A note on Tychonoff's Theorem and Axiom of Choice

Koji Nuida

Published: January 22, 2012
Last Update: January 6, 2013

Abstract

The aim of this note is to give some observation on a standard proof to deduce Axiom of Choice from Tychonoff's Theorem.

In this note, we basically deal with the axioms ZF^- of set theory, which means the Zermelo–Fraenkel set theory ZF except the Axiom of Foundation

$$\forall x(\exists y(y \in x) \to \exists y(y \in x \land \exists z(z \in x \land z \in y))) \ .$$

In this note, we say that a class $\mathcal{K}$ of sets is *downward closed* if, for any set $A \in \mathcal{K}$ and any set B for which there exists an injective map $B \hookrightarrow A$, it follows that $B \in \mathcal{K}$. Intuitively, this means that $\mathcal{K}$ is a class of cardinal numbers with the property that $|X| \leq |Y| \in \mathcal{K}$ implies $|X| \in \mathcal{K}$. For example, the classes Set of all sets, Finite of all finite sets, and $\mathsf{Countable}$ of all countable sets (i.e., sets A for which $|A| \leq \aleph_0$) are downward closed classes of sets.

On the other hand, we say that a class $\mathcal{T}$ of topological spaces is a *topological property* if, for any $X \in \mathcal{T}$ and any topological space Y which is homeomorphic to X, it follows that $Y \in \mathcal{T}$. Namely, we identify a topological property (in usual sense) with the class of all topological spaces having the property. For example, the classes Top of all topological spaces, T_1 of all T_1-spaces, and $\mathsf{Hausdorff}$ of all Hausdorff spaces are topological properties. A member of a topological property $\mathcal{T}$ is said to be a *$\mathcal{T}$-space*.

In what follows, we assume that $\mathcal{K}$ is a downward closed class of sets and $\mathcal{T}$ is a topological property. We define the following propositions:

AC($\mathcal{K}$) Let $\mathcal{A}$ be a family of non-empty sets with $\mathcal{A} \in \mathcal{K}$. Then there exists a choice function for $\mathcal{A}$, i.e., a map $f\colon \mathcal{A} \to \bigcup \mathcal{A}$ satisfying that $f(A) \in A$ for every $A \in \mathcal{A}$.

ACEq($\mathcal{K}$) The same as AC($\mathcal{K}$), except that all members of $\mathcal{A}$ are supposed to have equal cardinality.

AMC Let $\mathcal{A}$ be a family of non-empty sets. Then there exists a "multiple choice function" for $\mathcal{A}$, i.e., a map $f\colon \mathcal{A} \to 2^{\bigcup \mathcal{A}}$ satisfying that for each $A \in \mathcal{A}$, $f(A)$ is a finite non-empty subset of A.

AMCEq The same as AMC, except that all members of $\mathcal{A}$ are supposed to have equal cardinality.

$\mathrm{T}(\mathcal{T},\mathcal{K})$ Let $\mathcal{A}$ be a family of compact $\mathcal{T}$-spaces. Then any open cover $\mathcal{W}$ of the product topological space $\prod \mathcal{A}$ has a subcover $\mathcal{W}'$ with $\mathcal{W}' \in \mathcal{K}$.

$\mathrm{THomeo}(\mathcal{T},\mathcal{K})$ The same as $\mathrm{T}(\mathcal{T},\mathcal{K})$, except that all members of $\mathcal{A}$ are supposed to be homeomorphic to each other.

For example, $\mathrm{AC}(\mathsf{Set})$ is the Axiom of Choice (AC), $\mathrm{AC}(\mathsf{Countable})$ is the Axiom of Countable Choice (ACC), AMC is the Axiom of Multiple Choice (AMC), and $\mathrm{T}(\mathsf{Top},\mathsf{Finite})$ is the Tychonoff's Theorem. Note that $\mathrm{AC}(\mathsf{Finite})$ is a theorem of ZF^- (so is $\mathrm{ACEq}(\mathsf{Finite})$); roughly speaking, a finite number of selections can be unconditionally done simultaneously. Note also that all the above propositions are consequences of AC in ZF^-, since Tychonoff's Theorem can be proven in $\mathrm{ZF}^- + \mathrm{AC}$ (see Appendix below).

Now we describe a "pattern" of a proof (in ZF^-) to deduce AC from "Tychonoff-like" axioms, which is a slight modification of the standard proof to deduce AC from the original Tychonoff's Theorem:

> Let $\mathcal{A} = (A_i)_{i\in\Lambda}$ be a family of non-empty sets. First, choose a set p which does not belong to any A_i (by Russell's Paradox, the union $\bigcup_{i\in\Lambda} A_i$ does not contain all sets). Now we assume the following:
>
> (*) There exists a map which associates to each $i \in \Lambda$ a topological structure on $X_i := A_i \cup \{p\}$ with the property that (I) each X_i becomes a compact $\mathcal{T}$-space, and (II) there exists a map which associates to each $i \in \Lambda$ an open neighborhood U_i of p in X_i with $U_i \neq X_i$.
>
> We introduce a topological structure on each X_i as above. Now for each $i \in \Lambda$, let $\widetilde{U}_i$ denote the direct product of U_i and all X_j for $j \in \Lambda \setminus \{i\}$. Then $\mathcal{W} := (\widetilde{U}_i)_{i\in\Lambda}$ is a family of open subsets of $X := \prod_{i\in\Lambda} X_i$.
>
> Assuming the proposition $\mathrm{AC}(\mathcal{K})$, it follows that $\mathcal{W}$ does not have a subfamily $\mathcal{W}' = (\widetilde{U}_i)_{i\in\Lambda'}$ with the property that $\Lambda' \in \mathcal{K}$ and $\mathcal{W}'$ is an open cover of X. Indeed, for such a subfamily $\mathcal{W}'$, $\mathrm{AC}(\mathcal{K})$ implies that there exists an element $g \in \prod_{i\in\Lambda'}(X_i \setminus U_i)$, and now the element $f \in X$ defined by $f(i) = g(i)$ for $i \in \Lambda'$ and $f(i) = p$ for $i \in \Lambda \setminus \Lambda'$ does not belong to any member of $\mathcal{W}'$, a contradiction.
>
> By the above argument, assuming the proposition $\mathrm{T}(\mathcal{T},\mathcal{K})$ further, it follows that $\mathcal{W}$ is not an open cover of X. Namely, there exists an element $f \in X$ that does not belong to any $\widetilde{U}_i$ with $i \in \Lambda$. This means that we have $f(i) \notin U_i$, hence $f(i) \neq p$, for each $i \in \Lambda$; therefore f is an element of $\prod_{i\in\Lambda} A_i$. Hence AC holds.

By this argument, the combination of $\mathrm{AC}(\mathcal{K})$, $\mathrm{T}(\mathcal{T},\mathcal{K})$ and a certain condition ensuring the property (*) (if necessary) implies AC in ZF^-. We consider some special cases:

- When $\mathcal{T} = \mathsf{Top}$, to ensure (*) it suffices to define the open sets of each X_i as $\emptyset$, X_i and $\{p\}$. Indeed, the condition (I) is satisfied, while the condition (II) is also satisfied by defining $U_i = \{p\}$. As a result, AC($\mathcal{K}$) and T($\mathsf{Top}, \mathcal{K}$) imply AC. In particular, T($\mathsf{Top}, \mathsf{Finite}$) (i.e., Tychonoff's Theorem) implies AC; and AC($\mathsf{Countable}$) (i.e., ACC) and T($\mathsf{Top}, \mathsf{Countable}$) ("the product of compact spaces is a Lindelöf space") also imply AC. Note also that this argument proves stronger results such as that Tychonoff's Theorem for topological spaces *with precisely three open sets* implies AC.

- When $\mathcal{T} = \mathsf{T}_1$, to ensure (*) it suffices to first introduce the cofinite topology on each A_i and then attach to A_i a point p as an isolated point. Indeed, the cofinite topology is a compact T_1 topology, therefore the condition (*) is satisfied by choosing $U_i = \{p\}$ for (II) again. As a result, AC($\mathcal{K}$) and T($\mathsf{T}_1, \mathcal{K}$) imply AC. In particular, T($\mathsf{T}_1, \mathsf{Finite}$) (i.e., Tychonoff's Theorem for T_1-spaces) implies AC; and AC($\mathsf{Countable}$) (i.e., ACC) and T($\mathsf{T}_1, \mathsf{Countable}$) ("the product of compact T_1-spaces is a Lindelöf space") also imply AC. We emphasize that this definition of topology on X_i is adopted for a proof of AC from Tychonoff's Theorem in several books, but in fact the argument shows a stronger property that Tychonoff's Theorem *for T_1-spaces* implies AC (and, as mentioned above, a simpler choice of trivial (or indiscrete) topology on A_i instead of cofinite topology is enough to prove that Tychonoff's Theorem implies AC).

- [**Added: January 6, 2013**] When $\mathcal{T} = \mathsf{T}_4$ (The class of all T_4-spaces), in the same way as the above case $\mathcal{T} = \mathsf{Top}$, it suffices to define the open sets of each X_i as $\emptyset$, X_i and $\{p\}$ (note that this X_i has no pairs of non-empty and disjoint closed subsets, therefore the condition for T_4-spaces is automatically satisfied). Here I would like to emphasize that, in contrast to the case $\mathcal{T} = \mathsf{Top}$ where the cofinite topology is also applicable in the same way as the case $\mathcal{T} = \mathsf{T}_1$, for the present case $\mathcal{T} = \mathsf{T}_4$, the cofinite topology is not applicable (since it is not T_4), which would clarify the significance of the discrete topology.

- On the other hand, when $\mathcal{T} = \mathsf{Hausdorff}$, a similar strategy to first introduce a compact Hausdorff topology on each A_i and then attach an isolated point p *cannot succeed*. Indeed, if it is possible, then Tychonoff's Theorem for Hausdorff spaces (i.e., T($\mathsf{Hausdorff}, \mathsf{Finite}$)) could imply AC, but this has been proven as impossible. For an alternative strategy, here we introduce the discrete topology on each A_i, and then define X_i to be the [**Revised: January 6, 2013**] Alexandroff extension [**End of revision**] of A_i. In this case, the open neighborhoods of p in X_i are complements in X_i of finite subsets of A_i. The problem is that there is yet no clue to choose a distinguished open neighborhood of p in each X_i (except X_i itself). Now we introduce an additional axiom AMC, which enables us to choose a distinguished finite non-empty subset B_i of each A_i, hence an open neighborhood $U_i = X_i \setminus B_i$ of p in each X_i, as desired. (Note that AMC is known to be strictly weaker than AC in ZF$^-$.) As a result, the combination of AMC, AC($\mathcal{K}$) and T($\mathsf{Hausdorff}, \mathcal{K}$) implies AC in ZF$^-$. In particular, AMC and Tychonoff's Theorem for Hausdorff spaces imply AC; and AMC, ACC and "the product of compact Hausdorff spaces is a Lindelöf space" also imply AC.

Moreover, we consider further relaxation of the assumptions in the above argument. The key fact is the following:

Lemma 1. *Let $\mathcal{A}$ be a family of non-empty sets. Then there exists a non-empty set B satisfying that the sets $A \times B$ for $A \in \mathcal{A}$ have equal cardinality.*

Proof. We define $B := (\bigcup \mathcal{A})^{<\omega}$ (ω denoting the least infinite ordinal number). Let $A \in \mathcal{A}$. Then for each element $\xi = (a, (x_0, x_1, \dots, x_n))$ of $A \times B$, define $f(\xi) = (x_0, x_1, \dots, x_n, a) \in B$. We show that $f\colon A \times B \to B$ is injective. If $f((a, (x_0, \dots, x_n))) = f((a', (x'_0, \dots, x'_m)))$, then we have $(x_0, \dots, x_n, a) = (x'_0, \dots, x'_m, a')$, therefore $n = m$, $a = a'$ and $x_i = x'_i$ for every $0 \leq i \leq n$. Hence f is injective, therefore $|A \times B| \leq |B|$, while obviously $|B| \leq |A \times B|$ (since A is non-empty). Now Cantor–Bernstein–Schroeder Theorem implies that $|A \times B| = |B|$ for every $A \in \mathcal{A}$. □

By using Lemma 1, we modify the above pattern of a proof to deduce AC in the following manner:

1. For a family $\mathcal{A} = (A_i)_{i \in \Lambda}$ of non-empty sets, first choose a non-empty set B with the property that the sets $A'_i := A_i \times B \neq \emptyset$ for $i \in \Lambda$ have equal cardinality (by using Lemma 1).

2. Secondly, construct compact $\mathcal{T}$-spaces $X_i = A'_i \cup \{p\}$ and open neighborhoods $U_i \subsetneq X_i$ of p in the same way as (*), with an additional requirement that the X_i for $i \in \Lambda$ are homeomorphic to each other.

3. By assuming AC($\mathcal{K}$) or some weakened variant, prove that $\mathcal{W} = (\widetilde{U}_i)_{i \in \Lambda}$ does not have a subfamily $\mathcal{W}' = (\widetilde{U}_i)_{i \in \Lambda'}$ with the property that $\Lambda' \in \mathcal{K}$ and $\mathcal{W}'$ is an open cover of X.

4. Finally, by assuming THomeo($\mathcal{T}, \mathcal{K}$), deduce that $\mathcal{W}$ is not an open cover of X, yielding an element f of $\prod_{i \in \Lambda} A'_i$. Then we obtain an element of $\prod_{i \in \Lambda} A_i$ by taking the first component of each $f(i)$, $i \in \Lambda$.

In the special cases that $\mathcal{T} = \mathsf{Top}$ and $\mathcal{T} = \mathsf{T}_1$ discussed above, the definitions of topology on each X_i satisfy that, for each $i, j \in \Lambda$, the extension of a bijection $A'_i \to A'_j$ to a map $X_i \to X_j$ defined by $p \mapsto p$ gives a homeomorphism $X_i \to X_j$. Moreover, since now $U_i = \{p\}$, the components of the direct product $\prod_{i \in \Lambda'} (X_i \setminus U_i) = \prod_{i \in \Lambda'} A'_i$ have equal cardinality. Hence, for the choice of $\mathcal{T}$, the combination of weakened propositions ACEq($\mathcal{K}$) and THomeo($\mathcal{T}, \mathcal{K}$) also implies AC. In particular, Tychonoff's Theorem for *homeomorphic* T_1 spaces implies AC.

In contrast, for the other special case that $\mathcal{T} = \mathsf{Hausdorff}$, the modified argument proves that AMCEq, AC($\mathcal{K}$) and THomeo($\mathsf{Hausdorff}, \mathcal{K}$) imply AC, but *not* that AMCEq, ACEq($\mathcal{K}$) and THomeo($\mathsf{Hausdorff}, \mathcal{K}$) imply AC. This is because the finite subsets B_i of the sets A'_i obtained by applying AMCEq are not necessarily of the same size, therefore the direct product $\prod_{i \in \Lambda'} (X_i \setminus U_i)$ in the argument does not necessarily satisfy the hypothesis of ACEq($\mathcal{K}$). If AMCEq is strengthened in such a way that each component of the direct product will have a finite non-empty distinguished subset *of equal size*, then the strengthened variant of AMCEq and two axioms ACEq($\mathcal{K}$) and THomeo($\mathsf{Hausdorff}, \mathcal{K}$) imply AC. I do not know whether or not the combination of AMCEq, ACEq($\mathcal{K}$) and THomeo($\mathsf{Hausdorff}, \mathcal{K}$) can imply AC in ZF$^-$.

Appendix: Proof of Tychonoff's Theorem from Axiom of Choice

In this appendix, we show one of the standard proofs of Tychonoff's Theorem from Axiom of Choice, for the sake of clarifying that the proof indeed works in ZF$^-$. The proof is taken from Section 16 of [1].

First, we notice the following equivalent form of Axiom of Choice, called Tukey's Lemma. We prepare a terminology. We say that a family $\mathcal{F}$ of sets is of *finite character* if, for any set A, we have $A \in \mathcal{F}$ if and only if every finite subset of A belongs to $\mathcal{F}$. Then the following fact is known:

Theorem 1 (see e.g., [2, Exercise 11 in Chapter I])**.** *In* ZF$^-$*,* AC *is equivalent to the following proposition (*Tukey's Lemma*): For any family $\mathcal{F}$ of finite character and any $A \in \mathcal{F}$, there exists a maximal member (with respect to inclusion) of $\mathcal{F}$ containing A.*

We start the proof of Tychonoff's Theorem. Let $X = \prod_{i \in \Lambda} X_i$ be the product space of compact topological spaces X_i. It suffices to show that, for any family $\mathcal{F}$ of subsets of X having finite intersection property, the intersection of the family $\overline{\mathcal{F}} := \{\overline{A} \mid A \in \mathcal{F}\}$ is non-empty (where $\overline{A}$ denotes the closure of A). First, note that the collection of the families $\mathcal{F}$ satisfying the above condition is of finite character, therefore by using Tukey's Lemma, there exists a maximal family subject to this condition that contains a given family. Hence we may assume without loss of generality that a given family $\mathcal{F}$ itself is maximal.

For each $i \in \Lambda$, let $\mathcal{F}_i$ be the collection of the closures $\overline{\pi_i[A]}$ in X_i of the images $\pi_i[A]$ of all $A \in \mathcal{F}$ by the projection $\pi_i \colon X \to X_i$. Since $\mathcal{F}$ has finite intersection property, $\mathcal{F}_i$ also has finite intersection property (note that $\pi_i[\bigcap_{k=1}^n A_k] \subset \bigcap_{k=1}^n \pi_i[A_k] \subset \bigcap_{k=1}^n \overline{\pi_i[A_k]}$ for a finite number of $A_k \in \mathcal{F}$). Since X_i is compact, we have $\bigcap \mathcal{F}_i \neq \emptyset$; choose (by using AC) an element $p_i \in \bigcap \mathcal{F}_i$ for each $i \in \Lambda$. We show that the element $p = (p_i)_{i \in \Lambda}$ of X belongs to $\bigcap \overline{\mathcal{F}}$. This is equivalent to that each open neighborhood U of p in X intersects with every $A \in \mathcal{F}$. Moreover, it suffices to prove the claim for the case that U belongs to the open basis of X, namely there exist a finite subset Λ' of Λ and an open neighborhood U_i of p_i in X_i for each $i \in \Lambda'$ with the property that U is the direct product of the U_i for $i \in \Lambda'$ and X_i for $i \in \Lambda \setminus \Lambda'$.

For each $i \in \Lambda'$, let W_i denote the direct product of U_i and all X_j for $j \in \Lambda \setminus \{i\}$. Then we have $U = \bigcap_{i \in \Lambda'} W_i$. Now for each $A \in \mathcal{F}$, we have $p_i \in U_i \cap \overline{\pi_i[A]}$ by the choice of p, therefore $U_i \cap \pi_i[A] \neq \emptyset$. By the definition of W_i, this implies that $W_i \cap A \neq \emptyset$ for every $A \in \mathcal{F}$. Now, since $\mathcal{F}$ is maximal, we have the following properties:

- For a finite number of members A_k of $\mathcal{F}$, we have $\bigcap_k A_k \in \mathcal{F}$; this is because $\mathcal{F} \cup \{\bigcap_k A_k\}$ has finite intersection property as well as $\mathcal{F}$.
- For each $i \in \Lambda'$, $\mathcal{F} \cup \{W_i\}$ has finite intersection property; this is because, for a finite number of members A_k of $\mathcal{F}$, we have $\bigcap_k A_k \in \mathcal{F}$ by the above argument, therefore $W_i \cap \bigcap_k A_k \neq \emptyset$ as shown above.

Hence, since $\mathcal{F}$ is maximal, it follows that $W_i \in \mathcal{F}$ for every $i \in \Lambda'$. Now for each $A \in \mathcal{F}$, we have $\emptyset \neq A \cap \bigcap_{i \in \Lambda'} W_i = A \cap U$ by the finite intersection property of $\mathcal{F}$. Hence U intersects with every $A \in \mathcal{F}$, as desired. This completes the proof.

References

[1] H. Tanaka, "Axiom of Choice and Mathematics," Enlarged Edition (in Japanese), Yuseisha, 1999.

[2] K. Kunen, "SET THEORY, An introduction to independence proofs," Elsevier, 1980.