0^τ with respect to $S(W^\tau)$. Now for any $s \in S$, belonging by the hypothesis to an $I \in \tau\backslash S$ with $|W_I| < \infty$, the w_0^τ and $w_0(I)$ admit a reduced expression with respect to $S(W^\tau)$ and S ending with $w_0(I)$ and s, respectively, by Exchange Condition. Thus Proposition 5.8 implies that w_0^τ admits a reduced expression with respect to S ending with s. Since $s \in S$ is arbitrary, this means that W is finite and w_0^τ is the longest element of W (see Section 2.2), contradicting the hypothesis that W is infinite. Hence the claim follows. $\square$

Proof of Theorem 5.2 (2). Note that the graph $\tau\backslash\Gamma$ is connected. Since the hypothesis of (1) now fails, there is a path $I_0 I_1 \cdots I_r$ in $\tau\backslash\Gamma$, where $I_i \in \tau\backslash S$, such that $w_0(I_0) \in \mathrm{supp}^\tau(w)$ and $|W_{I_r}| = \infty$. By choosing the shortest possible path, we may assume that $|W_{I_i}| < \infty$ and $w_0(I_i) \notin \mathrm{supp}^\tau(w)$ for $1 \leq i \leq r-1$. Now Lemma 5.9 says that $I_0 I_1 \cdots I_{r-1}$ is also a path in Γ^τ, so by applying Lemma 2.7 (2) to the Coxeter system $(W^\tau, S(W^\tau))$, it is deduced that $w_0(I_{r-1}) \in \mathrm{supp}^\tau(w'ww'^{-1})$ where $w' = w_0(I_{r-1}) \cdots w_0(I_2) w_0(I_1) \in W^\tau$. Thus Proposition 5.8 implies that $\mathrm{supp}(w'ww'^{-1}) \subseteq S$ contains I_{r-1}, does not intersect I_r and is adjacent to I_r in Γ.

Take $s \in I_r$ adjacent to $\mathrm{supp}(w'ww'^{-1})$ in Γ. Now we show that, if W_{I_r} possesses an element u' of infinite order such that $s \in \mathrm{supp}(u'^k)$ for all $0 \neq k \in \mathbb{Z}$, then $u = w'^{-1}\tau^{-1}(u')w'$ is the desired element. Indeed, for $k \neq 0$, we have $\tau^{-1}(u')^k w'ww'^{-1}u'^{-k} \neq w'ww'^{-1}$ by the choice of u' and Lemma 2.7 (3) (note that $\tau^{-1}(u') \in W_{I_r}$), so, since $\tau(w') = w'$, we have

$$u^k w \tau(u)^{-k} = w'^{-1}\big(\tau^{-1}(u')^k w'ww'^{-1}u'^{-k}\big)w' \neq w.$$

Finally, we show the existence of such an element u', concluding the proof. Since $|W_{I_r}| = \infty$ and $I_r \in \tau\backslash S$ is a finite orbit, an irreducible component of W_{I_r}, therefore that containing s, is infinite. Now Theorem 4.1 implies that a Coxeter element of this component possesses the desired property. $\square$

6 Proof of the main theorem

This section is devoted to the proof of Theorem 3.1. First, note that the factor $W(\mathcal{O}_\rho)$ in the statement is $\rho(G)$-invariant, so the product $W(\mathcal{O}_\rho)\langle G_\rho\rangle$ of two subgroups of $W \rtimes G$ is indeed the semidirect product $W(\mathcal{O}_\rho) \rtimes \langle G_\rho\rangle$. This implies that, since $W(\mathcal{O}_\rho)$ is generated by involutions, the claim (2) follows immediately from (1). So we prove (1) below.

For the "only if" part, we assume that $wg \in (W \rtimes G)_{\mathrm{ACI}}$ and prove that $w \in W(\mathcal{O}_\rho)$ and $g \in G_\rho \cup \{1\}$. Now by (2.2) and Corollary 2.3 (2), we have

$$[H : Z_H(w'g')] < \infty \text{ for any } H \leq W \rtimes G \text{ and } w'g' \in \langle wg\rangle_{\lhd W \rtimes G}. \tag{6.1}$$

Note that $\rho_g(w) = w^{-1}$ and $g^2 = 1$ since $1 = (wg)^2 = w\rho_g(w) \cdot g^2$. We divide the proof into the following five steps.

Step 1: ρ_g maps each $W_I \in \mathcal{C}_W^{\mathrm{inf}}$ onto itself.

Assume contrary that ρ_g maps W_I onto an irreducible component other than W_I. Let $\pi : W \twoheadrightarrow W_I$ be the projection. Take $s \in I$ and put $a = swgs(wg)^{-1} \in \langle wg\rangle_{\lhd W \rtimes G}$. Then we have $a = sw\rho_g(s)w^{-1} \in W$, so $Z_{W_I}(a) = Z_{W_I}(\pi(a))$. Thus (6.1) implies that $\pi(a) \in W_I$ is almost central in W_I. However, the first assumption yields that $\pi(\rho_g(s)) = 1$, so $\pi(a) = s\pi(w)1\pi(w)^{-1} = s$, which is not almost central in W_I by Proposition 2.15. This is a contradiction.

Step 2: ρ_g is identity on every $W_I \in \mathcal{C}_W^{\mathrm{inf}}$.

Assume that the claim fails for W_I. Note that $\rho_g(W_I) = W_I$ by Step 1. Let $\pi : W \twoheadrightarrow W_I$ be the projection. Then we may assume without loss of generality that $\ell(\pi(w)) \leq \ell(\pi(uw\rho_g(u)^{-1}))$ for all $u \in W_I$; if this inequality fails, replace wg with another involution $u(wg)u^{-1} = uw\rho_g(u)^{-1} \cdot g$ in $\langle wg \rangle_{\lhd W \rtimes G}$, which is also almost central in $W \rtimes G$ by (6.1), and use the induction on $\ell(\pi(w))$.

Put $\tau = \rho_g|_I \in \operatorname{Aut}\Gamma_I$, which is assumed to be non-identity. Now if $\pi(w) = 1$, then we have $Z_{W_I}(wg) = Z_{W_I}(g) = {W_I}^{\tau}$ and so $[W_I : {W_I}^{\tau}] < \infty$ by (6.1), contradicting Theorem 5.1. Thus $\pi(w) \neq 1$.

We show that $\pi(w)$ is an involution in ${W_I}^{\tau}$. Let $s_1 \cdots s_n$ (where $n \geq 1$ and $s_i \in I$) be an arbitrary reduced expression of $\pi(w) \in W_I$. Then, since $\rho_g(w) = w^{-1}$ and $\rho_g(W_I) = W_I$, we have

$$\pi(w) = \pi(\rho_g(w)^{-1}) = \rho_g(\pi(w)^{-1}) = \tau(\pi(w)^{-1}) = \tau(s_n) \cdots \tau(s_1),$$

so $\ell(\pi(w)\tau(s_1)) < \ell(\pi(w))$, therefore Exchange Condition shows that $\pi(w) = s_1 \cdots \widehat{s_i} \cdots s_n \tau(s_1)$ for an index i. Now if $i \geq 2$, then $\pi(s_1 w \tau(s_1)^{-1}) = s_2 \cdots \widehat{s_i} \cdots s_n$ is shorter than $\pi(w)$, contradicting the minimality of $\ell(\pi(w))$. Thus we have $i = 1$ and $\pi(w) = s_2 \cdots s_n \tau(s_1)$. Since the original reduced expression $s_1 \cdots s_n$ is arbitrary, we can apply this argument to the new expression of $\pi(w)$. Iterating, we have

$$\pi(w) = s_3 \cdots s_n \tau(s_1)\tau(s_2) = \cdots = s_n \tau(s_1) \cdots \tau(s_{n-1}) = \tau(s_1) \cdots \tau(s_n).$$

Since $\pi(w) = s_1 \cdots s_n = \tau(s_n) \cdots \tau(s_1)$, the claim of this paragraph follows.

Now if $m_{s,\tau(s)} = \infty$ for some $s \in I$, then since $\tau^2 = \mathrm{id}_I$, Theorem 5.2 (2) (applied to $\pi(w)$) gives us an element $u \in W_I$ of infinite order such that

$$\pi(u^k wgu^{-k}(wg)^{-1}) = u^k \pi(w) \tau(u)^{-k} \pi(w)^{-1} \neq 1 \text{ for all } k \neq 0.$$

This means that $Z_{\langle u \rangle}(wg) = 1$, so $[\langle u \rangle : Z_{\langle u \rangle}(wg)] = \infty$, contradicting (6.1). On the other hand, if $m_{s,\tau(s)} < \infty$ for all $s \in I$, then the Coxeter group ${W_I}^{\tau}$ is infinite and irreducible by Theorem 5.2 (1). Now we have

$$Z_{{W_I}^{\tau}}(\pi(w)) = Z_{{W_I}^{\tau}}(w) = Z_{{W_I}^{\tau}}(wg),$$

which has finite index in ${W_I}^{\tau}$ by (6.1). Thus the non-identity involution $\pi(w) \in {W_I}^{\tau}$ is almost central in ${W_I}^{\tau}$, contradicting Proposition 2.15. Hence Step 2 is concluded.

Step 3: $w \in W_{\mathrm{fin}}$.

We show that $\pi(w) = 1$ for any $W_I \in \mathcal{C}_W^{\mathrm{inf}}$ with projection $\pi : W \twoheadrightarrow W_I$. Since ρ_g is identity on W_I (Step 2), we have $Z_{W_I}(wg) = Z_{W_I}(w) = Z_{W_I}(\pi(w))$ and so (by (6.1)) $\pi(w)$ is almost central in W_I. Now since $1 = \pi(w\rho_g(w)) = \pi(w)\rho_g(\pi(w)) = \pi(w)^2$, the claim follows from Proposition 2.15.

Step 4: $w \in W(\mathcal{O}_\rho)$.

Assume the contrary. Then there exist a $\rho^{\dagger}(G)$-orbit $\mathcal{O} \subseteq \mathcal{C}_W^{\mathrm{fin}}$ with infinite cardinality and $W_I \in \mathcal{O}$ (with projection $\pi_I : W \twoheadrightarrow W_I$) such that $\pi_I(w) \neq 1$. Fix the $\mathcal{O}$, and let $\mathcal{O}_0$ be the set of all such $W_I \in \mathcal{O}$, so $|\mathcal{O}_0| < \infty$.

We show that $\rho_h^{\dagger}(\mathcal{O}_0) = \mathcal{O}_0$, or equivalently $\rho_h^{\dagger}(\mathcal{O}_0) \subseteq \rho_h^{\dagger}(\mathcal{O}_0)$, for any $h \in Z_G(wg)$. Note that $\rho_h(w) = w$ since $wgh = hwg = \rho_h(w)hg$. Now if

$W_I \in \mathcal{O}_0$ and $\rho_h^\dagger(W_I) = W_J \notin \mathcal{O}_0$, then $\pi_J(\rho_h(w)) = \rho_h(\pi_I(w)) \neq 1$ and $\pi_J(w) = 1$, contradicting $\rho_h(w) = w$. Thus $\rho_h^\dagger(\mathcal{O}_0) \subseteq \mathcal{O}_0$ as desired.

Since $\mathcal{O}$ is an infinite $\rho^\dagger(G)$-orbit, we can choose infinitely many finite subsets $\mathcal{O}_1, \mathcal{O}_2, \dots$ of $\mathcal{O}$, irreducible components $W_{I_0}, W_{I_1}, W_{I_2}, \dots \in \mathcal{O}$ and elements $g_0, g_1, g_2, \dots \in G$ inductively, where we start with arbitrary $W_{I_0} \in \mathcal{O}_0$ and $g_0 = 1$, subject to the conditions $W_{I_k} \notin \bigcup_{i=0}^{k-1} \mathcal{O}_i$, $\rho_{g_k}^\dagger(W_{I_0}) = W_{I_k}$ and $\mathcal{O}_k = \rho_{g_k}^\dagger(\mathcal{O}_0) \ni W_{I_k}$ for all $k \geq 1$. Now if $i < j$ and $h \in Z_G(wg)$, then the previous paragraph implies that $\rho_{g_i h}^\dagger(W_{I_0}) = \rho_{g_i}^\dagger \rho_h^\dagger(W_{I_0}) \in \rho_{g_i}^\dagger(\mathcal{O}_0) = \mathcal{O}_i$, while $\rho_{g_j}^\dagger(W_{I_0}) = W_{I_j} \notin \mathcal{O}_i$, so we have $g_j \neq g_i h$. Thus all the g_i belong to distinct cosets in $G/Z_G(wg)$, while $[G : Z_G(wg)] < \infty$ by (6.1). This contradiction yields the claim.

Step 5: $g \in G_\rho \cup \{1\}$.

Note that $g^2 = 1$. Since $hwg = \rho_h(w) \cdot hg$ for $h \in G$, we have $Z_G(wg) \subseteq Z_G(g)$ and so g is almost central in G by (6.1). From now, we check (3.1).

By Step 4, the union $\mathcal{O}$ of a finite number of some $\rho^\dagger(G)$-orbits with finite cardinalities satisfies that $w \in W(\mathcal{O})$. Since $|\mathcal{O}| < \infty$, it suffices to show that ρ_g is identity on all $W_I \in \mathcal{O}' = \mathcal{C}_W \smallsetminus \mathcal{O}$ except a finite number of finite irreducible components. Now $\mathcal{O}'$ is $\rho_g^\dagger$-invariant as well as its complement $\mathcal{O}$, while $W(\mathcal{O}') \subseteq Z_W(w)$, so $Z_{W(\mathcal{O}')}(wg) = Z_{W(\mathcal{O}')}(g)$ is the fixed-point subgroup $W(\mathcal{O}')^\tau$ (where $\tau = \rho_g|_{W(\mathcal{O}')}$). Since $Z_{W(\mathcal{O}')}(wg)$ has finite index in $W(\mathcal{O}')$ by (6.1), the claim follows from Theorem 5.1.

Hence the "only if" part has been proved. From now, we prove the other part; so we assume that $w \in W(\mathcal{O}_\rho)$, $g \in G_\rho \cup \{1\}$ and wg is an involution, and prove that wg is almost central in $W \rtimes G$. By the choice of w, there are a finite number of finite $\rho^\dagger(G)$-orbits in $\mathcal{C}_W^{\mathrm{fin}}$ such that their union $\mathcal{O}$ satisfies that $w \in W(\mathcal{O})$. Now note that

$$Z_{W \rtimes G}(wg) \supseteq Z_{W \rtimes G}(w) \cap Z_{W \rtimes G}(g) \supseteq (Z_W(w) Z_G(w)) \cap (Z_W(g) Z_G(g)),$$

so it suffices to show that both $Z_W(w)Z_G(w)$ and $Z_W(g)Z_G(g)$ have finite index in $W \rtimes G$ (see (2.3)). Moreover, Lemma 2.4 reduces the claim to the following four claims:

Step 6: $Z_W(w)$ has finite index in W.

This follows since w lies in the finite direct factor $W(\mathcal{O})$ of W.

Step 7: $Z_G(w)$ has finite index in G.

Since $W(\mathcal{O})$ is finite and $\rho(G)$-invariant, the action gives rise to a homomorphism ρ' from G to the finite group $\operatorname{Aut} W(\mathcal{O})$. Now $\ker \rho'$ is contained in $Z_G(w)$ (since $w \in W(\mathcal{O})$) and has finite index in G, proving the claim.

Step 8: $Z_W(g)$ has finite index in W.

This is trivial if $g = 1$. If $g \in G_\rho$, then the property (3.1) and Theorem 5.1 imply that the fixed-point subgroup $W^{\rho_g} = Z_W(g)$ by ρ_g has finite index in W, as desired.

Step 9: $Z_G(g)$ has finite index in G.

This is obvious from the choice of g.

Hence the proof of Theorem 3.1 is concluded.

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