Locally Finite Continuations and Coxeter Groups of Infinite Ranks

Bernhard Mühlherr[1] and Koji Nuida[23]

[1] Mathematisches Institut, Justus-Liebig-Universität Gießen, Germany
Bernhard.Muehlherr@math.uni-giessen.de
[2] Graduate School of Information Science and Technology,
The University of Tokyo, Japan
nuida@mist.i.u-tokyo.ac.jp
[3] Information Technology Research Institute, National Institute of
Advanced Industrial Science and Technology, Japan

September 12, 2019

Abstract

An involution r in a Coxeter group W is called an intrinsic reflection of W if $r \in S^W$ for each Coxeter generating set S of W. In recent joint work with R. B. Howlett [13] we determined all intrinsic reflections in finitely generated Coxeter groups. In the present paper we extend this result to the infinite rank case. An important tool in [13] is the notion of the finite contiuation of an involution that is only meaningful for finitely generated Coxeter groups. Here we introduce the locally finite continuation for any subset of an arbitrary group which enables us to deal with Coxeter groups of infinite rank. We apply our result to show that certain classes of Coxeter groups are reflection independent and we investigate rigidity of 2-spherical Coxeter systems of arbitrary ranks.

Keywords: Coxeter groups; isomorphism problem; intrinsic reflections

MSC number: 20F55 (primary)

1 Introduction

Let (W, S) be a Coxeter system. Then the rank of (W, S) is the cardinality of S and elements of the form $s^w := w^{-1}sw$, where $w \in W$ and $s \in S$, are called reflections of (W, S). A subset R of W is called a Coxeter generating set of W if (W, R) is a Coxeter system. An involution $t \in W$ is called an *intrinsic reflection* of W if it is a reflection of (W, R) for every Coxeter generating set R of W. In this paper we provide a complete solution to the following problem.

Problem: Let (W, S) be a Coxeter system and let $s \in S$. Give, in terms of the Coxeter graph of (W, S), conditions that are necessary and sufficient to ensure that s is an intrinsic reflection of W.

In [13] a complete solution of this problem has been provided in the case where (W, S) has finite rank in order to provide the final step to the characterization of strongly rigid Coxeter systems of finite rank. A Coxeter system (W, S) is called strongly rigid if any Coxeter generating set of W is conjugate to S in W. An important class in the context of strong rigidity is the class of 2-spherical Coxeter systems. A Coxeter system (W, S) is called 2-spherical if st has finite order for all s and t in S, and a 2-spherical Coxeter system is called strongly 2-spherical, if it is irreducible and non-spherical. In [11] and [5] (see also [6]) it was shown that each strongly 2-spherical Coxeter system of finite rank is strongly rigid. If the assumption of finite rank is dropped this is no longer true. A most interesting counterexample is provided by the Coxeter group of finitary permutations on a countable set. This group admits two Coxeter generating sets that even have non-isomorphic Coxeter graphs, namely the graphs A_∞ and $A_{\pm\infty}$ (see Figure 1 in Section 4.1 below for their definitions). Our solution to the problem above will enable us to show, that this is actually the only strongly 2-spherical Coxeter system (W,S) admitting a Coxeter generating set R such that the Coxeter graphs of (W,S) and (W,R) are not isomorphic. Here is the precise statement of our result.

Rigidity-Theorem: Let (W,S) be a strongly 2-spherical Coxeter system (of arbitrary rank) and let R be a Coxeter generating set of W. Then the following hold.

(a) We have $S^W = R^W$; in particular, (W,S) is reflection independent (in the sense of [1]).

(b) If (W,S) is not of type A_∞ nor $A_{\pm\infty}$, then S and R are locally conjugate. This means that there exists a bijection $\alpha\colon S \to R$ such that there exists for each finite subset K of S an element $c_K \in W$ with $\alpha(t) = t^{c_K}$ for all $t \in K$; in particular, (W,S) is rigid (in the sense of [3]).

(c) If there exists a finite subset J of S such that $\langle J\rangle$ is an infinite group and such that $[s,J] \neq 1_W$ for all $s \in S$, then $R = S^w$ for some $w \in W$; in particular (W,S) is strongly rigid.

Assertion (a) of the Rigidity-Theorem is Proposition 14.2 and the proofs of Assertions (b) and (c) are given at the end of the final section. Note that it is not hard to construct examples of strongly 2-spherical Coxeter systems (W,S) having a Coxeter generating set R that is not conjugate, but only locally conjugate to S in W. This happens for instance in Coxeter systems whose Coxeter graph is an infinite tree.

We now come back to the discussion of our solution of the problem above. We recall that a complete solution has been provided in [13] in the finite rank case. A lot of the arguments used in that paper do not need the assumption of finite rank. However, one of the key tools used in [13] is the notion of the finite continuation of a finite order element that had been introduced in [11] for Coxeter systems of finite rank. This concept relies on several important features of Coxeter systems of finite rank that are known to be no longer available in the infinite rank case. In the finite rank case there are maximal finite subgroups and all of them are parabolic subgroups, and moreover, the intersection of any set of parabolic subgroups is again a parabolic subgroup. Our main observation is that the principal ideas behind the definition of the finite continuation can nevertheless be applied also to Coxeter systems of infinite rank by using the *locally finite continuation*. One of the main ingredients in this modification is the notion of a locally parabolic subgroup of a Coxeter system of arbitrary rank introduced by the second author in [19]. Indeed, it is basically the content of that paper that provides an appropriate framework for making the ideas of [11] and [13] work also in the infinite rank case.

As already mentioned the main result of the present paper is a solution to the problem formulated above without any restriction on the rank of the Coxeter system (W,S). Its precise statement requires some preparation and it will be given in Section 2. As our main result is a generalization of Theorem 1 in [13] we will comment in this section in more detail about the additional work that has to be done. In Section 3 we introduce the locally finite continuation of a subset of an arbitrary group G. Assuming the axiom of choice, the main advantage compared to the finite continuation is that any finite subgroup is contained in a maximal locally finite subgroup. We provide some observations about basic properties of the locally finite continuation and we discuss in particular global and local approaches which turn out to be equivalent. Although we apply the locally finite continuation exclusively to Coxeter groups of infinite rank in the present paper, it is worthwhile to point out that the definition makes sense for arbitrary groups. It is conceivable that the locally finite continuation provides a valuable tool for studying the isomorphism problem for other classes of groups. But we do not know of any concrete example.

In Section 4 we collect several preliminary results on Coxeter groups that are needed later on. We recall in particular the classification of locally finite Coxeter groups and also the main results of [19] on locally parabolic subgroups.

In [11] the finite continuation of a fundamental reflection of a Coxeter system of finite rank is described. Here we describe the locally finite continuation of a fundamental reflection of a Coxeter system of arbitrary rank. This is done in Sections 5 to 9. The statement of the main result in [11] and its proof given there are rather technical. It involves the definition of (half-)focuses and C_3-neighbors that cannot be avoided. As our result generalizes the main result of [11] we have to deal with all these technicalities as well. However, we are able to take advantage of some tedious considerations made in [11] by just reducing certain parts of our proof to the finite rank case. On the other hand, we have to deal with new phenomenons in our reasoning that are due to the fact, that the list of irreducible locally finite Coxeter systems is slightly longer than the list of irreducible finite Coxeter systems. This is however not visible in the final description of the locally finite continuation of a reflection. As it turns out there are "no surprises" when comparing it to the main result in [11].

In Section 13 we will complete the proof of Theorem 2.2. This is our main result that provides a complete solution to the problem formulated above. As already pointed out, the main tool for establishing it is our description of the locally finite continuation of a fundamental reflection. We essentially follow the strategy of [13]. However, there are some additional ingredients needed with respect to the reasoning in [13]. Since the Krull–Remak–Schmidt Theorem is no longer applicable in our new context, we will also need some facts about direct decompositions of locally finite Coxeter groups which we provide in Section 10. On the other hand, the proof in [13] for the finite rank case used some technical lemmas summarized in Section 11 of [13] about some relations between the centers of finite continuations and the intrinsic reflections. We will extend them in Section 11 to the infinite rank case with locally finite continuations. Furthermore, we have to provide a complete solution of the problem above in the case where (W, S) is a locally finite Coxeter system. This will be done in Section 12.

In the final two sections we provide some applications of our complete solution to the aforementioned problem. A Coxeter group W is called reflection independent if any two Coxeter generating sets for W define the same set of reflections in W. We give in Section 14 two natural classes of Coxeter groups of arbitrary rank where all groups in the class are proven to be reflection independent. In particular, we prove Part (a) of our Rigidity-Theorem as the main result of that section. Then Section 15 is devoted to proving the remaining Parts (b) and (c) of our Rigidity-Theorem.

2 Statement of the Main Result

Here we briefly summarize the background information needed to state the main result of this paper, assuming the basic knowledge of Coxeter systems; see Section 4 for omitted details. The *odd graph* of a Coxeter system (W, S) is the graph with vertex set S where two distinct vertices s, t are joined if and only if st has odd order in W. The *presentation graph* $\Pi(S)$ of (W, S) is the graph with vertex set S where two distinct vertices s, t are joined if and only if st has finite order. An *odd component* of (W, S) is a connected component of the odd graph of (W, S). For any subset $J \subseteq S$, we define

$$\begin{aligned} J^{\perp} &= \{s \in S \mid st \text{ has order two for any } t \in J\} \ , \\ J^{\infty} &= \{s \in S \mid st \text{ has infinite order for any } t \in J\} \ . \end{aligned}$$

We say that a generator $s \in S$ is *right-angled* if $S = \{s\} \cup \{s\}^{\perp} \cup \{s\}^{\infty}$. Following the previous paper [13], for any odd component M of S, we define $E(M) = S \setminus M^{\infty}$, and we define $C_0(M)$ to be the irreducible component of $E(M)$ that contains M. We say that M is *mutable* if at least one irreducible component of $E(M)$ other than $C_0(M)$ is of (-1)-type (i.e., generating a (finite) subgroup with nontrivial center). The next definition has also appeared in [13].

Definition 2.1. Let (W, S) be a Coxeter system. A pair $(z, a) \in S \times S$ is called a *blowing down pair* for (W, S) if the following conditions hold:

1. z is right-angled in (W, S) and $a \in \{z\}^{\perp}$;
2. the component C of $\{z\}^{\perp}$ containing a is of type $I_2(n)$ or D_n for some odd integer $n \geq 3$, and $b := \rho_C a \rho_C \neq a$ where ρ_C denotes the longest element of the Coxeter system $(\langle C \rangle, C)$;
3. for each connected component U of $\Pi(\{z\}^{\infty})$, either $U \subseteq \{a\}^{\infty}$ or $U \subseteq \{b\}^{\infty}$.

Our main result is Theorem 2.2 below. Here we say that a Coxeter system (W, S) is *spherical* if W is a finite group, and (W, S) is *locally spherical* if $(\langle I \rangle, I)$ is spherical for any finite subset I of S. On the other hand, we postpone the definition of the notion of *C_3-neighbors* until Section 7 since the definition is too complicated to be included here; we only note that the notion of C_3-neighbors is defined solely in terms of the Coxeter graphs.

Theorem 2.2. *Let (W, S) be a Coxeter system of arbitrary rank, and let $s \in S$. Let M be the odd component of (W, S) containing s, and let $J = C_0(M)$. If the odd component M is mutable then s is not an intrinsic reflection of W. If M is not mutable then the following hold:*

1. *if $J = \{s\}$ then s is an intrinsic reflection of W if and only if there is no $t \in S$ such that (s, t) is a blowing down pair;*

2. *if $J \neq \{s\}$ and J is of (-1)-type then s is an intrinsic reflection of W if and only if J is of type C_2, H_3, E_7, or $I_2(4k)$ for some $k \geq 2$;*

3. *if J is spherical but not of (-1)-type then s is an intrinsic reflection of W if and only if J is not of type A_5;*

4. *if J is not spherical then s is an intrinsic reflection of W if and only if either J is locally spherical or M has no C_3-neighbors.*

Theorem 2.2 is actually a synthesis of Propositions 2.3, 2.4, and 2.5 below. In order to prove Theorem 2.2 we will have to show Proposition 2.5 since Propositions 2.3 and 2.4 are already available in published form.

Proposition 2.3. *Let (W, S) be a Coxeter system of arbitrary rank, and let $s \in S$ be a right-angled generator. If the odd component $\{s\}$ is mutable then s is not an intrinsic reflection of W. If $\{s\}$ is not mutable then s is an intrinsic reflection of W if and only if there is no $t \in S$ such that (s, t) is a blowing down pair.*

Proof. Proposition 2.3 is the main result of our previous paper [16]. □

Proposition 2.4. *Let (W, S) be a Coxeter system of arbitrary rank, and let $s \in S$. Let M be the odd component of (W, S) containing s, and let $J = C_0(M)$. If the odd component M is mutable then s is not an intrinsic reflection of W. If M is not mutable, then s is not an intrinsic reflection of W if one of the following conditions is satisfied:*

1. *$J \neq \{s\}$ and J is of (-1)-type except type C_2, H_3, E_7, or $I_2(4k)$ with $k \geq 2$;*

2. *J is of type A_5;*

3. *J is not locally spherical and M has a C_3-neighbor.*

Proof. Proposition 2.4 is an immediate consequence of Proposition 9.1 in our joint paper [13] with R. B. Howlett. Note that the proof of Proposition 9.1 in [13] does not require the finiteness of the rank of (W, S), as already pointed out after the proof of Proposition 9.1 in loc. cit.. □

Proposition 2.5. *Let (W, S) be a Coxeter system of arbitrary rank, and let $s \in S$. Let M be the odd component of (W, S) containing s, and let $J = C_0(M)$. Suppose that M is not mutable. Then s is an intrinsic reflection of W if one of the following conditions is satisfied:*

1. *J is of type C_2, H_3, E_7, or $I_2(4k)$ for some $k \geq 2$;*

2. *J is locally spherical and is not of (-1)-type nor of type A_5;*

3. *J is not locally spherical and M has no C_3-neighbors.*

On the proof of Proposition 2.5: This proposition is proved in Section 12 of [13] for Coxeter systems of *finite* ranks and the assumption of finite rank is essential for the arguments given there. In view of what has been said so far on the proof of Theorem 2.2, our task is to give a proof of Proposition 2.5 for Coxeter systems of arbitrary ranks. It will be accomplished in Section 13.

3 Locally Finite Continuations in Arbitrary Groups

In this section, we introduce the notion of the locally finite continuation of a subset of an arbitrary group, and study its basic properties.

3.1 The Definitions

We recall a terminology from group theory: We say that a group G is *locally finite*, if any finite subset of G generates a finite subgroup of G. As already indicated in the introduction there is a "local" and a "global" definition for the locally finite continuation of any subset of a group. We start with the "local version".

Definition 3.1. Let G be an arbitrary group, and let $X \subseteq G$. We define the *locally finite continuation* of the subset X in G, denoted by $\mathrm{LFC}_G(X)$, as

$$\begin{aligned}&\mathrm{LFC}(X) = \mathrm{LFC}_G(X)\\ &:= \{g \in G \mid \text{if } X \subseteq Z \subseteq G \text{ and } \langle Z\rangle \text{ is locally finite, then } \langle Z \cup \{g\}\rangle \text{ is also locally finite}\}\ .\end{aligned}$$

Note that $\mathrm{LFC}(X) = G$ if $\langle X\rangle$ is not a locally finite group. Therefore we will consider $\mathrm{LFC}(X)$ almost always for subsets X of G that generate a locally finite group. When $X = \{w\}$, we often denote $\mathrm{LFC}(X)$ simply by $\mathrm{LFC}(w)$.

We now come to the "global" definition of the finite continuation of a subset of a group.

Definition 3.2. Let G be an arbitrary group, and let $X \subseteq G$. We define the set $\mathrm{LFC}^\dagger(X) = \mathrm{LFC}^\dagger_G(X)$ to be the intersection of all the maximal locally finite subgroups H of G containing X. (When there is no such H, $\mathrm{LFC}^\dagger_G(X)$ is defined to be the whole group G.)

In order to clarify the logical implication relations between the two definitions, we recall the following standard terminology:

Definition 3.3. We say that a partially ordered set $(P, \preceq)$ satisfies the *maximality condition*, if for any $p \in P$, there exists a $q \in P$ that is maximal with respect to $\preceq$ and satisfies $p \preceq q$.

In our argument below, this definition is applied only to the case of a set of some subsets of a given set, with inclusion as the partial ordering. In particular, we consider the partially ordered set of all the locally finite subgroups of a group G containing a given subset X; in this context the maximality condition is always satisfied (assuming the Axiom of Choice, or equivalently Zorn's Lemma).

Now we establish the equivalence of the "local version" and the "global version" of the locally finite continuation.

Proposition 3.4. *Let G be an arbitrary group, and let $X \subseteq G$.*

1. *We have* $\mathrm{LFC}(X) \subseteq \mathrm{LFC}^\dagger(X)$.
2. *If the set of all the locally finite subgroups of G containing X satisfies the maximality condition (in particular, if the Axiom of Choice holds), then we have* $\mathrm{LFC}(X) = \mathrm{LFC}^\dagger(X)$.

Proof. For the first assertion, let $g \in \mathrm{LFC}(X)$ and let H be any maximal locally finite subgroup of G containing X. As $g \in \mathrm{LFC}(X)$ and H is locally finite, the group $\langle H \cup \{g\}\rangle$ is locally finite. Since H is a maximal locally finite subgroup of G and $H \subseteq \langle H \cup \{g\}\rangle$ we have $\langle H \cup \{g\}\rangle = H$. We conclude that $g \in H$ which yields $g \in \mathrm{LFC}^\dagger(X)$ and finishes the proof of the first assertion.

For the second assertion, let $g \in \mathrm{LFC}^\dagger(X)$. We have to show that $g \in \mathrm{LFC}(X)$. Let $Z \subseteq G$ be a subset containing X and generating a locally finite subgroup. By the maximality condition in the hypothesis, there is a maximal locally finite subgroup H of G satisfying $X \subseteq \langle Z\rangle \subseteq H$. Now the fact $g \in \mathrm{LFC}^\dagger(X)$ implies that $g \in H$; therefore $\langle Z \cup \{g\}\rangle$ is contained in H and is locally finite as well as H. Hence $g \in \mathrm{LFC}(X)$ and we are done. □

As the equality $\mathrm{LFC}(X) = \mathrm{LFC}^\dagger(X)$ above is important in our argument below, hereafter we always assume (as usual) the Axiom of Choice without mentioning.

3.2 Relations with Finite Continuations

Replacing "locally finite subgroup" with "finite subgroup" in the two definitions for locally finite continuations leads us to two versions of *finite continuations* of a subset X in a group G. Namely, we define

$$\begin{aligned}&\mathrm{FC}(X) = \mathrm{FC}_G(X)\\ &:= \{g \in G \mid \text{if } X \subseteq Z \subseteq G \text{ and } \langle Z \rangle \text{ is finite, then } \langle Z \cup \{g\} \rangle \text{ is also finite}\}\ ,\end{aligned}$$

and we define $\mathrm{FC}^\dagger(X) = \mathrm{FC}^\dagger_G(X)$ to be the intersection of all the maximal finite subgroups H of G containing X (again, it is defined to be the whole group G when there is no such H). We recall that the finite continuation has been defined in [11] for any element w of finite order in a finitely generated Coxeter group W. The definition given in loc. cit. corresponds to $\mathrm{FC}^\dagger_W(w)$. Now essentially the same arguments as the ones given in the proof of Proposition 3.4 yield the equivalence of the two versions of finite continuations.

Proposition 3.5. *Let G be an arbitrary group, and let $X \subseteq G$.*

1. *We have* $\mathrm{FC}(X) \subseteq \mathrm{FC}^\dagger(X)$.
2. *If the set of all the finite subgroups of G containing X satisfies the maximality condition, then we have* $\mathrm{FC}(X) = \mathrm{FC}^\dagger(X)$.

Comparing Proposition 3.5 with Proposition 3.4, the equivalence in Proposition 3.5 requires a non-trivial assumption on the maximality condition, while the equivalence in Proposition 3.4 always holds. This shows an advantage of locally finite continuations against finite continuations. Moreover, if $X \subseteq G$ generates an infinite subgroup then we have $\mathrm{FC}(X) = G$ by definition; while we have the following result for the other case:

Proposition 3.6. *Let G be an arbitrary group, and let $X \subseteq G$. If $\langle X \rangle$ is finite, then* $\mathrm{LFC}(X) = \mathrm{FC}(X)$.

Proof. First, let $g \in \mathrm{LFC}(X)$, and let $Z \subseteq G$ be any subset that contains X and generates a finite subgroup. Then $\langle Z \rangle$ is locally finite, and as $g \in \mathrm{LFC}(X)$ we have that $\langle Z \cup \{g\} \rangle$ is locally finite. On the other hand, since $\langle Z \rangle$ is finite, Z and hence also $Z \cup \{g\}$ is a finite set. We conclude that $\langle Z \cup \{g\} \rangle$ is finite. This shows that $g \in \mathrm{FC}(X)$, therefore we have $\mathrm{LFC}(X) \subseteq \mathrm{FC}(X)$.

Secondly, let $g \in \mathrm{FC}(X)$, and let $Z \subseteq G$ be any subset that contains X and generates a locally finite subgroup. We have to show that $\langle Z \cup \{g\} \rangle$ is locally finite; for this purpose, it suffices to check that $\langle Z' \cup \{g\} \rangle$ is finite for any finite subset Z' of Z. Since $\langle X \rangle$ is finite by the assumption, X is a finite set, so is $Z'' := Z' \cup X \subseteq Z$. Since $\langle Z \rangle$ is locally finite, it follows that $\langle Z'' \rangle$ is finite. Now the fact $g \in \mathrm{FC}(X)$ implies that $Z'' \cup \{g\}$ generates a finite subgroup, so does its subset $Z' \cup \{g\}$, as desired. Hence $\langle Z \cup \{g\} \rangle$ is locally finite. This shows that $g \in \mathrm{LFC}(X)$, therefore we have $\mathrm{FC}(X) \subseteq \mathrm{LFC}(X)$, hence $\mathrm{LFC}(X) = \mathrm{FC}(X)$. □

Owing to the properties above, in this paper we focus on locally finite continuations rather than finite continuations.

3.3 General Properties

In this subsection we provide some basic results on locally finite continuations in arbitrary groups.

Lemma 3.7. *For an arbitrary group G and a subset X of G, the following hold:*

1. *If $X \subseteq Y \subseteq G$, then $X \subseteq \mathrm{LFC}(X) \subseteq \mathrm{LFC}(Y)$;*
2. $\mathrm{LFC}(X) = \mathrm{LFC}(\langle X \rangle)$*;*
3. $\mathrm{LFC}(X)$ *is a subgroup of G;*
4. *if σ is an isomorphism from G to another group G', then* $\mathrm{LFC}_{G'}(\sigma(X)) = \sigma(\mathrm{LFC}_G(X))$.

Proof. These assertions are all obvious by considering the corresponding assertions for $\mathrm{LFC}^{\dagger}$ owing to the second assertion of Proposition 3.4. (We just note that the assertions can also be directly proved for LFC by straightforward arguments even without the Axiom of Choice.) $\square$

Lemma 3.8. *Let G be an arbitrary group, and let $X \subseteq G$. Then we have $\mathrm{LFC}(\mathrm{LFC}(X)) = \mathrm{LFC}(X)$. Moreover, if $\langle X \rangle$ is locally finite, then $\mathrm{LFC}(X)$ is also locally finite.*

Proof. The first assertion is again obvious by considering the corresponding assertion for $\mathrm{LFC}^{\dagger}$ owing to the second assertion of Proposition 3.4 (and can also be directly proved even without the Axiom of Choice). For the second assertion, let $x_1, \dots, x_n$ be in $\mathrm{LFC}(X)$. By the hypothesis that $\langle X \rangle$ is locally finite, the definition of $\mathrm{LFC}(X)$ implies recursively that, for each $i = 1, \dots, n$, $X \cup \{x_1, \dots, x_i\}$ generates a locally finite subgroup. Hence, its finite subset $\{x_1, \dots, x_n\}$ generates a finite subgroup. This proves the second assertion. $\square$

Lemma 3.9. *Let $H \leq G$ be two groups, and let $X \subseteq H$. Then we have $\mathrm{LFC}_G(X) \cap H \subseteq \mathrm{LFC}_H(X)$.*

Proof. Let $x \in \mathrm{LFC}_G(X) \cap H$ and let Z be any subset of H that contains X and generates a locally finite subgroup of H. Then we have $X \subseteq Z \subseteq G$ and $\langle Z \rangle$ is locally finite, therefore the fact $x \in \mathrm{LFC}_G(X)$ implies that $\langle Z \cup \{x\} \rangle$ is also locally finite. Hence we have $x \in \mathrm{LFC}_H(X)$. $\square$

Lemma 3.10. *Let G be an arbitrary group, and let $X \subseteq G$ be a subset that generates a locally finite subgroup. Let H be a subgroup of G, and suppose that $g \in H$ for any element $g \in G$ for which $\langle X \cup \{g\} \rangle$ is a locally finite subgroup. Then we have $\mathrm{LFC}_G(X) = \mathrm{LFC}_H(X)$.*

Proof. First note that $X \subseteq H$ by the assumption. Now for any $x \in \mathrm{LFC}_G(X)$, the fact that $\langle X \rangle$ is locally finite implies that $\langle X \cup \{x\} \rangle$ is also locally finite. Therefore we have $x \in H$ by the assumption, and $x \in \mathrm{LFC}_G(X) \cap H \subseteq \mathrm{LFC}_H(X)$ by Lemma 3.9. Hence we have $\mathrm{LFC}_G(X) \subseteq \mathrm{LFC}_H(X)$.

Conversely, let $x \in \mathrm{LFC}_H(X)$, and let $Z \subseteq G$ be any subset that contains X and generates a locally finite subgroup. Now for any $z \in Z$, the subgroup $\langle X \cup \{z\} \rangle \leq \langle Z \rangle$ is locally finite, therefore $z \in H$ by the assumption. Hence we have $Z \subseteq H$, and now the fact $x \in \mathrm{LFC}_H(G)$ implies that $\langle Z \cup \{x\} \rangle$ is locally finite. It follows that $x \in \mathrm{LFC}_G(X)$, therefore we have $\mathrm{LFC}_H(X) \subseteq \mathrm{LFC}_G(X)$. This concludes the proof. $\square$

Lemma 3.11. *Let a group G be the direct sum of a family of groups $\{G_i\}_{i \in \Lambda}$, with projection maps $\pi_i \colon G \to G_i$. If $X \subseteq G$ and $\langle X \rangle$ is locally finite, then $\mathrm{LFC}_G(X)$ is the direct sum of $\mathrm{LFC}_{G_i}(\pi_i(X))$ over all $i \in \Lambda$.*

Proof. First note that, a subgroup H of G is locally finite if and only if $\pi_i(H) \leq G_i$ is locally finite for every $i \in \Lambda$. Now suppose that $x \in G$, $\pi_i(x) \in \mathrm{LFC}_{G_i}(\pi_i(X))$ for every $i \in \Lambda$, $X \subseteq Z \subseteq G$ and $\langle Z \rangle$ is locally finite. Then for each $i \in \Lambda$, $\langle \pi_i(Z) \rangle$ is locally finite, and the fact $\pi_i(x) \in \mathrm{LFC}_{G_i}(\pi_i(X))$ implies that $\langle \pi_i(Z \cup \{x\}) \rangle$ is also a locally finite subgroup of G_i. Now the fact mentioned at the beginning of the proof implies that $\langle Z \cup \{x\} \rangle$ is locally finite, therefore we have $x \in \mathrm{LFC}_G(X)$. This shows that $\mathrm{LFC}_G(X)$ includes the direct sum of $\mathrm{LFC}_{G_i}(\pi_i(X))$ over all $i \in \Lambda$.

Conversely, let $x \in \mathrm{LFC}_G(X)$, and suppose that $i \in \Lambda$, $\pi_i(X) \subseteq Z \subseteq G_i$ and $\langle Z \rangle$ is locally finite. Then, by naturally regarding Z as a subset of G, we have $\langle \pi_i(X \cup Z) \rangle = \langle Z \rangle$ (since $\pi_i(X) \subseteq Z$) which is locally finite, while for any other index $j \neq i$, we have $\langle \pi_j(X \cup Z) \rangle = \langle \pi_j(X) \rangle$ which is locally finite by the fact that $\langle X \rangle$ is locally finite. Now the fact mentioned at the beginning of the proof implies that $\langle X \cup Z \rangle$ is locally finite, so is $\langle X \cup Z \cup \{x\} \rangle$ by the fact $x \in \mathrm{LFC}_G(X)$. Therefore, $\langle \pi_i(X \cup Z \cup \{x\}) \rangle = \langle Z \cup \{\pi_i(x)\} \rangle$ is also locally finite. This shows that $\pi_i(x) \in \mathrm{LFC}_{G_i}(\pi_i(X))$ for every $i \in \Lambda$, therefore $\mathrm{LFC}_G(X)$ is included in the direct sum of $\mathrm{LFC}_{G_i}(\pi_i(X))$ over all $i \in \Lambda$. This concludes the proof. $\square$

4 Preliminaries on Coxeter Systems

This section summarizes some definitions and facts about Coxeter systems. In this paper, (W, S) usually denotes a Coxeter system, where S is a Coxeter generating set of a Coxeter group W. Namely, W admits a presentation of the form

$$W = \langle S \mid (st)^{m_{s,t}} = 1 \text{ for all } s, t \in S \text{ with } m_{s,t} < \infty \rangle \ ,$$

where the indices $m_{s,t}$ satisfy that $m_{s,t} = m_{t,s} \in \{1, 2, \dots\} \cup \{\infty\}$ and we have $m_{s,t} = 1$ if and only if $s = t$. The cardinality $|S|$ of S is called the *rank* of (W, S), which may be infinite unless otherwise specified. We refer to the book [14] for basic definitions and facts for Coxeter systems that are implicit in this paper. We write $o(g)$ to denote the order of an element g of a group; then we have $o(st) = m_{s,t}$ for any $s, t \in S$. We say that (W, S) is *irreducible*, if S is non-empty and the Coxeter graph (also known as Coxeter diagram) for (W, S) is connected. (Recall that two generators $s, t \in S$ are joined in the Coxeter graph by an edge with label $m_{s,t}$ if and only if $m_{s,t} \geq 3$; the edge label is frequently omitted when $m_{s,t} = 3$). In this case we also say that S is *irreducible*. A *reflection* in W with respect to S, or a *reflection* of (W, S), is an element of the set $S^W = \{s^w = w^{-1}sw \mid s \in S, w \in W\}$.

For any subset $I \subseteq S$, the subgroup $\langle I \rangle$ of W generated by I is called a *visible subgroup* of W. It is a Coxeter group with Coxeter generating set I, and the Coxeter graph for $(\langle I \rangle, I)$ is the full subgraph of the Coxeter graph for (W, S) restricted to the vertex set I. We also have that $\langle I \rangle \cap \langle J \rangle = \langle I \cap J \rangle$ for any subsets $I, J \subseteq S$. See e.g., [14] for these properties of visible subgroups. On the other hand, any subgroup of the form $\langle I \rangle^w = w^{-1}\langle I \rangle w$ with $I \subseteq S$ and $w \in W$ is called a *parabolic subgroup* of W.

We say that a subset $I \subseteq S$ is a *direct factor* of S, if $[I, S \setminus I] = 1$; and $I \subseteq S$ is an *irreducible component* of S, if I is an irreducible direct factor of S. In the corresponding cases we also say that $(\langle I \rangle, I)$ is a *direct factor* and an *irreducible component* of (W, S), respectively.

4.1 Spherical and Locally Spherical Coxeter Systems

A Coxeter system (W, S) is called *spherical*, if W is a finite group. In this case we also say that S is *spherical*. Here we introduce a generalization of this notion that is meaningful in the infinite rank cases:

Definition 4.1. We say that a Coxeter system (W, S) is *locally spherical*, if W is a locally finite group. In this case we also say that S is *locally spherical*.

We note that S is locally spherical if and only if every finite subset of S is spherical. They are characterized as follows.

Proposition 4.2. *A Coxeter system (W, S) is locally spherical if and only if, each irreducible component of (W, S) is either spherical or of one of the four types A_∞, $A_{\pm\infty}$, $B_\infty = C_\infty$, and D_∞ with infinite rank defined by the Coxeter graphs in Figure 1.*

Proof. This is Proposition 1 of [19]. □

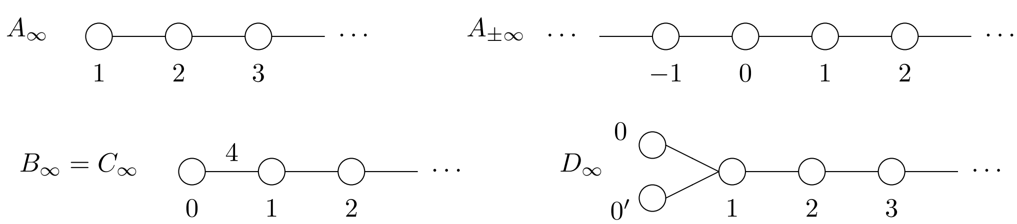

Figure 1: Locally spherical irreducible Coxeter systems

The following three lemmas on Coxeter systems of type A, C, and D will be used in Section 15.

Lemma 4.3. *Let (W, S) be a Coxeter system and let*

$$m := \max\{|U| \mid U = \langle T \rangle \text{ with } T \subseteq S^W, |T| \leq 4\} \ .$$

Then:

1. $m = 60$ *if* (W, S) *is of type* A_∞ *or* $A_{\pm\infty}$*;*

2. $m = 384$ *if* (W, S) *is of type* C_∞*;*

3. $m = 192$ *if* (W, S) *is of type* D_∞*.*

Proof. Let $U \leq W$ be a subgroup generated by at most four reflections of (W, S). Then U is a subgroup generated by at most four reflections in a visible subgroup $\langle J \rangle$ where $(\langle J \rangle, J)$ is of type A_n (respectively, C_n, D_n) for some integer $n \geq 4$. Thus the assertions follow from the corresponding statements for Coxeter systems of type A_n, C_n, and D_n with $4 \leq n < \infty$ where they are easily verified. □

In the following two lemmas we recall (without proofs) elementary facts about finite Coxeter systems of type C and D, respectively. The labellings of the Coxeter graphs of type $C_{\ell+1}$ and $D_{\ell+2}$ are given by the corresponding subgraphs in Figure 1.

Lemma 4.4. *Let* (W, S) *be a Coxeter system of type* C_{n+1} *where* $S = \{s_0, s_1, \dots, s_n\}$*. Let* $2 \leq k \leq n$ *and* $T := \{t_0, t_1, \dots, t_k\} \subseteq S^W$*, and suppose that* $(\langle T \rangle, T)$ *is a Coxeter system of type* C_{k+1}*. Then there exists* $w \in W$ *satisfying* ${t_i}^w = s_i$ *for all* $i \in \{0, 1, \dots, k\}$*.*

Lemma 4.5. *Let* (W, S) *be a Coxeter system of type* D_{n+2} *where* $S = \{s_0, s_{0'}, s_1, \dots, s_n\}$*. Let* $2 \leq k < n$ *and* $T := \{t_0, t_{0'}, t_1, \dots, t_k\} \subseteq S^W$*, and suppose that* $(\langle T \rangle, T)$ *is a Coxeter system of type* D_{k+2}*. Then there exists* $w \in W$ *satisfying* ${t_i}^w = s_i$ *for all* $i \in \{0, 0', 1, \dots, k\}$*.*

For any spherical subset I of S, the unique longest element in $\langle I \rangle$ is denoted by ρ_I. We say that a Coxeter system (W, S) is of (-1)*-type*, if it is spherical and ρ_S is in the center $Z(W)$ of W. In this case we also say that S is of (-1)*-type*. For an irreducible Coxeter system (W, S), the center $Z(W)$ is trivial if (W, S) is not of (-1)-type; and $Z(W)$ is the group of order two generated by ρ_S if (W, S) is of (-1)-type. Moreover, for a Coxeter system (W, S), it is of (-1)-type if and only if it is spherical and each irreducible component of (W, S) is of (-1)-type.

We need the following result by Richardson [20] on involutions in Coxeter groups.

Proposition 4.6. *Let* (W, S) *be a Coxeter system. Each involution in* W *is conjugate to an element of the form* ρ_I *for a subset* $I \subseteq S$ *of* (-1)*-type. On the other hand, if* $I, J \subseteq S$ *are of* (-1)*-type and* ρ_I *is conjugate in* W *to* ρ_J*, then* I *and* J *are conjugate in* W *and in particular of the same type.*

4.2 On Supports of Elements

Let (W, S) be a Coxeter system. By the fact $\langle I \rangle \cap \langle J \rangle = \langle I \cap J \rangle$ for $I, J \subseteq S$ mentioned above, for any element $w \in W$, the *support* $\mathrm{supp}(w)$ of w is uniquely defined as the smallest (finite) subset $I \subseteq S$ satisfying $w \in \langle I \rangle$. We note that $\mathrm{supp}(w^{-1}) = \mathrm{supp}(w)$ for any $w \in W$. The following property is deduced immediately from the definition of the support and is used later.

Lemma 4.7. *Let* (W, S) *be a Coxeter system,* $w \in W$*,* $I \subseteq S$*, and let* $s \in \mathrm{supp}(w) \setminus I$*. Then we have* $s \in \mathrm{supp}(uw)$ *and* $s \in \mathrm{supp}(wu)$ *for any* $u \in \langle I \rangle$*, and in particular* $s \in \mathrm{supp}(w^u)$ *for any* $u \in \langle I \rangle$*.*

We shall need also the following fact which is Assertion (2) of Lemma 2.9 in [18].

Lemma 4.8. *Let* (W, S) *be a Coxeter system. If* $w \in W$*,* $s \in S \setminus \mathrm{supp}(w)$ *and* s *is adjacent to the set* $\mathrm{supp}(w)$ *in the Coxeter graph of* (W, S)*, then we have* $\mathrm{supp}(sws) = \mathrm{supp}(w) \cup \{s\}$*.*

4.3 On Conjugate Visible Subgroups

Let $I \subseteq S$ and $s \in S \setminus I$, and let I_s denote the irreducible component of $I \cup \{s\}$ containing s. If I_s is spherical, then we define $v[s, I] = \rho_{I_s} \rho_{I_s \setminus \{s\}} \in \langle I_s \rangle$. This element has the property that $v[s, I] I v[s, I]^{-1} \subseteq I \cup \{s\}$; and by putting $K := (I \cup \{s\}) \setminus I_s$, we have $K \subseteq v[s, I] I v[s, I]^{-1}$ and $v[s, I] I v[s, I]^{-1} \setminus K$ is the union of some spherical irreducible components of $v[s, I] I v[s, I]^{-1}$. The following result was proved in [12] for finite Coxeter groups and generalized in [7] to arbitrary Coxeter groups.

Lemma 4.9. *Let $I, J \subseteq S$. Then $\langle I \rangle$ and $\langle J \rangle$ are conjugate in W if and only if I is conjugate to J in W, i.e., $J = wIw^{-1}$ for some $w \in W$. Moreover, if $w \in W$ and $J = wIw^{-1}$, then there are a finite sequence $J = J_0, J_1, \dots, J_n = I$ of subsets of S and a finite sequence of elements $s_1, \dots, s_n$ of S with the properties that $v[s_i, J_i]$ is defined and $v[s_i, J_i]J_i v[s_i, J_i]^{-1} = J_{i-1}$ for each $i = 1, \dots, n$, and $v[s_1, J_1] \cdots v[s_n, J_n]$ is the element in $w\langle I \rangle$ of minimal length.*

By Lemma 4.9, if two visible subgroups $\langle I \rangle$ and $\langle J \rangle$ are conjugate, then the subsets $I, J \subseteq S$ are conjugate, in particular they have the same cardinality. Due to this, the *type* and the *rank* of a parabolic subgroup G are uniquely defined to be the type and the rank, respectively, of any visible subgroup conjugate to G.

4.4 On Parabolic Subgroups

In this subsection we collect several facts about parabolic subgroups of Coxeter systems.

Lemma 4.10. *Let (W, S) be a Coxeter system, $J, K \subseteq S$, and let $w \in W$. Suppose that $w^{-1}Jw \subseteq \langle K \rangle$. Then we have $(wu)^{-1}J(wu) \subseteq K$ for some $u \in \langle K \rangle$.*

Proof. This is Lemma 3.3 in [13]. □

Lemma 4.11. *Let (W, S) be a Coxeter system, $I \subseteq S$, and let G be a parabolic subgroup of W. Then $\langle I \rangle \cap G$ is a parabolic subgroup of $\langle I \rangle$. Moreover, if $|I|$ is finite and $\langle I \rangle \not\subseteq G$, then the rank of $\langle I \rangle \cap G$ is strictly smaller than the rank $|I|$ of $\langle I \rangle$.*

Proof. This is an immediate consequence of Corollary 7 in [10]. □

Let (W, S) be a Coxeter system and let $X \subseteq W$. Lemma 4.11 implies that, if X is contained in some finite-rank parabolic subgroup of W, then the intersection of all parabolic subgroups of W containing X is again a parabolic subgroup of W; it is called the *parabolic closure* of X and is denoted by $\mathrm{P}(X)$. We also use the notation $\mathrm{P}(x)$ when $X = \{x\}$. Two special cases for the above are the cases where X is a finite set or (W, S) is of finite rank. On the other hand, in general, it is known that the intersection of all the parabolic subgroups of W containing X is not necessarily a parabolic subgroup of W; see Example 1 of [19].

Lemma 4.12. *Let (W, S) be a Coxeter system, and let $I \subseteq S$ be a subset of (-1)-type. Then $\mathrm{P}(\rho_I) = \langle I \rangle$.*

Proof. First, we have $\mathrm{P}(\rho_I) \leq \langle I \rangle$ since $\rho_I \in \langle I \rangle$. Now Lemma 4.11 implies that $\mathrm{P}(\rho_I)$ is a parabolic subgroup of $\langle I \rangle$, therefore $\rho_I \in \mathrm{P}(\rho_I) = w\langle K \rangle w^{-1}$ for some $K \subseteq I$ and $w \in \langle I \rangle$. Since I is of (-1)-type, ρ_I is central in $\langle I \rangle$, therefore we have $\rho_I \in \langle K \rangle$. This implies that $K = I$ and $\mathrm{P}(\rho_I) = w\langle I \rangle w^{-1} = \langle I \rangle$. □

The following result is due to Tits; see e.g., Theorem 4.5.3 of [2] for a proof.

Proposition 4.13. *Let (W, S) be a Coxeter system, and let G be a finite subgroup of W. Then G is contained in a finite parabolic subgroup of W.*

Corollary 4.14. *Let (W, S) be a Coxeter system and let $X \subseteq W$. If $\langle X \rangle$ is finite, then $\mathrm{P}(X)$ is defined and is also a finite subgroup of W.*

We also have the following well-known result; here we give a proof for the sake of completeness.

Proposition 4.15. *Let (W, S) be a Coxeter system of finite rank. Then any finite subgroup of W is contained in a maximal finite subgroup of W, and any maximal finite subgroup of W is a parabolic subgroup of W.*

Proof. For the former assertion, let H be a finite subgroup of W. By Lemma 4.11 and the assumption, the ranks of parabolic subgroups in W are bounded by the rank of (W, S) which is finite. Therefore, by Proposition 4.13, there exists a finite parabolic subgroup P of W containing H that has the maximal rank subject to this condition. Now if G is a finite subgroup of W and P is properly contained in G, then G is contained in a finite parabolic subgroup P' of W by Proposition 4.13, and the rank of P' should be strictly larger than that of P by Lemma 4.11. This contradicts the choice of P. Hence P is a maximal finite subgroup of W, proving the former assertion. The latter assertion follows immediately from Proposition 4.13. □

Moreover, by utilizing the parabolic closure, we show the following property used in our argument later. For any spherical subset I of S, we define

$$\mathcal{E}(I) := \{s \in S \mid |\langle J \cup \{s\}\rangle| < \infty \text{ for some } J \subseteq S \text{ conjugate in } W \text{ to } I\} \ . \tag{1}$$

Lemma 4.16. *Let (W,S) be a Coxeter system. If $I \subseteq S$, $g \in W$ and $\langle I \cup \{g\}\rangle$ is finite, then $g \in \langle \mathcal{E}(I)\rangle$.*

Proof. Due to the assumption, the parabolic closure of $I \cup \{g\}$ is defined and is finite by Corollary 4.14. We have $w^{-1}Iw \subseteq w^{-1}\mathrm{P}(I \cup \{g\})w = \langle J\rangle$ for some spherical $J \subseteq S$ and $w \in W$. By Lemma 4.10, we have $(wu)^{-1}I(wu) = K$ for some $K \subseteq J$ and $u \in \langle J\rangle$. Now we have $|\langle K \cup \{s\}\rangle| \leq |\langle J\rangle| < \infty$ for each $s \in J$, therefore we have $J \subseteq \mathcal{E}(I)$ and $u \in \langle \mathcal{E}(I)\rangle$. On the other hand, we can choose $v \in \langle I\rangle$ in a way that $(wu)^{-1}v$ is the element in $(wu)^{-1}\langle I\rangle$ of minimal length and admits a decomposition $(wu)^{-1}v = v[s_1, J_1] \cdots v[s_n, J_n]$ as in Lemma 4.9 (where K plays the role of J). Now a recursive argument for $i = n, n-1, \dots, 1$ implies the following: J_i is conjugate to I in W; J_i is spherical (recall that $J_n = I$ is spherical by the assumption); the irreducible component of $J_i \cup \{s_i\}$ containing s_i is spherical; and $J_i \cup \{s_i\}$ is spherical. Therefore, we have $J_i \cup \{s_i\} \subseteq \mathcal{E}(I)$ for each $i = 1, \dots, n$; hence we have $(wu)^{-1}v \in \langle \mathcal{E}(I)\rangle$, while we have $u \in \langle \mathcal{E}(I)\rangle$ as above and $v \in \langle \mathcal{E}(I)\rangle$ since $I \subseteq \mathcal{E}(I)$. Therefore we have $w \in \langle \mathcal{E}(I)\rangle$, concluding the proof. $\square$

4.5 Reflection Subgroups and Locally Parabolic Subgroups

The following property is fundamental, and can be deduced by considering the case $|J| = 1$ in Lemma 4.10.

Proposition 4.17. *Let (W,S) be a Coxeter system, and let $I \subseteq S$. Then we have $S^W \cap \langle I\rangle = I^{\langle I\rangle}$.*

A subgroup G of W is called a *reflection subgroup* of W (with respect to S), if G is generated by reflections in W (with respect to S). It was shown in [8, 9] that, for any reflection subgroup G, there is a distinguished subset S_G of G with the properties that (G, S_G) is a Coxeter system, the root system of (G, S_G) is a subset of the root system of (W,S), and the decomposition of the root system of (G, S_G) into positive roots and negative roots coincides with that for (W,S). Based on the existence of the distinguished generating set S_G for G, the following notion was introduced in Definition 1 of [19].

Definition 4.18. Let (W,S) be a Coxeter system. We say that a subgroup G of W is a *locally parabolic subgroup* of W, if G is a reflection subgroup of W and any finite subset of the Coxeter generating set S_G of G is conjugate in W to a subset of S.

We have the following two fundamental facts about locally parabolic subgroups.

Proposition 4.19. *Let (W,S) be a Coxeter system. Any parabolic subgroup (possibly of infinite rank) of W is a locally parabolic subgroup of W.*

Proof. This is an immediate consequence of Lemma 6 in [19]. $\square$

Proposition 4.20. *Let (W,S) be a Coxeter system. The intersection of an arbitrarily large (possibly infinite) family of locally parabolic subgroups of W is again a locally parabolic subgroup of W.*

Proof. This is Theorem 1 of [19]. $\square$

Moreover, a combination of Theorems 2 and 3 in [19] yields the following property, which is an analogy of Proposition 4.15.

Proposition 4.21. *Let (W,S) be a Coxeter system. Then any locally finite subgroup of W is contained in a maximal locally finite subgroup of W, and any maximal locally finite subgroup of W is a locally parabolic subgroup of W.*

We note that, in comparison to Proposition 4.15 for finite subgroups in W, Proposition 4.21 for locally finite subgroups in W holds without any restriction on the Coxeter system (W,S). This fact, combined with Proposition 4.20, yields the following definition, which is an analogy to the parabolic closures.

Definition 4.22. Let (W, S) be a Coxeter system, and let $X \subseteq W$. We define the *locally parabolic closure* $\mathrm{LP}(X)$ of X to be the intersection of all locally parabolic subgroups of W containing X. We also use the notation $\mathrm{LP}(x)$ when $X = \{x\}$.

The arguments above are enough for showing that $\mathrm{LP}(X)$ is locally parabolic provided $\langle X \rangle$ is locally finite. In fact, a slightly stronger statement holds by Theorem 2 of [19].

Proposition 4.23. *Let (W, S) be a Coxeter system, and let $X \subseteq W$. If $\langle X \rangle$ is locally finite, then $\mathrm{LP}(X)$ is a locally finite, locally parabolic subgroup of W.*

We also have the following result, which shows that the locally parabolic closure coincides with the parabolic closure provided the latter is effectively defined:

Proposition 4.24. *Let (W, S) be a Coxeter system, and let $X \subseteq W$. If X is contained in some finite-rank parabolic subgroup of W, then we have $\mathrm{LP}(X) = \mathrm{P}(X)$ and it is a parabolic subgroup of finite rank. The assumption is in particular satisfied when $\langle X \rangle$ is a finite group.*

Proof. The former assertion is a part of Lemma 10 of [19], and the latter remark follows from Proposition 4.13. □

5 Locally Finite Continuations in Coxeter Groups

Based on the results in the previous sections, here we give two results on locally finite continuations in Coxeter groups that are of significant importance.

Proposition 5.1. *Let (W, S) be a Coxeter system, and let $X \subseteq W$. If $\langle X \rangle$ is locally finite, then the locally finite continuation $\mathrm{LFC}(X) = \mathrm{LFC}_W(X)$ of X in W is a locally parabolic subgroup of W; hence $\mathrm{LFC}(X)$ admits (by definition of locally parabolic subgroups) a Coxeter generating set that consists of some reflections in W with respect to S.*

Proof. Under the assumption, Proposition 4.21 implies that the set $\mathrm{LFC}^{\dagger}(X)$ defined in Definition 3.2 is the intersection of locally parabolic subgroups of W, and it is again locally parabolic by Proposition 4.20. Now the assertion follows from Proposition 3.4. □

Proposition 5.2. *Let (W, S) be a Coxeter system of finite rank, and let $w \in W$ be an element of finite order. Then $\mathrm{LFC}(w)$ is equal to the intersection of all the maximal finite subgroups of W containing w, the latter being the definition of the finite continuation of w in [11].*

Proof. In our notation, the finite continuation of w in [11] is equal to $\mathrm{FC}^{\dagger}(w)$ introduced in Section 3.2. Now by the finite-rank assumption on (W, S), we have $\mathrm{FC}(w) = \mathrm{FC}^{\dagger}(w)$ by Proposition 3.5 and Proposition 4.15. Since now $\mathrm{LFC}(w) = \mathrm{FC}(w)$ by Proposition 3.6, the assertion holds. □

We also have the following two properties of locally finite continuations in Coxeter groups.

Proposition 5.3. *Let (W, S) be a Coxeter system, and let $I \subseteq S$ be a subset of (-1)-type. Then we have $\langle I \rangle \leq \mathrm{LFC}(\rho_I) = \mathrm{LFC}(\langle I \rangle)$.*

Proof. Proposition 5.1 implies that $\mathrm{LP}(\rho_I) \leq \mathrm{LFC}(\rho_I)$, while we have $\mathrm{P}(\rho_I) = \mathrm{LP}(\rho_I) = \langle I \rangle$ by Lemma 4.12 and Proposition 4.24. Therefore we have $\langle I \rangle \leq \mathrm{LFC}(\rho_I)$. Now by Lemma 3.7 and Lemma 3.8, we have $\mathrm{LFC}(\langle I \rangle) \leq \mathrm{LFC}(\mathrm{LFC}(\rho_I)) = \mathrm{LFC}(\rho_I) \leq \mathrm{LFC}(\langle I \rangle)$, therefore $\mathrm{LFC}(\rho_I) = \mathrm{LFC}(\langle I \rangle)$. □

Proposition 5.4. *Let (W, S) be a Coxeter system, and let $I \subseteq S$ be a spherical subset. Then we have $\mathrm{LFC}_W(I) = \mathrm{LFC}_{\langle \mathcal{E}(I) \rangle}(I)$, where $\mathcal{E}(I) \subseteq S$ is defined by Equation (1) before Lemma 4.16.*

Proof. In this case, Lemma 4.16 implies that the subgroup $H := \langle \mathcal{E}(I) \rangle$ satisfies the assumption in Lemma 3.10. Now the assertion follows immediately from Lemma 3.10. □

Moreover, the following result enables us to reduce a significant part of the analysis of intrinsic reflections in a Coxeter group W to analyzing intrinsic reflections in locally finite continuations. This result is an analogy of Lemma 9.3 (ii) in [13] shown for finite continuations in finite rank cases.

Lemma 5.5. *Let (W,S) be a Coxeter system of arbitrary rank, and let r be a reflection in W with respect to S. If r is an intrinsic reflection of* $\mathrm{LFC}(r)$ *(which is a Coxeter group by Proposition 5.1), then r is also an intrinsic reflection of W.*

Proof. Let $R \subseteq W$ and suppose that (W,R) is a Coxeter system. Now we note that the definition of $\mathrm{LFC}(r)$ does not depend on the choice of the Coxeter generating set of W. By Proposition 5.1 applied to (W,R) instead of (W,S), the group $\mathrm{LFC}(r)$ admits a Coxeter generating set consisting of reflections in W with respect to R. Since r is an intrinsic reflection of $\mathrm{LFC}(r)$ by the assumption, r is conjugate to an element of the aforementioned generating set for $\mathrm{LFC}(r)$, which is a reflection with respect to R as mentioned above. This implies that r is also a reflection with respect to R. This completes the proof. ☐

Owing to Lemma 5.5, it is worthwhile for our goal to determine the structure of the locally finite continuations of reflections in Coxeter groups of arbitrary ranks. The following Sections 6 to 9 are devoted to this purpose.

6 Decomposition of Locally Finite Continuations of Reflections

Let (W,S) be a Coxeter system. In order to determine the locally finite continuation $\mathrm{LFC}(r) = \mathrm{LFC}_W(r)$ of a reflection r in W, in this section we establish a certain decomposition result which allows us to concentrate on an "essential" part of the locally finite continuation.

6.1 Definitions

Here we introduce and summarize some relevant definitions and notation mostly following the previous paper [13]. The *odd graph* $\Omega(S)$ of a Coxeter system (W,S) is the subgraph of the Coxeter graph of (W,S) with the same vertex set S and in which two distinct vertices $s,t \in S$ are joined by an edge if and only if $o(st)$ is finite and odd. We may write $\Omega(S)$ as $\Omega(W,S)$ when we emphasize the group W. An *odd component* of S is the vertex set of a connected component of $\Omega(S)$. It is well-known that two generators $s,t \in S$ are in the same odd component of S if and only if they are conjugate in W. Now by virtue of Lemma 3.7 (4) applied to the inner automorphisms of W, the problem of determining $\mathrm{LFC}(r)$ for reflections r is reduced to the case where $r \in S$, and to determine $\mathrm{LFC}(r)$ for a given $r \in S$, we may consider (instead of r) any element $s \in S$ in the same odd component of S as r that is convenient for our argument.

For any subset $J \subseteq S$, we define

$$J^{\perp} = \{s \in S \mid o(st) = 2 \text{ for any } t \in J\} \ ,$$
$$J^{\infty} = \{s \in S \mid o(st) = \infty \text{ for any } t \in J\} \ .$$

Now let M be an odd component of S. We define the *even closure* $E(M)$ of M in S by $E(M) := S \setminus M^{\infty}$. Note that M is an irreducible subset of $E(M)$. We define the *M-principal component* of $E(M)$, denoted by $C_0(M)$, to be the irreducible component of $E(M)$ containing M. We call the other irreducible components of $E(M)$ the *M-subsidiary components* of $E(M)$.

6.2 The Decomposition

Let (W,S) be a Coxeter system. Let M be an odd component of S and let $s \in M$. We note that the set $E(M)$ is equal to the set $\mathcal{E}(\{s\})$ defined by Equation (1) preceding Lemma 4.16. Now combining Proposition 5.4 and Lemma 3.11 yields the following:

Lemma 6.1. *Let (W,S) be a Coxeter system, M be an odd component of S, and let $s \in M$. Then* $\mathrm{LFC}_W(s)$ *is the direct sum of* $\mathrm{LFC}_{\langle C_0(M) \rangle}(s)$ *and* $\mathrm{LFC}_{\langle J \rangle}(1)$ *over all the M-subsidiary components J of $E(M)$.*

Hence, the problem of determining $\mathrm{LFC}_W(s)$ is reduced to determining $\mathrm{LFC}_{\langle C_0(M)\rangle}(s)$ and also determining $\mathrm{LFC}_{\langle J\rangle}(1)$ for any irreducible subset J of S. The former case is studied in the subsequent sections. For the latter case, it follows immediately that we have $\mathrm{LFC}_{\langle J\rangle}(1) = \langle J\rangle$ if J is locally spherical, as $\langle J\rangle$ itself is locally finite in this case. Therefore, it suffices to consider the J that is not locally spherical.

The key step for dealing with this case is the following proposition, which is stated in a somewhat generalized form for the sake of later references (see Section 4.2 for the definition of the support $\mathrm{supp}(w)$ of an element w):

Lemma 6.2. *Let (W, S) be a Coxeter system, and let $w \in W$. Then the following two conditions are equivalent:*

1. *There exists a finite subgroup G of W satisfying $|\langle G \cup \{w\}\rangle| = \infty$.*
2. *There exists an irreducible component J of S that is not locally spherical and satisfies $J \cap \mathrm{supp}(w) \neq \emptyset$.*

Proof. First we show that the negation of Condition 2 implies the negation of Condition 1. Let $J_1, \dots, J_k$ be the irreducible components of S having non-empty intersection with the finite set $\mathrm{supp}(w)$, therefore $w \in \prod_{i=1}^{k} \langle J_i\rangle$. By the negation of Condition 2, each $\langle J_i\rangle$ is locally finite. Put $K := S \setminus \bigcup_{i=1}^{k} J_i$. Let π_{J_i} and π_K denote the projection maps $W \to W_{J_i}$ and $W \to W_K$, respectively. Now we suppose that $G \leq W$ is finite, and we show that $\langle G \cup \{w\}\rangle$ is also finite. By the choice of $J_1, \dots, J_k$, $\langle G \cup \{w\}\rangle$ is contained in the direct product of $\pi_K(G)$ and $\langle \pi_{J_i}(G) \cup \{\pi_{J_i}(w)\}\rangle$ for $i = 1, \dots, k$. Since G is finite, $\pi_K(G)$ is also finite; while for each i, $\pi_{J_i}(G) \cup \{\pi_{J_i}(w)\}$ is a finite subset of locally finite group $\langle J_i\rangle$, therefore $\langle \pi_{J_i}(G) \cup \{\pi_{J_i}(w)\}\rangle$ is also finite. This implies that $\langle G \cup \{w\}\rangle$ is finite as well. Hence the negation of Condition 1 holds.

Secondly, we show that Condition 2 implies Condition 1. Let π_J be the projection map $W \to W_J$. Now if G is a finite subgroup of W_J and $\langle G \cup \{\pi_J(w)\}\rangle$ is not finite, then $\langle G \cup \{w\}\rangle$ is not finite since $\pi_J(\langle G \cup \{w\}\rangle) = \langle G \cup \{\pi_J(w)\}\rangle$. Due to this fact, we may assume without loss of generality that $S = J$, i.e., S is irreducible and is not locally spherical, and moreover that $w \neq 1$. Now we can take a finite irreducible subset K of S that is non-spherical and satisfies $w \in \langle K\rangle$. Since K is finite, Proposition 4.15 implies that there is a subset L of K for which $\langle L\rangle$ is a maximal finite subgroup of $\langle K\rangle$. This $\langle L\rangle$ satisfies Condition 1 when $w \notin \langle L\rangle$; from now, we consider the other case $w \in \langle L\rangle$. Since L is spherical and K is non-spherical, L is properly contained in K. Since K is irreducible, there is a finite sequence $s_1, \dots, s_\ell$ of mutually distinct elements of $K \setminus \mathrm{supp}(w)$ satisfying that $s_1 \notin \mathrm{supp}(w)^{\perp}$, $s_i \notin \mathrm{supp}(w) \cup \{s_1, \dots, s_{i-1}\}$ and $s_i \notin (\mathrm{supp}(w) \cup \{s_1, \dots, s_{i-1}\})^{\perp}$ for every $2 \leq i \leq \ell$, and $s_\ell \notin L$. Put $u := s_\ell s_{\ell-1} \cdots s_2 s_1 \in \langle K\rangle$. Then by recursively applying Lemma 4.8, we have $s_\ell \in \mathrm{supp}(uwu^{-1})$, therefore $uwu^{-1} \notin \langle L\rangle$. This implies that $w \notin G := u^{-1}\langle L\rangle u$, while G is a maximal finite subgroup of $\langle K\rangle$ as well as $\langle L\rangle$, therefore the subgroup G satisfies Condition 1. This completes the proof. □

Corollary 6.3. *Let (W, S) be an irreducible Coxeter system that is not locally spherical. Then we have* $\mathrm{LFC}_W(1) = 1$.

Proof. For any non-trivial element $w \in W$, Condition 2 in Lemma 6.2 is satisfied (with $J = S$), therefore Lemma 6.2 implies that there is a finite subgroup G of W for which $H := \langle G \cup \{w\}\rangle$ is not finite. In particular, H is not locally finite. Hence we have $w \notin \mathrm{LFC}_W(1)$ by Definition 3.1, therefore the assertion holds. □

Now by combining Corollary 6.3 to Lemma 6.1 and the argument in the paragraph next to Lemma 6.1, we have the following decomposition result on the locally finite continuation.

Theorem 6.4. *Let (W, S) be a Coxeter system, M be an odd component of S, and let $s \in M$. Let $\Sigma(M)$ be the union of all locally spherical M-subsidiary components of $E(M)$. Then* $\mathrm{LFC}_W(s) = \mathrm{LFC}_{\langle C_0(M)\rangle}(s) \times \langle \Sigma(M)\rangle$.

Since $\mathrm{LFC}_{\langle C_0(M)\rangle}(s) = \langle C_0(M)\rangle$ if $C_0(M)$ is locally spherical, it suffices due to Theorem 6.4 to determine $\mathrm{LFC}_{\langle C_0(M)\rangle}(s)$ for the case where $C_0(M)$ is not locally spherical.

7 Focuses, Half-Focuses and C_3-Neighbors

To study the "essential" component $\mathrm{LFC}_{\langle C_0(M)\rangle}(s)$ of $\mathrm{LFC}_W(s)$ mentioned in the previous section, in this section we summarize definitions of some relevant objects, which have played a central role in the preceding work [11] for the finite rank cases by Franzsen, Howlett and the first author of this paper. According to the argument in the previous section, here we put the following assumption throughout this section:

Assumption. Throughout this section, we suppose that (W, S) is a Coxeter system, M is an odd component of S, $s \in M$, $S = C_0(M)$, and S is not locally spherical. (In particular, (W, S) is irreducible and non-spherical.)

7.1 The Definitions

We now record the definitions of a focus, a half-focus and a C_3-neighbor of an odd component which has been introduced in Definitions 4, 5 and 6 of [11]. Due to the assumption on (W, S) in this section, the definitions given below are slightly different from those given in [11]. But it is straightforward to check that they are equivalent to those in [11] if (W, S) is as in the assumption above. We also note that there are redundancies in our set of axioms. These redundancies will facilitate our reasoning later on at various places and will not be a disadvantage for us. We also give a remark on notations: For a subset J of S, we let $\Omega(J)$ denote the odd graph of the Coxeter system $(\langle J\rangle, J)$ (see Section 6.1).

Definition 7.1. Let $a \in M$ and $b \in S \setminus M$. We say that (a, b) is a *focus* of M in S, if the following conditions are all satisfied:

1. The odd graph $\Omega(M)$ is a tree and if $\{x, y\} \subseteq M$ is an edge of $\Omega(M)$, then $o(xy) = 3$.
2. For each $c \in M$, the subset $C[b..c] \subseteq S$ consisting of b and all vertices in the unique path in $\Omega(M)$ joining c and a is of type C_k, where $k = |C[b..c]|$. (In particular $o(ab) = 4$.)
3. If $c, d \in M$, $c \notin C[b..d]$ and $d \notin C[b..c]$, then $o(cd) = \infty$.
4. If $t \in S \setminus (M \cup \{b\})$, $c \in M$ and $o(tc) < \infty$, then $o(td) = 2$ for every $d \in C[b..c]$ (in particular $o(tb) = o(tc) = 2$ in this case).

Figure 2 shows an example of a focus. In the figure, (a, b) is a focus of $M = S \setminus \{b, t\}$.

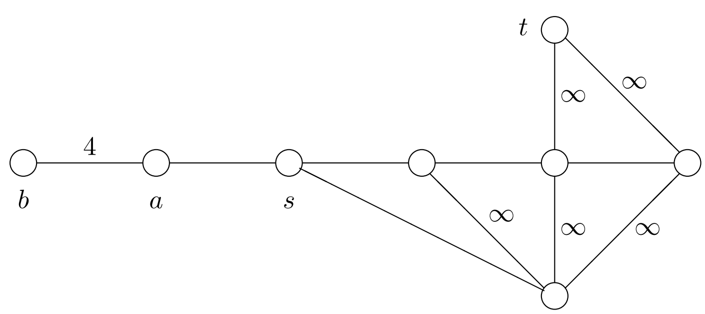

Figure 2: Example of a focus; here (a, b) is a focus of $M = S \setminus \{b, t\}$

Definition 7.2. Let $a, b \in M$ and suppose that $o(ab) = 2$. We say that $\{a, b\}$ is a *half-focus* of M in S, if the following conditions are all satisfied:

1. We have $o(ta) = o(tb) \in \{2, 3\}$ for every $t \in S \setminus \{a, b\}$ and $o(ta) = o(tb) = 2$ for every $t \in S \setminus M$. (In particular, the transposition that switches a and b is an automorphism of the Coxeter graph of (W, S).)

2. The odd graph $\Omega(M \setminus \{b\})$ is a tree and if $\{x, y\} \subseteq M$ is an edge of $\Omega(M)$, then $o(xy) = 3$.
3. For each $c \in M \setminus \{a, b\}$, the subset $D[a, b..c] \subseteq S$ consisting of b and all vertices in the unique path in $\Omega(M \setminus \{b\})$ joining c and a is of type D_k, where $k = |D[a, b..c]|$ (here type D_3 is regarded as type A_3), and the Coxeter graph for $(\langle D[a, b..c]\rangle, D[a, b..c])$ admits a non-trivial graph automorphism that switches a and b.
4. If $c, d \in M \setminus \{a, b\}$, $c \notin D[a, b..d]$ and $d \notin D[a, b..c]$, then $o(cd) = \infty$.
5. If $t \in S \setminus M$, $c \in M \setminus \{a, b\}$ and $o(tc) < \infty$, then $o(td) = 2$ for every $d \in D[a, b..c]$ (in particular $o(ta) = o(tc) = 2$ in this case).

Figure 3 shows an example of a half-focus. In the figure, $\{a, b\}$ is a half-focus of $M = S \setminus \{t, t'\}$.

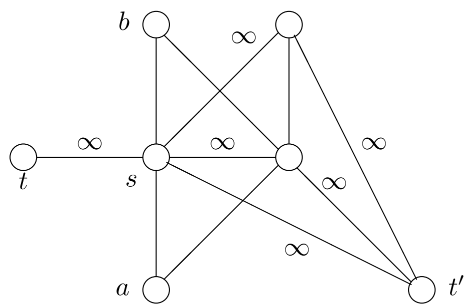

Figure 3: Example of a half-focus; here $\{a, b\}$ is a half-focus of $M = S \setminus \{t, t'\}$

Definition 7.3. Let $b \in S \setminus M$. We say that b is a C_3-*neighbor* of M in S, if we have $o(bt) \in \{2, 4\}$ for every $t \in M$, $o(bt) = 2$ for every $t \in S \setminus (M \cup \{b\})$, and for each $c \in M$ with $o(bc) = 4$, there exists an $a = a(b; c) \in M$ satisfying the followings:

1. We have $o(ca) = 3$, $o(ba) = 2$, and $o(ct) = \infty$ for every $t \in M \setminus \{a, c\}$.
2. For each $t \in S \setminus (M \cup \{b\})$, we have either $o(tc) = \infty$ or $o(tb) = o(tc) = o(ta) = 2$.

Figure 4 shows an example of a C_3-neighbor. In the figure, we put $s := a$, and b is a C_3-neighbor of $M = \{c, a, a', c'\}$, with $a(b; c) = a$ and $a(b; c') = a'$.

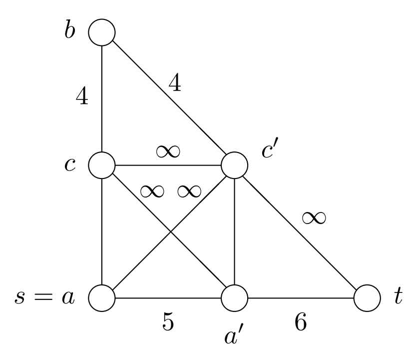

Figure 4: Example of a C_3-neighbor; here b is a C_3-neighbor of $M = \{c, a, a', c'\}$, with $a(b; c) = a$ and $a(b; c') = a'$

7.2 Some Properties

From now, we give some basic properties for the objects defined above, which will be used in our argument below.

Lemma 7.4. *Let $\{a,b\}$ be a half-focus of M in S. Then the set, say N, of neighbors of a in the odd graph $\Omega(M)$ is non-empty and equal to the set of neighbors of b in $\Omega(M)$. Moreover, we have $o(at) = o(bt) = 3$ for any $t \in N$, $D[b, a..c] = D[a, b..c]$ for each $c \in M \setminus \{a,b\}$, and we have $o(tt') = \infty$ for any distinct $t, t' \in N$. Finally, $\Omega(M \setminus \{a\})$ is a tree.*

Proof. This is easily deduced by the conditions in the definition of a half-focus (note that the assertion $N \neq \emptyset$ follows from the assumptions that $\{a,b\} \subseteq M$, $o(ab) = 2$ and $\Omega(M)$ is connected). □

Lemma 7.5. *Let $\{a,b\} \subseteq M$ be a half-focus of M in S. Suppose that $c \in M$, $o(ac) = 3 = o(bc)$, and $M = \{a,b,c\}$. Then there exists $e \in S$ satisfying $o(ae) = 2 = o(be)$ and $o(ec) = \infty$.*

Proof. Recall that $S = C_0(M)$ and S is not locally spherical by the assumption in this section; in particular, S is irreducible and $S \neq M$ (note that now M is of type A_3 by the assumption of this lemma). Hence there is a neighbor $e \in S \setminus M$ of the set M in the Coxeter graph of (W,S). Now Condition 1 in the definition of a half-focus implies that $o(ea) = o(eb) = 2$, therefore e must be adjacent to c in the graph, meaning that $o(ec) \geq 3$. As $e \in S \setminus M$, now Condition 5 (with $t = e$) implies that $o(ec) = \infty$ (otherwise we must have $o(ec) = 2$ which yields a contradiction), therefore this e satisfies the conditions in the assertion. This concludes the proof. □

Definition 7.6. Let $t \in M$. We say that t is *well-positioned* in S, if one of the following two conditions is satisfied:

- There is no C_3-neighbor of M in S.
- There are a C_3-neighbor b of M in S and an element $c \in M$ satisfying that $o(bc) = 4$ and t is the element $a = a(b;c)$ of M specified by the condition for b being a C_3-neighbor of M in S.

Lemma 7.7. *Let $t \in M$ be a well-positioned element in S. Then t commutes with every C_3-neighbor of M in S.*

Proof. The assertion is trivial if no C_3-neighbor of M in S exists; from now, we consider the other case. Let $b \in S \setminus M$ and $c \in M$ be as in the condition for t being well-positioned in S; hence $t = a := a(b;c)$. Then $t = a$ commutes with b by the definition of $a(b;c)$. On the other hand, for any other C_3-neighbor b' of M in S, we have $o(b'c) \in \{2,4\}$ since b' is a C_3-neighbor of M in S, therefore $o(b'b) = o(b'c) = o(b'a) = 2$ since b is a C_3-neighbor of M in S. Hence $t = a$ commutes with every C_3-neighbor of M in S, concluding the proof. □

8 s-Principal Coxeter Systems

This section summarizes some graph-theoretic arguments related to the objects introduced in the previous section, which are used in our proof of the main result. In this section, we put the same assumption as the previous section:

Assumption. Throughout this section, we suppose that (W,S) is a Coxeter system, M is an odd component of S, $s \in M$, $S = C_0(M)$, and S is not locally spherical.

8.1 s-Principal Subsets of S

First of all, we introduce the following terminology:

Definition 8.1. We say that a subset $I \subseteq S$ is *s-principal*, if I is not locally spherical, $s \in I$, $M \cap I$ is the odd component of I containing s, and I is the $(M \cap I)$-principal component of I.

We note that, due to the assumption above, the whole set S is also s-principal in the sense above.

The rest of this section is devoted to establish a kind of results, which intuitively says the following: If there are certain finitely many elements that are relevant to our argument, and if the whole set S admits some property about (in)existence of focuses, half-focuses or C_3-neighbors of M, then we can obtain a *finite* s-principal subset of S that involves all the finitely many relevant elements and also inherits from S the same property about (in)existence of focuses, half-focuses or C_3-neighbors. First we introduce some "closure" operations in order to obtain a new finite s-principal subset of S.

Definition 8.2. Let $I \subseteq S$ be a finite subset with $I \cap M \neq \emptyset$. We say that a subset $J \subseteq S$ is an *odd connection* of I, if J is finite, $I \subseteq J$, $J \setminus I \subseteq M$, and the odd graph $\Omega(M \cap J)$ is connected.

We note that an odd connection of such a set I always exists, since $\Omega(M)$ is connected.

Definition 8.3. Let $I \subseteq S$ be a finite non-empty subset. For each $t \in I \setminus M$, let $t_0 = t, t_1, \dots, t_k$ be a shortest path in the Coxeter graph from t to some element t_k of M. Moreover, for each $i = 0, 1, \dots, k-1$, let x_i be an element of M with $o(t_i x_i) < \infty$. Then we define an *even connection* of I to be the finite subset of S obtained by adding to I the elements $t_1, \dots, t_k$ and $x_0, \dots, x_{k-1}$ specified above for each element $t \in I \setminus M$.

We note that an even connection of such a set I always exists, since $S = C_0(M)$. Now the following property is deduced immediately from the definitions of odd connections and even connections:

Lemma 8.4. *Let $I \subseteq S$ be a finite non-spherical subset that contains s. Then an odd connection of an even connection of I is a finite s-principal subset of S. In particular, any finite subset of S is contained in a finite s-principal subset of S.*

Definition 8.5. For any finite subset I of S, we denote by $\mathcal{P}(I)$ the set of all finite s-principal subsets of S that contain I (which is non-empty by Lemma 8.4).

8.2 (Half-)Focuses and s-Principal Sets

Lemma 8.6. *Let $I \subseteq J \subseteq S$ be two s-principal subsets of S, let (a, b) be a focus of $M \cap I$ in I and let $a', b' \in J$. Then $\{a', b'\}$ is not a half-focus of $M \cap J$ in J and if (a', b') is a focus of $M \cap J$ in J, then $(a, b) = (a', b')$.*

Proof. As (a, b) is a focus of $M \cap I$ in I, we have that $b \in I \subseteq J$ and $b \notin M \cap I$, therefore $b \notin M$ and $b \in J \setminus (M \cap J)$. As J is s-principal, $a \in M \cap J$ and $o(ab) = 4$, it follows from Conditions 1 and 5 for a half-focus, that there cannot be a half-focus of $M \cap J$ in J. Therefore the first assertion holds and we assume from now on that (a', b') is a focus of $M \cap J$ in J. We have to show that $(a, b) = (a', b')$.

Assume, for the contrary, that $b \neq b'$. Then by Condition 4 for the focus (a', b') of $M \cap J$ in J with $t := b \in J \setminus ((M \cap J) \cup \{b'\})$ and $c := a \in M \cap J$, we have $o(ba) = 2$, which contradicts the property $o(ab) = 4$ mentioned above. Hence we have $b = b'$.

Now assume, for the contrary, that $a \neq a'$. Then by Condition 2 for the focus (a', b) of $M \cap J$ in J with $c := a \in M \cap J$, the set $C[b..a]$ is of type C_k with some $k \geq 3$, therefore we have $o(ab) = 2$, which contradicts the property $o(ab) = 4$ again. Hence we have $a = a'$. This concludes the proof. □

Proposition 8.7. *Let $I \subseteq J \subseteq S$ be two s-principal subsets of S, let $\{a, b\}$ be a half-focus of $M \cap I$ in I and let $a', b' \in J$. Then (a', b') is not a focus of $M \cap J$ in J and if $\{a', b'\}$ is a half-focus of $M \cap J$ in J, then $\{a, b\} = \{a', b'\}$.*

Proof. In the proof, for $K \in \{I, J\}$ we denote by $N_K(t)$ the set of neighbors of $t \in M \cap K$ in the graph $\Omega(M \cap K)$. By Lemma 7.4 applied to the half-focus $\{a, b\}$ of $M \cap I$ in I, we have $N_I(a) = N_I(b)$ and there exists an element c of $M \cap I$ with $o(ac) = 3 = o(bc)$. Note that $o(ab) = 2$ by the definition for the half-focus $\{a, b\}$ of $M \cap I$ in I. From now on, we fix this common neighbor c of a and b throughout this proof.

First, assume for the contrary that (a', b') is a focus of $M \cap J$ in J. As $\Omega(M \cap J)$ is a tree by Condition 1 for the focus (a', b') of $M \cap J$ in J, it follows that $N_I(a) = N_I(b)$ consists of a single element, therefore $N_I(a) = N_I(b) = \{c\}$. Now if $M \cap I = \{a, b, c\}$, then by Lemma 7.5 applied to the half-focus $\{a, b\}$ of $M \cap I$ in I, there exists $e \in I \setminus (M \cap I) \subseteq J \setminus (M \cap J)$ with $o(ae) = o(be) = 2$ and $o(ec) = \infty$. We have $e \neq b'$, as otherwise Condition 2 for the focus (a', b') of $M \cap J$ in J implies that $o(et) \in \{2, 4\}$ for any $t \in M \cap J$, and in particular $o(ec) \in \{2, 4\}$ which is a contradiction. Moreover, as (a, c, b) forms a path in the tree $\Omega(M \cap J)$, we must have either $c \in C[b'..a]$ or $c \in C[b'..b]$; we assume by symmetry that $c \in C[b'..a]$. Then Condition 4 for the focus (a', b') of $M \cap J$ in J (applied to $t := e$ and the set $C[b'..a]$) implies that $o(ec) = 2$. This is a contradiction. Hence we have $M \cap I \neq \{a, b, c\}$.

Now, as $\Omega(M \cap I)$ is connected and $N_I(a) = N_I(b) = \{c\}$ (see above), there exists an element d of $N_I(c)$ other than a and b. By Condition 3 for the half-focus $\{a, b\}$ of $M \cap I$ in I, we have $D[a, b..d] = \{a, c, b, d\}$, $o(ad) = o(bd) = 2$ and $o(cd) = 3$. Now by the shape of the subgraph $\Omega(\{a, c, b, d\})$ of the tree $\Omega(M \cap J)$, there must be two elements x, y among $\{a, b, d\}$ satisfying both $x \notin C[b'..y]$ and $y \notin C[b'..x]$. Then we have $o(xy) = \infty$ by Condition 3 for the focus (a', b') of $M \cap J$ in J, while $o(xy) = 2$ by the choice of x and y. This is a contradiction. Hence (a', b') is not a focus of $M \cap J$ in J, as desired.

From now on, we suppose that $\{a', b'\}$ is a half-focus of $M \cap J$ in J and prove that $\{a, b\} = \{a', b'\}$. First, by Lemma 7.4 applied to the half-focus $\{a', b'\}$ of $M \cap J$ in J, we have $N_J(a') = N_J(b')$. Now assume for the contrary that $\{a, b\} \neq \{a', b'\}$. By symmetry, we assume without loss of generality that $b' \notin \{a, b\}$. Now we have $o(ab) = 2$ (see the first paragraph of the proof) and $a, b \in (M \cap I) \setminus \{b'\} \subseteq (M \cap J) \setminus \{b'\}$. This implies that $\{a, b\} \not\subseteq N_J(b')$, as otherwise we must have $o(ab) = \infty$ by Lemma 7.4 applied to the half-focus $\{a', b'\}$ of $M \cap J$ in J (with $t := a$ and $t' := b$) which is a contradiction. In particular $b' \neq c$. Moreover, there cannot be an element t of $N_I(a) = N_I(b)$ different from c and b', as otherwise $\{c, t\} \subseteq N_I(a) \cap N_I(b) \subseteq N_J(a) \cap N_J(b)$ and $\Omega((M \cap J) \setminus \{b'\})$ contains a cycle but this contradicts Condition 2 (saying that $\Omega((M \cap J) \setminus \{b'\})$ is a tree) for the half-focus $\{a', b'\}$ of $M \cap J$ in J. Hence we have $N_I(a) = N_I(b) = \{c\}$.

As $o(ab) = 2$ and $a, b \in (M \cap J) \setminus \{b'\}$ (see above), by Condition 4 for the half-focus $\{a', b'\}$ of $M \cap J$ in J, we must have either $b \neq a'$ and $a \in D[a', b'..b]$ (including the case $a = a'$), or $a \neq a'$ and $b \in D[a', b'..a]$ (including the case $b = a'$). By symmetry, we assume without loss of generality that $b \neq a'$ and $a \in D[a', b'..b]$. Now the set $D[a', b'..b] \setminus \{b'\}$ forms a path from b to a' in $\Omega((M \cap J) \setminus \{b'\})$ that contains a, while $c \neq b'$ and $c \in N_I(a) \cap N_I(b) \subseteq N_J(a) \cap N_J(b)$. As $\Omega((M \cap J) \setminus \{b'\})$ is acyclic, it follows that the first three vertices of this path $D[a', b'..b] \setminus \{b'\}$ must be (b, c, a). In particular, we have $c \in D[a', b'..b]$.

Now Condition 5 for the half-focus $\{a', b'\}$ of $M \cap J$ in J (applied to the $D[a', b'..b]$) implies that there cannot be an element $t \in I \setminus (M \cap I) \subseteq J \setminus (M \cap J)$ with $o(tb) = 2$ and $o(tc) = \infty$. Hence we have $M \cap I \neq \{a, b, c\}$ by Lemma 7.5 applied to the half-focus $\{a, b\}$ of $M \cap I$ in I. As $\Omega(M \cap I)$ is connected and $N_I(a) = N_I(b) = \{c\}$ as shown above, there must be an element d of $N_I(c)$ other than a and b. Now if such d were different from b', then by the shape of $D[a', b'..b]$ described above, we have $d \neq a'$ and $D[a', b'..d] = (D[a', b'..b] \setminus \{b\}) \cup \{d\}$, therefore $b \notin D[a', b'..d]$ and $d \notin D[a', b'..b]$. As $b, d \notin \{a', b'\}$ (see above), we have $o(bd) = \infty$ by Condition 4 for the half-focus $\{a', b'\}$ of $M \cap J$ in J, which contradicts Condition 1 for the half-focus $\{a, b\}$ of $M \cap I$ in I as $d \in I \setminus \{a, b\}$. Hence any such d must be b', which implies that $N_I(c) = \{a, b, b'\}$ and therefore $b' \in I$, $a = a'$ and $D[a', b'..b] = \{b, c, a, b'\}$ is of type D_4.

Now by both Condition 2 and Lemma 7.4 for the half-focuses $\{a, b\}$ of $M \cap I$ in I and $\{a, b'\}$ of $M \cap J$ in J, respectively, it follows that both $\Omega((M \cap I) \setminus \{b\})$ and $\Omega((M \cap I) \setminus \{b'\})$ are acyclic, and $N_I(b) = N_I(a) = N_I(b')$. Hence we must have $N_I(b) = N_I(a) = N_I(b') = \{c\}$, while $N_I(c) = \{a, b, b'\}$ as above, therefore $M \cap I = \{b, c, a, b'\}$ which is of type D_4. On the other hand, I is irreducible and not locally spherical as I is an s-principal subset of S. Hence we have $I \neq M \cap I$ and there is an element $t \in I \setminus (M \cap I) \subseteq J \setminus (M \cap J)$ with $t \notin (M \cap I)^{\perp}$. Now we have $o(tb) = 2$ by Condition 1 for the half-focus $\{a, b\}$ of $M \cap I$ in I, therefore $t \in D[a', b'..b]^{\perp} = (M \cap I)^{\perp}$ (recall that now $D[a', b'..b] = M \cap I$) by Condition 5 for the half-focus $\{a', b'\}$ of $M \cap J$ in J. This is a contradiction. Hence we have $\{a, b\} = \{a', b'\}$. This concludes the proof. $\square$

Now we start to give the results mentioned at the beginning of Section 8.1. We recall Definition 8.5 for the set $\mathcal{P}(I)$, which is in particular non-empty.

Lemma 8.8. *Let $a \in M$ and $b \in S \setminus M$, and suppose that (a, b) is a focus of M in S. Then for any finite subset I of S, there exists a finite s-principal subset $I^\dagger$ of S satisfying that $a \in M \cap I^\dagger$, $b \in I^\dagger$, $I \subseteq I^\dagger$, and (a, b) is a focus of $M \cap I^\dagger$ in $I^\dagger$.*

Proof. First we replace I with some (finite) set in $\mathcal{P}(I \cup \{a, b\})$, which does not affect the assertion. Now since $\Omega(M)$ is a tree by the definition of a focus, its connected subgraph $\Omega(M \cap I)$ is also a tree. Moreover, now for each $c \in M \cap I$, the set $C[b..c]$ with respect to the pair of I and $M \cap I$ coincides with that with respect to the pair of S and M. Therefore, all the other conditions for (a, b) being a focus of $M \cap I$ in I follow immediately from the corresponding conditions for (a, b) being a focus of M in S. Hence the assertion holds. □

Lemma 8.9. *Let $a, b \in M$, and suppose that $\{a, b\}$ is a half-focus of M in S. Then for any finite subset I of S, there exists a finite s-principal subset $I^\dagger$ of S satisfying that $a, b \in M \cap I^\dagger$, $I \subseteq I^\dagger$, and $\{a, b\}$ is a half-focus of $M \cap I^\dagger$ in $I^\dagger$.*

Proof. First we replace I with some set in $\mathcal{P}(I \cup \{a, b\})$, which does not affect the assertion. Moreover, since $\Omega(M \setminus \{b\})$ is connected by the definition of a half-focus, we can add finitely many elements of M to I in a way that $\Omega((M \cap I) \setminus \{b\})$ becomes connected; we note that this process does not violate the property of I being s-principal. Now since $\Omega(M \setminus \{b\})$ is a tree by the definition of a half-focus, its connected subgraph $\Omega((M \cap I) \setminus \{b\})$ is also a tree. Moreover, now for each $c \in (M \cap I) \setminus \{a, b\}$, the set $D[a, b..c]$ with respect to the pair of I and $M \cap I$ coincides with that with respect to the pair of S and M. Therefore, all the other conditions for $\{a, b\}$ being a half-focus of $M \cap I$ in I follow immediately from the corresponding conditions for $\{a, b\}$ being a half-focus of M in S. Hence the assertion holds. □

Lemma 8.10. *Suppose that there is no focus and no half-focus of M in S. Then for any finite subset I of S, there exists a finite s-principal subset $I^\dagger$ of S satisfying that $I \subseteq I^\dagger$ and there is no focus and no half-focus of $M \cap I^\dagger$ in $I^\dagger$.*

Proof. First we replace I with some set in $\mathcal{P}(I)$, which does not affect the assertion. We note that the assertion holds if there is no focus and no half-focus of $M \cap I$ in I.

First, we consider the case that there is a focus (a, b) of $M \cap I$ in I. By the assumption that (a, b) is not a focus of M in S, some of the conditions in the definition is not satisfied. Now we can take a finite subset J of S in the following manner to ensure that, for $I^\dagger \in \mathcal{P}(I \cup J)$, (a, b) is not a focus of $M \cap I^\dagger$ in $I^\dagger$:

- If the odd graph $\Omega(M)$ is not a tree, then $\Omega(M)$ contains a cycle since $\Omega(M)$ is connected. Now we set J to be the vertex set of this cycle; then the same cycle violates Condition 1 for (a, b) to be a focus of $M \cap I^\dagger$ in $I^\dagger$.
- If an element $c \in M$ violates Condition 2 for (a, b) to be a focus of M in S, then we set $J := C[b..c]$. Now c also violates Condition 2 for (a, b) to be a focus of $M \cap I^\dagger$ in $I^\dagger$.
- If two elements $c, d \in M$ violate Condition 3 for (a, b) to be a focus of M in S, then we set $J := C[b..c] \cup C[b..d]$. Now c and d also violate Condition 3 for (a, b) to be a focus of $M \cap I^\dagger$ in $I^\dagger$.
- Finally, if two elements $t \in S \setminus (M \cup \{b\})$ and $c \in M$ violate Condition 4 for (a, b) to be a focus of M in S, then we set $J := C[b..c] \cup \{t\}$. Now t and c also violate Condition 4 for (a, b) to be a focus of $M \cap I^\dagger$ in $I^\dagger$.

Moreover, Lemma 8.6 (applied to $J := I^\dagger$) implies that there is no focus of $M \cap I^\dagger$ in $I^\dagger$ different from (a, b) and there is no half-focus of $M \cap I^\dagger$ in $I^\dagger$. Hence there is no focus and no half-focus of $M \cap I^\dagger$ in $I^\dagger$, as desired.

Similarly, we consider the other case that there is a half-focus $\{a, b\}$ of $M \cap I$ in I. By the assumption that $\{a, b\}$ is not a half-focus of M in S, some of the conditions in the definition is not satisfied. Now we can also take a finite subset J of S in the following manner to ensure that, for $I^\dagger \in \mathcal{P}(I \cup J)$, $\{a,b\}$ is not a half-focus of $M \cap I^\dagger$ in $I^\dagger$:

- If an element $t \in S \setminus M$ violates Condition 1 for $\{a,b\}$ to be a half-focus of M in S, then we let $J := \{t\}$.
- If $\Omega(M \setminus \{b\})$ is not connected, then we let J be the set of two elements of M that are in distinct connected components of $\Omega(M \setminus \{b\})$.
- If $\Omega(M \setminus \{b\})$ contains a cycle, then we let J be the vertex set of this cycle.
- If an element $c \in M \setminus \{b\}$ violates Condition 3 for $\{a,b\}$ to be a half-focus of M in S, then we let $J := D[a, b..c]$.
- If elements $c, d \in M \setminus \{a,b\}$ violate Condition 4 for $\{a,b\}$ to be a half-focus of M in S, then we let $J := D[a, b..c] \cup D[a, b..d]$.
- Finally, if elements $t \in S \setminus M$ and $c \in M \setminus \{b\}$ violate Condition 5 for $\{a,b\}$ to be a half-focus of M in S, then we let $J := \{t\} \cup D[a, b..c]$.

Moreover, Proposition 8.7 (applied to $J := I^\dagger$) implies that there is no half-focus of $M \cap I^\dagger$ in $I^\dagger$ different from $\{a,b\}$ and there is no focus of $M \cap I^\dagger$ in $I^\dagger$. Hence there is no focus and no half-focus of $M \cap I^\dagger$ in $I^\dagger$, as desired. This concludes the proof. □

8.3 C_3-Neighbors and s-Principal Sets

For the next result, recall from Section 7.2 the definition of a well-positioned element.

Lemma 8.11. *Let $b \in S \setminus M$, and suppose that b is a C_3-neighbor of M in S. Let $s_0 \in M$ be a well-positioned element in S. Then for any finite subset I of S, there exists a finite s-principal subset $I^\dagger$ of S satisfying that $s_0 \in M \cap I^\dagger$, $b \in I^\dagger$, $I \subseteq I^\dagger$, s_0 is well-positioned in $I^\dagger$, and b is a C_3-neighbor of $M \cap I^\dagger$ in $I^\dagger$.*

Proof. Let $b' \in S \setminus M$ and $c' \in M$ be the elements specified in the condition for s_0 being well-positioned in S. We replace I with some set in $\mathcal{P}(I \cup \{s_0, b, b', c'\})$, which does not affect the assertion.

First, we add to I an element $a'' = a(b'; c'')$ specified in the definition of a C_3-neighbor for M in S for each of (finitely many) generators $c'' \in (M \cap I) \setminus \{c'\}$ with $o(b'c'') = 4$. Since $o(c''a'') = 3$ for any such pair (c'', a''), the odd graph $\Omega(M \cap I)$ remains connected during this process, therefore this process does not violate the property of I being s-principal. Moreover, since $o(b'a'') = 2$ for any such pair (c'', a''), this process does not generate any further element $\overline{c} \in M \cap I$ with $o(b'\overline{c}) = 4$. Therefore, all the conditions for b' being a C_3-neighbor of $M \cap I$ in I and for s_0 being the element $a(b'; c')$ with respect to I follow immediately from the corresponding conditions for b' being a C_3-neighbor of M in S and for s_0 being the element $a(b'; c')$ with respect to S. Now the assertion holds when $b' = b$; from now, we consider the other case $b' \neq b$.

In this case, we moreover add to I an element $a = a(b; c) \in M$ specified in the definition of a C_3-neighbor for M in S for each of (finitely many) generators $c \in M \cap I$ with $o(bc) = 4$. Since $o(ca) = 3$ for any such pair (c, a), the odd graph $\Omega(M \cap I)$ remains connected during this process, therefore this process does not violate the property of I being s-principal. Moreover, the definition of a C_3-neighbor implies that we have $o(ba) = 2$ and $o(b'c) = o(b'a) = 2$ for any such pair (c, a), therefore this process does not generate any further element $\overline{c} \in M \cap I$ with $o(b\overline{c}) = 4$ or $o(b'\overline{c}) = 4$. Hence, b' is still a C_3-neighbor of $M \cap I$ in I, s_0 is still well-positioned in I, and all the conditions for b being a C_3-neighbor of $M \cap I$ in I follow immediately from the corresponding conditions for b being a C_3-neighbor of M in S. This concludes the proof. □

Lemma 8.12. *Let $b \in S \setminus M$, and suppose that b is not a C_3-neighbor of M in S. Then for any finite subset I of S, there exists an s-principal subset $I^\dagger$ of S satisfying that $b \in I^\dagger$, $I \subseteq I^\dagger$, and b is not a C_3-neighbor of $M \cap I^\dagger$ in $I^\dagger$.*

Proof. At the beginning we add the element b to the set I. Now, since b is not a C_3-neighbor, some of the conditions for a C_3-neighbor of M in S is not satisfied. First, if $o(bt) \notin \{2,4\}$ for some $t \in M$ or if $o(bt) \neq 2$ for some $t \in S \setminus (M \cup \{b\})$, then we add to I the element t. Secondly, suppose that for some $c \in M$ with $o(bc) = 4$, the number of elements $d \in M \setminus \{c\}$ with $o(cd) < \infty$ is either zero or at least two. In this case, we add to I the element c and, in the latter situation above, also add to I two distinct elements $d_1, d_2 \in M \setminus \{c\}$ with $o(cd_1), o(cd_2) < \infty$. Finally, suppose that for some $c \in M$ with $o(bc) = 4$, there is precisely one element $a \in M \setminus \{c\}$ with $o(ca) < \infty$, and for this element a, we have one of the following properties; $o(ca) \neq 3$; $o(ba) \neq 2$; or there is an element $t \in S \setminus (M \cup \{b\})$ satisfying that $o(tc) < \infty$ and at least one of $o(tb)$, $o(tc)$, and $o(ta)$ is not equal to 2. In this case, we add to I the elements c, a and, in the last situation above, also add to I the element t. We note that, since b is not a C_3-neighbor of M in S, some of the situations considered above indeed happens.

After the process of expanding the set I as above, and by taking some $I^\dagger \in \mathcal{P}(I)$ further, the process above implies that, the elements that were added to the set I also violate the corresponding condition for b being a C_3-neighbor of $M \cap I^\dagger$ in $I^\dagger$. This concludes the proof. □

9 On the Main Part of the Locally Finite Continuation

In this section, we state and prove the following result, which in combination with Theorem 6.4 determines the structure of the locally finite continuation of a reflection in a Coxeter group of arbitrary rank:

Theorem 9.1. *Let (W, S) be a Coxeter system, M be an odd component of S, and $s \in M$. Suppose that $S = C_0(M)$ and S is not locally spherical. Then precisely one of the following situations happens:*

1. *There is a focus (a, b) of M in S. Now we have $\mathrm{LFC}(a) = \langle a, b \rangle$, $o(ab) = 4$, and $S^W \cap \mathrm{LFC}(a)$ contains no element that commutes with a and is not conjugate to a in $\mathrm{LFC}(a)$.*
2. *There is a half-focus $\{a, b\}$ of M in S. Now we have $\mathrm{LFC}(a) = \langle a, b \rangle$, $o(ab) = 2$, and any element of $S^W \cap \mathrm{LFC}(a)$ is conjugate to a in W.*
3. *There are no focuses and no half-focuses of M in S. Now there is an element $s_0 \in M$ that is well-positioned in S (see Section 7.2 for the terminology); and for any such element s_0, $\mathrm{LFC}(s_0)$ is an elementary abelian 2-group generated by s_0 and all the C_3-neighbors of M in S, and s_0 is the unique element of $S^W \cap \mathrm{LFC}(s_0)$ that is conjugate to s_0 in W.*

Before proving the theorem, we state an immediate corollary of Theorem 9.1 and Theorem 6.4.

Corollary 9.2. *Let (W, S) be an arbitrary Coxeter system, and let $M \subseteq S$ be an odd component of S. Then there exists an element $s \in M$ satisfying the following conditions:*

- *$\mathrm{LFC}_W(s)$ is a visible subgroup of W; say, $\mathrm{LFC}_W(s) = \langle I \rangle$ for $I \subseteq S$.*
- *Let $\Sigma(M)$ denote the union of all locally spherical M-subsidiary components of $E(M)$. Then we have $\Sigma(M) \subseteq I$, $I \setminus \Sigma(M)$ is a direct factor of I, and $\langle I \setminus \Sigma(M) \rangle = \mathrm{LFC}_{\langle C_0(M) \rangle}(s)$.*
- *If $C_0(M)$ is locally spherical, then $I \setminus \Sigma(M) = C_0(M)$. If $C_0(M)$ is not locally spherical, then one of the following conditions is satisfied;*
  1. *$I \setminus \Sigma(M)$ is of type C_2;*
  2. *$I \setminus \Sigma(M)$ is of type $A_1 \times A_1$ and the two generators in $I \setminus \Sigma(M)$ are conjugate to each other in W;*
  3. *$I \setminus \Sigma(M)$ consists of s and all the C_3-neighbors of M in $C_0(M)$, and all elements of $I \setminus \Sigma(M)$ commute with each other.*

In particular, the locally finite continuation of a reflection has the following property, which is obvious by the definition of finite continuations in the finite rank cases studied in [11] but is never obvious solely by the definition of locally finite continuations.

Corollary 9.3. *Let (W, S) be an arbitrary Coxeter system. Then the locally finite continuation $\mathrm{LFC}_W(r)$ of any reflection r in W is always (not just locally parabolic, but also) a parabolic subgroup of W.*

We also have another consequence of the structure of the locally finite continuation determined by Theorem 9.1 and Theorem 6.4; this is a straightforward generalization of Corollary 9.7 in [13] originally shown for finite continuations in finite rank cases, and the same proof as [13] works for the present case.

Corollary 9.4. *Let (W, S) be a Coxeter system of arbitrary rank, and let M be an odd component of (W, S). Let $s \in M$, and suppose that $\mathrm{LFC}(s) = \langle J \rangle$ for some $J \subseteq S$. If $Z(\mathrm{LFC}(s)) = \langle s \rangle$, then each component of $(\langle J \rangle, J)$ of rank at least two is an M-subsidiary component of $E(M)$.*

Now we start to prove Theorem 9.1. The important starting point is that the theorem specialized to the finite rank cases has been proved in [11] (though the description of the statement is slightly different) and we can efficiently utilize the result for the finite rank cases in our argument below:

Proposition 9.5. *Theorem 9.1 is true for the cases where (W, S) has finite rank.*

Proof. This is an immediate consequence of the main result (Theorem 7) of [11] for the finite rank cases. □

In the rest of this section, we put the same assumption as Theorem 9.1 and also focus on the infinite rank cases:

Assumption. In the rest of this section, we suppose that (W, S) is a Coxeter system, M is an odd component of S, $s \in M$, $S = C_0(M)$, $|S| = \infty$, and S is not locally spherical.

Lemma 9.6. *If (a, b) is a focus (respectively, $\{a, b\}$ is a half-focus) of M in S, then we have $\mathrm{LFC}(a) = \langle a, b \rangle$.*

Proof. First, to show that $b \in \mathrm{LFC}(a)$, let $Z \subseteq W$ be any subset that contains a and generates a locally finite subgroup. In order to show that $\langle Z \cup \{b\} \rangle$ is locally finite (which will imply $b \in \mathrm{LFC}(a)$ by Definition 3.1), it suffices to prove that $|\langle Z' \cup \{b\} \rangle| < \infty$ for any finite subset Z' of Z. We may assume without loss of generality that $a \in Z'$. Now there is a finite subset I of S satisfying $Z' \subseteq \langle I \rangle$, and for this I, we can take a finite subset $I^\dagger \subseteq S$ as in Lemma 8.8 (respectively, Lemma 8.9). Then, since (a, b) is a focus (respectively, $\{a, b\}$ is a half-focus) of $M \cap I^\dagger$ in $I^\dagger$, Theorem 9.1 for the finite rank cases implies that $b \in \mathrm{LFC}_{\langle I^\dagger \rangle}(a)$. Since $a \in Z' \subseteq \langle I \rangle \subseteq \langle I^\dagger \rangle$ and $\langle Z' \rangle$ is finite by the choice of Z, it follows that $\langle Z' \cup \{b\} \rangle$ is also finite, as desired. Hence we have $b \in \mathrm{LFC}(a)$, therefore $\langle a, b \rangle \subseteq \mathrm{LFC}(a)$.

On the other hand, let $w \in \mathrm{LFC}(a)$. We can take a finite subset $I \subseteq S$ satisfying $w \in \langle I \rangle$, and for this I, we can take a finite subset $I^\dagger \subseteq S$ as in Lemma 8.8 (respectively, Lemma 8.9). Then, since (a, b) is a focus (respectively, $\{a, b\}$ is a half-focus) of $M \cap I^\dagger$ in $I^\dagger$, Theorem 9.1 for the finite rank cases implies that $\mathrm{LFC}_{\langle I^\dagger \rangle}(a) = \langle a, b \rangle$, and we moreover have $w \in \mathrm{LFC}(a) \cap \langle I^\dagger \rangle \subseteq \mathrm{LFC}_{\langle I^\dagger \rangle}(a) = \langle a, b \rangle$ by Lemma 3.9. Hence we have $\mathrm{LFC}(a) \subseteq \langle a, b \rangle$ and therefore $\mathrm{LFC}(a) = \langle a, b \rangle$, concluding the proof. □

By virtue of Lemma 9.6, hereafter we focus on the case where there is no focus and no half-focus of M in S. We write $t^G = \{t^w \mid w \in G\}$ for any $t \in W$ and $G \leq W$.

Lemma 9.7. *Suppose that there is no focus and no half-focus of M in S. If $s_0 \in M$ is a well-positioned element in S, and if b is a C_3-neighbor of M in S, then we have $b \in \mathrm{LFC}(s_0)$.*

Proof. Let $Z \subseteq W$ be any subset that contains s_0 and generates a locally finite subgroup. In order to show that $\langle Z \cup \{b\} \rangle$ is locally finite (which will imply $b \in \mathrm{LFC}(s_0)$), it suffices to prove that $|\langle Z' \cup \{b\} \rangle| < \infty$ for any finite subset Z' of Z. We may assume without loss of generality that $s_0 \in Z'$. Now there is a finite subset I of S satisfying $Z' \subseteq \langle I \rangle$, and for this I, we can take a finite subset $I^\dagger \subseteq S$ as in Lemma 8.11. Then, since s_0 is well-positioned in $I^\dagger$ and b is a C_3-neighbor of $M \cap I^\dagger$ in $I^\dagger$, Theorem 9.1 for the finite rank cases implies that $b \in \mathrm{LFC}_{\langle I^\dagger \rangle}(s_0)$. Since $s_0 \in Z' \subseteq \langle I \rangle \subseteq \langle I^\dagger \rangle$ and $\langle Z' \rangle$ is finite by the choice of Z, it follows that $\langle Z' \cup \{b\} \rangle$ is also finite, as desired. Hence we have $b \in \mathrm{LFC}(s_0)$. This concludes the proof. □

Lemma 9.8. *Suppose that there is no focus and no half-focus of M in S. Let $s_0 \in M$ be a well-positioned element in S, and let J denote the set of all C_3-neighbors of M in S. Then we have* $\mathrm{LFC}(s_0) = \langle J \cup \{s_0\} \rangle$.

Proof. Since $\langle J \cup \{s_0\} \rangle \subseteq \mathrm{LFC}(s_0)$ by Lemma 9.7, it suffices to show that $\mathrm{LFC}(s_0) \subseteq \langle J \cup \{s_0\} \rangle$. Assume for the contrary that $\mathrm{LFC}(s_0) \not\subseteq \langle J \cup \{s_0\} \rangle$. Since $\mathrm{LFC}(s_0)$ is a reflection subgroup of W by Proposition 5.1, there is a reflection $r \in \mathrm{LFC}(s_0) \setminus \langle J \cup \{s_0\} \rangle$.

By Lemma 8.10, there is a finite s-principal subset $I \subseteq S$ satisfying that $s_0 \in M \cap I$, r is a reflection of $(\langle I \rangle, I)$, and there is no focus and no half-focus of $M \cap I$ in I. Now Theorem 9.1 for the finite rank cases implies the following: There is a well-positioned element $s_1 \in M \cap I$ in I, and $\mathrm{LFC}_{\langle I \rangle}(s_1)$ is an elementary abelian 2-group generated by s_1 and the set (say, J_1) of all C_3-neighbors of $M \cap I$ in I. In particular, all vertices in $\{s_1\} \cup J_1$ are mutually isolated in the Coxeter graph of $(\langle I \rangle, I)$, and we have $J_1 \cap M = \emptyset$ by the definition for C_3-neighbors of $M \cap I$ in I.

As $\Omega(M \cap I)$ is connected and $s_0, s_1 \in M \cap I$, there is an element $w \in \langle M \cap I \rangle$ with $s_1 = {s_0}^w$. Now for any $t \in \{r\} \cup (J \cap I)$, we have $t^w \in \mathrm{LFC}(s_0)^w = \mathrm{LFC}(s_1)$ by Lemma 3.7 (4), and $t^w \in \mathrm{LFC}(s_1) \cap \langle I \rangle \subseteq \mathrm{LFC}_{\langle I \rangle}(s_1) = \langle \{s_1\} \cup J_1 \rangle$ by Lemma 3.9. Hence by Proposition 4.17, the reflection t^w of (W, S) is also a reflection of $(\langle \{s_1\} \cup J_1 \rangle, \{s_1\} \cup J_1)$. By the shape of $\{s_1\} \cup J_1$ mentioned above, it follows that $t^w \in \{s_1\} \cup J_1$, and in particular $t^w \in J_1$ since $t \neq s_0$. Moreover, when $t \in J \cap I$, we also have $t \in \mathrm{supp}(t^w)$ by Lemma 4.7 since $t \notin M$ (by the definition of C_3-neighbors of M in S) and $w \in \langle M \rangle$, therefore we have $t^w = t$. Now since $r \notin J$, we have $r^w \neq t^w = t$ for any $t \in J \cap I$, therefore $r^w \notin J \cap I$ and (since $r^w \in I$) we have $r^w \notin J$; that is, $b_0 := r^w \in J_1 \subseteq I \setminus M$ is not a C_3-neighbor of M in S. Note that $o(b_0 s_1) = 2$ by the shape of $\{s_1\} \cup J_1$ mentioned above.

By Lemma 8.12, there is a finite s-principal subset $K \subseteq S$ satisfying that $I \subseteq K$ and b_0 is not a C_3-neighbor of $M \cap K$ in K. Now we have $b_0 \in \mathrm{LFC}(s_1) \cap \langle K \rangle \subseteq \mathrm{LFC}_{\langle K \rangle}(s_1)$ by Lemma 3.9. Moreover, by Theorem 9.1 for the finite rank cases, one of the following three possibilities as in the theorem happens where $(\langle K \rangle, K)$, $M \cap K$, and s_1 play the roles of (W, S), M, and s, respectively:

- There is a focus (a, b) of $M \cap K$ in K; and $K^{\langle K \rangle} \cap \mathrm{LFC}_{\langle K \rangle}(a)$ contains no element that commutes with a and is not conjugate to a in $\mathrm{LFC}_{\langle K \rangle}(a)$. In this case, as $\Omega(M \cap K)$ is connected and $s_1, a \in M \cap K$, there is an element $u \in \langle M \cap K \rangle$ with $a = {s_1}^u$. Now we have ${b_0}^u \in \mathrm{LFC}_{\langle K \rangle}(s_1)^u = \mathrm{LFC}_{\langle K \rangle}(a)$ by Lemma 3.7 (4); we have $o({b_0}^u a) = o(b_0 s_1) = 2$; and ${b_0}^u$ is a reflection of $(\langle K \rangle, K)$ and is not conjugate to a (as $b_0 \in S \setminus M$ is not conjugate to $s_1 \in M$). The existence of such an element ${b_0}^u$ yields a contradiction.

- There is a half-focus $\{a, b\}$ of $M \cap K$ in K; and any element of $K^{\langle K \rangle} \cap \mathrm{LFC}_{\langle K \rangle}(a)$ is conjugate to a in $\langle K \rangle$. In this case, as $\Omega(M \cap K)$ is connected and $s_1, a \in M \cap K$, there is an element $u \in \langle M \cap K \rangle$ with $a = {s_1}^u$. Now we have ${b_0}^u \in \mathrm{LFC}_{\langle K \rangle}(s_1)^u = \mathrm{LFC}_{\langle K \rangle}(a)$ by Lemma 3.7 (4); and ${b_0}^u$ is a reflection of $(\langle K \rangle, K)$ and is not conjugate to a (as $b_0 \in S \setminus M$ is not conjugate to $s_1 \in M$). The existence of such an element ${b_0}^u$ yields a contradiction.

- There are no focuses and no half-focuses of $M \cap K$ in K; there is an element $s_2 \in M \cap K$ that is well-positioned in K; $\mathrm{LFC}_{\langle K \rangle}(s_2)$ is an elementary abelian 2-group generated by s_2 and the set (say, J_2) of all the C_3-neighbors of $M \cap K$ in K. In this case, as $\Omega(M \cap K)$ is connected and $s_1, s_2 \in M \cap K$, there is an element $u \in \langle M \cap K \rangle$ with $s_2 = {s_1}^u$. Now we have ${b_0}^u \in \mathrm{LFC}_{\langle K \rangle}(s_1)^u = \mathrm{LFC}_{\langle K \rangle}(s_2)$ by Lemma 3.7 (4). By a similar argument based on the property $\mathrm{LFC}_{\langle K \rangle}(s_2) = \langle \{s_2\} \cup J_2 \rangle$ and Proposition 4.17 used above, it follows that ${b_0}^u$ is a reflection of $(\langle \{s_2\} \cup J_2 \rangle, \{s_2\} \cup J_2)$ and therefore is an element of J_2 (note that ${b_0}^u \neq s_2$ as $b_0 \neq s_1$). Moreover, as $b_0 \notin M$ and $u \in \langle M \rangle$, we have $b_0 \in \mathrm{supp}({b_0}^u)$ by Lemma 4.7 and therefore $b_0 = {b_0}^u \in J_2$. This contradicts the fact that b_0 is not a C_3-neighbor of $M \cap K$ in K.

Hence a contradiction happens in any possible case, therefore we have $\mathrm{LFC}(s_0) \subseteq \langle J \cup \{s_0\} \rangle$, as desired. This concludes the proof. □

Proof of Theorem 9.1. First of all, we mention that two or more cases among the three cases in Theorem 9.1 cannot happen simultaneously. It is obvious that the third case (with no focus nor half-focus) cannot be consistent with the first case (with a focus) nor the second case (with a half-focus). On the other hand, when the first case happens, Lemma 8.6 (applied to $I = J := S$) implies that there cannot be a half-focus, preventing the second case. This proves the assertion in this paragraph.

If there is a focus (a, b) of M in S, then we have $\mathrm{LFC}(a) = \langle a, b\rangle$ by Lemma 9.6. Moreover, we have $o(ab) = 4$ by Condition 2 for the focus (a, b) of M in S, therefore $(\mathrm{LFC}(a), \{a, b\})$ is a Coxeter system of type C_2. Now an exhaustive search shows that the reflections in $\mathrm{LFC}(a)$ that commute with a are a and a^b, both being conjugate to a in $\mathrm{LFC}(a)$. Hence we are in the first case of Theorem 9.1.

If there is a half-focus $\{a, b\}$ of M in S, then we have $\mathrm{LFC}(a) = \langle a, b\rangle$ by Lemma 9.6. Moreover, by the definition for the half-focus $\{a, b\}$ of M in S, we have $o(ab) = 2$ and $a, b \in M$. It also follows that the reflections in $\mathrm{LFC}(a)$ are a and b, both being conjugate to a in W as $a, b \in M$. Hence we are in the second case of Theorem 9.1.

Finally, we consider the case where there is no focus and no half-focus of M in S. First we show that there exists an element of M that is well-positioned in S. By definition, this is trivial if there is no C_3-neighbor of M in S. On the other hand, suppose that there is a C_3-neighbor b of M in S. Now by definition, a well-positioned element exists if $o(bc) = 4$ for some $c \in M$. Assume for the contrary that $o(bc) \neq 4$ for any $c \in M$. Then by the definition for the C_3-neighbor b of M in S, we have $S \setminus (M \cup \{b\}) \subseteq \{b\}^{\perp}$ and moreover $M \subseteq \{b\}^{\perp}$, therefore $S \setminus \{b\} \subseteq \{b\}^{\perp}$. This contradicts the assumption that S is irreducible (note that $M \neq \emptyset$ and $b \in S \setminus M$, therefore $S \neq \{b\}$). Hence there exists an element s_0 of M that is well-positioned in S.

For any such s_0, Lemma 9.8 implies that $\mathrm{LFC}(s_0) = \langle J \cup \{s_0\}\rangle$ where J is the set of all C_3-neighbors of M in S. By the definition of C_3-neighbors of M in S, we have $J \subseteq S \setminus M$ and therefore $o(bb') = 2$ for any distinct $b, b' \in J$. Moreover, we have $J \subseteq \{s_0\}^{\perp}$ by Lemma 7.7. This implies that all elements of $J \cup \{s_0\}$ commute with each other and hence $\mathrm{LFC}(s_0) = \langle J \cup \{s_0\}\rangle$ is an elementary abelian 2-group. Finally, this structure of $\mathrm{LFC}(s_0)$ implies that the reflections in $\mathrm{LFC}(s_0)$ are the elements of $J \cup \{s_0\}$; while no element of J is conjugate to s_0 as $J \subseteq S \setminus M$ and $s_0 \in M$. Hence s_0 is the unique reflection in $\mathrm{LFC}(s_0)$ that is conjugate to s_0. Summarizing, we are in the third case of Theorem 9.1. This concludes the proof of Theorem 9.1. □

10 On Direct Sum Decompositions of Coxeter Groups

In this section, we summarize some properties of direct sum decompositions of Coxeter groups of arbitrary rank. We note that the Krull–Remak–Schmidt Theorem is in general not applicable here since the Coxeter groups are not necessarily of finite rank.

Proposition 10.1. *Let (W, S) be an irreducible Coxeter system of arbitrary rank. If (W, S) is not of type C_k, $I_2(2k)$ for odd integer $k \geq 3$, E_7 nor H_3, then W is directly indecomposable as abstract group. If (W, S) is of type C_k (respectively, $I_2(2k)$) for odd integer $k \geq 3$, then W is isomorphic to $W_1 \times W_2$ where W_1 and W_2 are Coxeter groups of types A_1 and D_k (respectively, $I_2(k)$). If (W, S) is of type E_7 or H_3, then we have $|Z(W)| = 2$, $W = Z(W) \times W^+$ where W^+ denotes the normal subgroup of W of even-length elements, and W^+ is not a Coxeter group. All the possibilities of direct product decompositions of W are listed above.*

Proof. This follows from the results obtained in [17]. More presicsely, the Proposition is a combination of Theorem 2.17, Lemma 2.18, and Theorem 3.3 in [17]. □

We introduce some notation that is used in the rest of this section. Let $\mathcal{G}$ denote the class of nontrivial groups consisting of irreducible Coxeter groups that are directly indecomposable (see Proposition 10.1 above) and also subgroups W^+ of Coxeter groups W of type E_7 or H_3. Proposition 10.1 implies that any group in the class $\mathcal{G}$ is directly indecomposable and any Coxeter group is decomposed into the direct sum of some groups in the class $\mathcal{G}$. On the other hand, for a family $\{G_\lambda\}_{\lambda\in\Lambda}$ of groups in the class $\mathcal{G}$, we define

$$\Lambda_{A_1} = \{\lambda \in \Lambda \mid G_\lambda \text{ is a Coxeter group of type } A_1\} \ ,$$
$$\Lambda_{-1} = \{\lambda \in \Lambda \mid G_\lambda \text{ is a Coxeter group of } (-1)\text{-type}\} \ .$$

Then a straightforward translation of Theorem 3.9 in [17] (combined with Lemma 3.7 in the same paper) applied to the current situation yields the following result.

Proposition 10.2. *Let $G = \bigoplus_{\lambda \in \Lambda} G_\lambda$ and $G' = \bigoplus_{\lambda' \in \Lambda'} G'_{\lambda'}$ be direct sums of groups $G_\lambda, G'_{\lambda'}$ in the class $\mathcal{G}$. Let $\pi_\lambda \colon G \to G_\lambda$ ($\lambda \in \Lambda$) and $\pi'_{\lambda'} \colon G' \to G'_{\lambda'}$ ($\lambda' \in \Lambda'$) be the projections. Let $f \colon G \xrightarrow{\sim} G'$ be a group isomorphism from G onto G'. Then:*

1. *There is a bijection $\varphi \colon \Lambda \to \Lambda'$ satisfying $G_\lambda \simeq G'_{\varphi(\lambda)}$ for all $\lambda \in \Lambda$. Moreover, for each $\lambda \in \Lambda \setminus \Lambda_{A_1}$, the map $g_\lambda \colon G_\lambda \to G'_{\varphi(\lambda)}$ given by $g_\lambda(w) = \pi'_{\varphi(\lambda)}(f(w))$ for $w \in G_\lambda$ is an isomorphism.*
2. *There is a homomorphism $g_Z \colon G \to Z(G')$ satisfying*
$$f(w) = \begin{cases} g_\lambda(w) g_Z(w) & \text{if } w \in G_\lambda \text{ with } \lambda \in \Lambda \setminus \Lambda_{A_1} \\ g_Z(w) & \text{if } w \in G_\lambda \text{ with } \lambda \in \Lambda_{A_1} \end{cases}$$
and $\pi'_{\varphi(\lambda)}(g_Z(G_\lambda)) = 1$ for all $\lambda \in \Lambda \setminus \Lambda_{A_1}$.
3. *If $\Lambda_{-1} \subseteq \Lambda^\dagger \subseteq \Lambda$, then we have $\Lambda'_{-1} \subseteq \varphi(\Lambda^\dagger)$ and $f(\bigoplus_{\lambda \in \Lambda^\dagger} G_\lambda) = \bigoplus_{\lambda' \in \varphi(\Lambda^\dagger)} G'_{\lambda'}$.*

Proposition 10.2 implies the following result, which is also of independent interest.

Corollary 10.3. *Let (W,S) be an arbitrary Coxeter system with $|Z(W)| = 1$. Then a direct sum decomposition of W into irreducible Coxeter groups is unique. More precisely, for any two decompositions $W = \bigoplus_{\lambda \in \Lambda} V_\lambda$ and $W = \bigoplus_{\lambda' \in \Lambda'} V'_{\lambda'}$ of W into direct sums of subgroups V_λ and $V'_{\lambda'}$ all of those are irreducible Coxeter groups, there is a bijection $\varphi \colon \Lambda \to \Lambda'$ satisfying $V_\lambda = V'_{\varphi(\lambda)}$ for all $\lambda \in \Lambda$.*

Proof. Since the center of W is trivial by the hypothesis, none of the Coxeter groups $V_\lambda, V'_{\lambda'}$ are of (-1)-type. Hence all of V_λ and $V'_{\lambda'}$ are directly indecomposable by Proposition 10.1 and therefore belong to the class $\mathcal{G}$. To apply Proposition 10.2 for $G = G' = W$, $G_\lambda = V_\lambda$, $G'_{\lambda'} = V'_{\lambda'}$, and $f = \mathrm{id}_W$, we note that $\Lambda_{-1} = \emptyset$ since W has trivial center. Therefore, for the bijection $\varphi \colon \Lambda \to \Lambda'$ yielded by Proposition 10.2 (1), for each $\lambda \in \Lambda$, Proposition 10.2 (3) applied to $\Lambda^\dagger = \{\lambda\}$ implies that $V_\lambda = V'_{\varphi(\lambda)}$ (recall that now $f = \mathrm{id}_W$). This completes the proof. □

Proposition 10.2 also implies the following fact that we shall need later.

Corollary 10.4. *Let (W,S) be a Coxeter system of arbitrary rank. Let $I \subseteq S$ be an irreducible component of (-1)-type, and suppose that any other irreducible component of S is not of (-1)-type. Suppose moreover that the following conditions are satisfied:*

- *I is not of type C_k nor $I_2(2k)$ with odd integer $k \geq 3$.*
- *If I is of type A_1, then S has no component of type D_k or $I_2(k)$ with odd integer $k \geq 3$.*

Let $W = \bigoplus_{\lambda \in \Lambda} V_\lambda$ be any decomposition into direct sum of subgroups V_λ that are irreducible Coxeter groups. Then we have $\langle I \rangle = V_\lambda$ for some $\lambda \in \Lambda$.

Proof. First of all, from the decomposition $W = \bigoplus_{\lambda \in \Lambda} V_\lambda$ we obtain a direct sum decomposition of the form $W = \bigoplus_{\lambda' \in \Lambda'} V'_{\lambda'}$ into subgroups belonging to the class $\mathcal{G}$ by subdividing each component V_λ that is directly decomposable into its direct factors. Namely, each $V'_{\lambda'}$ is either some of V_λ where V_λ is directly indecomposable, or a nontrivial direct factor of some V_λ where V_λ is directly decomposable.

We consider first the case where I is not of type E_7 nor H_3. Then by the hypothesis and Proposition 10.1, all the irreducible components of W belong to the class $\mathcal{G}$. Now we apply Proposition 10.2 to the case where $f = \mathrm{id}_W$ and the set $\Lambda^\dagger$ in Proposition 10.2 (3) indicates the unique irreducible component $\langle I \rangle$ of (-1)-type. It follows that $\langle I \rangle = V'_{\lambda'}$ for some $\lambda' \in \Lambda'$. The assertion in this case holds if $V'_{\lambda'}$ is equal to some V_λ. Assume for the contrary that $V'_{\lambda'}$ is not equal to any of V_λ, therefore $V'_{\lambda'}$ is a nontrivial direct factor of some V_λ. Since $\langle I \rangle = V'_{\lambda'}$ has nontrivial center, Proposition 10.1 implies that, the only possibility is that I is of type A_1 and V_λ is of type C_k, $I_2(2k)$ with odd integer $k \geq 3$, E_7, or H_3. We write $V_\lambda = V'_{\lambda'} \times V'_{\lambda''}$ with $\lambda'' \in \Lambda'$. Now Proposition 10.2 (3) applied to the set $\Lambda^\dagger$ indicating the pair of two direct factors $V'_{\lambda'}$ and

$V'_{\lambda''}$ implies that we have $V'_{\lambda'} \times V'_{\lambda''} = \langle I \rangle \times \langle J \rangle$ for some irreducible component J of S. This implies that $\langle J \rangle \simeq V'_{\lambda''}$, while $V'_{\lambda''}$ is either a Coxeter group of type D_k or $I_2(k)$ with odd integer $k \geq 3$, or isomorphic to ${W_0}^+$ for a Coxeter group W_0 of type E_7 or H_3. This contradicts the hypothesis; note that two irreducible spherical Coxeter groups of different types cannot be isomorphic, as shown in e.g., Proposition 6.1 of [13]. Hence the assertion holds in this case.

Now we consider the other case where I is of type E_7 or H_3. By Proposition 10.1, we have $\langle I \rangle = \langle \rho_I \rangle \times \langle I \rangle^+$ and both $\langle \rho_I \rangle$ and $\langle I \rangle^+$ belong to the class $\mathcal{G}$ as well as the other irreducible components of (W, S). We apply Proposition 10.2 to the case where $f = \mathrm{id}_W$. By Proposition 10.2 (1), $\langle I \rangle^+$ is isomorphic to some $V'_{\lambda'_1}$, therefore this $V'_{\lambda'_1}$ is not a Coxeter group. Hence, $V'_{\lambda'_1}$ must be a nontrivial direct factor of some V_{λ_1}, therefore this V_{λ_1} has the same type as $\langle I \rangle$ and we have $V_{\lambda_1} = V'_{\lambda'_2} \times V'_{\lambda'_1}$ for some $\lambda'_2 \in \Lambda'$ with $V'_{\lambda'_2} = Z(V_{\lambda_1})$ being the group of order two. Now by the hypothesis that $\langle I \rangle$ is the unique irreducible component of (W, S) of (-1)-type, no factors V_λ with $\lambda \in \Lambda$ other than V_{λ_1} are of (-1)-type; in particular, no such V_λ is of the same type as $\langle I \rangle$. Then it follows that $V'_{\lambda'_1}$ is the only factor $V'_{\lambda'}$ isomorphic to $\langle I \rangle^+$ and $V'_{\lambda'_2}$ is the only factor $V'_{\lambda'}$ isomorphic to $\langle \rho_I \rangle$. Now the bijection yielded by Proposition 10.2 (1) maps the factors $\langle \rho_I \rangle$ and $\langle I \rangle^+$ to $V'_{\lambda'_2}$ and $V'_{\lambda'_1}$, respectively; then we have $\langle I \rangle = \langle \rho_I \rangle \times \langle I \rangle^+ = V'_{\lambda'_2} \times V'_{\lambda'_1} = V_{\lambda_1}$ by Proposition 10.2 (3) for the set $\Lambda^\dagger$ indicating the pair of $\langle \rho_I \rangle$ and $\langle I \rangle^+$. This completes the proof. □

The following technical Lemma will be used in the proof of Proposition 11.2 below.

Lemma 10.5. *Let $G = \bigoplus_{\lambda \in \Lambda} G_\lambda$ be the direct sum of nontrivial, directly indecomposable groups G_λ. Suppose that the center of G is trivial. Moreover, suppose that $G = H_1 \times H_2$, $H_1 \neq 1$, and H_1 is directly indecomposable. Then there is an index $\lambda \in \Lambda$ satisfying that $H_1 = G_\lambda$ and $H_2 = \bigoplus_{\lambda' \neq \lambda} G_{\lambda'}$.*

Proof. By the hypothesis, all of the groups G_λ, H_1, and H_2 have trivial center. For any direct factor V of G, let π_V denote the projection $G \to V$.

Since $H_1 \neq 1$, there is an index $\lambda_0 \in \Lambda$ with $\pi_{H_1}(G_{\lambda_0}) \neq 1$. We write $G' = \bigoplus_{\lambda \neq \lambda_0} G_\lambda$, therefore $G = G_{\lambda_0} \times G'$. Now H_1 is a central product of $\pi_{H_1}(G_{\lambda_0})$ and $\pi_{H_1}(G')$ (i.e., $\pi_{H_1}(G_{\lambda_0})$ and $\pi_{H_1}(G')$ commute with each other, $H_1 = \pi_{H_1}(G_{\lambda_0})\pi_{H_1}(G')$, and $\pi_{H_1}(G_{\lambda_0}) \cap \pi_{H_1}(G') \subseteq Z(H_1)$), and the fact $Z(H_1) = 1$ implies moreover that $H_1 = \pi_{H_1}(G_{\lambda_0}) \times \pi_{H_1}(G')$. Since H_1 is directly indecomposable by the hypothesis, and $\pi_{H_1}(G_{\lambda_0}) \neq 1$ as above, it follows that $H_1 = \pi_{H_1}(G_{\lambda_0})$ and $\pi_{H_1}(G') = 1$, therefore $G' \leq H_2$.

Similarly, for each $\lambda \in \Lambda$, G_λ is a central product of $\pi_{G_\lambda}(H_1)$ and $\pi_{G_\lambda}(H_2)$, and the fact $Z(G_\lambda) = 1$ implies that $G_\lambda = \pi_{G_\lambda}(H_1) \times \pi_{G_\lambda}(H_2)$. Since G_λ is directly indecomposable by the hypothesis, there is a unique index $i_\lambda \in \{1, 2\}$ satisfying $G_\lambda = \pi_{G_\lambda}(H_{i_\lambda})$ and $\pi_{G_\lambda}(H_{3-i_\lambda}) = 1$. Now for each $\lambda \neq \lambda_0$, we have $1 \neq G_\lambda \leq G' \leq H_2$, therefore $\pi_{G_\lambda}(H_2) \neq 1$. This implies that $i_\lambda \neq 1$ and hence $i_\lambda = 2$, therefore we have $\pi_{G_\lambda}(H_1) = 1$ for every $\lambda \neq \lambda_0$. Hence we have $\pi_{G'}(H_1) = 1$ and $H_1 \leq G_{\lambda_0}$.

Moreover, the fact $1 \neq H_1 \leq G_{\lambda_0}$ implies that $\pi_{G_{\lambda_0}}(H_1) \neq 1$. Therefore, we have $i_{\lambda_0} \neq 2$ and $i_{\lambda_0} = 1$. Hence we have $\pi_{G_{\lambda_0}}(H_2) = 1$ and $H_2 \leq G'$, therefore $H_2 = G'$. Now the facts $G = H_1 \times H_2 = G_{\lambda_0} \times G'$, $H_1 \leq G_{\lambda_0}$, and $H_2 = G'$ imply that $H_1 = G_{\lambda_0}$. This completes the proof. □

11 On Centers of Locally Finite Continuations

In this section, we give generalizations of the results in Section 11 of [13] originally shown for finite rank cases to the present case of arbitrary rank. First, the following lemma is a counterpart of Lemma 11.1 in [13].

Lemma 11.1. *Let (W, S) be a Coxeter system of arbitrary rank and let $s \in S$. Let $R \subseteq W$ be a Coxeter generating set for W. Then there exist a Coxeter generating set R_1 for W and $L \subseteq K \subseteq R_1$ with the following properties:*

1. *$R_1 = R^w$ for some $w \in W$;*
2. *$\mathrm{LFC}(s) = \langle K \rangle$;*

3. $(\langle L \rangle, L)$ *is of* (-1)*-type and* $s = \rho_L$.

Proof. If (W,S) is of finite rank and we consider $\mathrm{FC}(s)$ instead of $\mathrm{LFC}(s)$ then the resulting statement is precisely Lemma 11.1 of [13]. Now a careful reading of the proof of that lemma in [13] shows that, the only typical properties of $\mathrm{FC}(s)$ used in the proof are that $s \in \mathrm{FC}(s)$ and that $\mathrm{FC}(s)$ is a parabolic subgroup with respect to any Coxeter generating set for W. Those typical properties needed in the proof are also possessed by $\mathrm{LFC}(s)$ owing to Corollary 9.3, and the finiteness of the rank of (W,S) was used in the proof of Lemma 11.1 in [13] only for defining $\mathrm{FC}(s)$ and was not used in any other place during the proof of that lemma. This implies that our statement here concerning $\mathrm{LFC}(s)$ instead of $\mathrm{FC}(s)$ is also true. $\square$

The next proposition is a counterpart of Proposition 11.2 in [13].

Proposition 11.2. *Let* (W,S) *be a Coxeter system of arbitrary rank, and let* $s \in S$. *Suppose that* $Z(\mathrm{LFC}(s)) = \langle s \rangle$. *Then one of the following holds:*

1. s *is an intrinsic reflection of* W*;*

2. s *is a right-angled generator in* S.

Proof. If (W,S) is of finite rank and we consider $\mathrm{FC}(s)$ instead of $\mathrm{LFC}(s)$ then the resulting statement is precisely Proposition 11.2 of [13]. Now a careful reading of the proof of that proposition in [13] shows that, the typical properties of $\mathrm{FC}(s)$ and the finiteness assumption on the rank of (W,S) are used only at the following points during the proof of that proposition in [13]:

- The original proof used the fact that $\mathrm{FC}(s)$ is a parabolic subgroup of W. The same property is possessed by $\mathrm{LFC}(s)$ as shown in Corollary 9.3.
- The original proof used Lemma 11.1 and Corollary 9.7 of the same paper [13] stated for finite continuations in finite rank cases. Their generalizations to locally finite continuations in arbitrary rank cases hold as Lemma 11.1 and Corollary 9.4 above, respectively.
- The original proof considered the following situation: $J' \subseteq J \subseteq S$, $r \in J$, $\mathrm{FC}(r) = \langle r \rangle \times \langle J' \rangle$, and $\langle J' \rangle$ has trivial center; a certain subgroup of W denoted by X is a nontrivial direct factor of a certain irreducible spherical Coxeter group, and X is not a Coxeter group of type A_1; and X is also a direct factor of $\langle J' \rangle$. The original proof for finite rank cases concluded that this X coincides with an irreducible component of $(\langle J' \rangle, J')$, by using a variation of Krull–Remak–Schmidt Theorem (Proposition 10.4 of [13]). On the other hand, the same conclusion follows in the present case (where we consider $\mathrm{LFC}(r)$ instead of $\mathrm{FC}(r)$) from Lemma 10.5 above, since by Proposition 10.1 the group X as well as all irreducible components of $(\langle J' \rangle, J')$ are directly indecomposable.

As a consequence, essentially the same proof as Proposition 11.2 of [13] shows that our statement here is also true. $\square$

The next proposition is a counterpart of Proposition 11.3 in [13].

Proposition 11.3. *Let* (W,S) *be a Coxeter system of arbitrary rank, and let* $s \in S$. *Suppose that* $Z(\mathrm{LFC}(s)) = \langle s,t \rangle$ *for some* $t \neq s$ *with the property that* t *is conjugate to* s *in* W. *Then* s *is an intrinsic reflection of* W.

Proof. If (W,S) is of finite rank and we consider $\mathrm{FC}(s)$ instead of $\mathrm{LFC}(s)$ then the resulting statement is precisely Proposition 11.3 of [13]. Now a careful reading of the proof of that proposition in [13] shows that, only the typical properties of $\mathrm{FC}(s)$ and the finiteness assumption on the rank of (W,S) used during the proof is that, the original proof used Lemma 11.1 of the same paper [13] stated for finite continuations in finite rank cases. Its generalization to locally finite continuations in arbitrary rank cases holds as Lemma 11.1 above. As a consequence, essentially the same proof as Proposition 11.3 of [13] shows that our statement here is also true. $\square$

12 The Locally Spherical Cases

In this section, we study intrinsic reflections of locally spherical Coxeter groups. First we consider the case that (W,S) is an irreducible and locally spherical Coxeter system. As the finite rank cases have been discussed in Section 6 of [13], it remains to consider the infinite rank cases; that is, (W,S) is of type A_∞, $A_{\pm\infty}$, C_∞, or D_∞.

Here we give some definitions and relevant properties. Let (W,S) be an arbitrary Coxeter system and let $s \in S$. Let $W^{\perp s}$ denote the subgroup of W generated by all reflections other than s itself that commute with s. This group $W^{\perp s}$ generated by reflections is, as mentioned in Section 4.5, a Coxeter group with a certain Coxeter generating set. Let $(W^{\perp s})_{\mathrm{fin}}$ denote the direct sum of all the spherical components of the Coxeter group $W^{\perp s}$; it is known (by e.g., Theorem 3.4 of [17]) that the subset $(W^{\perp s})_{\mathrm{fin}}$ of $W^{\perp s}$ does not depend on the choice of the Coxeter generating set for $W^{\perp s}$. Then Theorem 3.7 of [18] shows (in our terminology) the following:

Proposition 12.1. *Let (W,S) be an arbitrary Coxeter system, and let $s \in S$. Suppose that the subgroup $(W^{\perp s})_{\mathrm{fin}}$ of W is either trivial or generated by a single reflection conjugate to s in W. Then s is an intrinsic reflection of W.*

On the other hand, the result of [4] by Brink and Howlett enables us to determine the structure of $W^{\perp s}$ for Coxeter systems (W,S) of type A_∞, $A_{\pm\infty}$, C_∞, and D_∞ in the following manner, where we denote by s_i the generator in S numbered as i in Figure 1.

Proposition 12.2. *Let (W,S) be a Coxeter system of type A_∞, $A_{\pm\infty}$, C_∞, or D_∞ as in Figure 1.*

1. *For type A_∞, we have $W^{\perp s_1} = \langle S \setminus \{s_1, s_2\}\rangle$ which is of type A_∞.*
2. *For type $A_{\pm\infty}$, we have $W^{\perp s_0} = \langle (S \setminus \{s_{-1}, s_0, s_1\}) \cup \{s_{-1}s_1s_0s_1s_{-1}\}\rangle$ which is of type $A_{\pm\infty}$.*
3. *For type C_∞, we have $W^{\perp s_0} = \langle (S \setminus \{s_0, s_1\}) \cup \{s_1s_0s_1\}\rangle$ which is of type C_∞, and $W^{\perp s_1} = \langle s_0s_1s_0\rangle \times \langle (S \setminus \{s_0, s_1, s_2\}) \cup \{s_2s_1s_0s_1s_2\}\rangle$ which is of type $A_1 \times C_\infty$.*
4. *For type D_∞, we have $W^{\perp s_0} = \langle s_{0'}\rangle \times \langle (S \setminus \{s_0, s_{0'}, s_1\}) \cup \{s_1s_{0'}s_0s_1s_2s_1s_0s_{0'}s_1\}\rangle$ which is of type $A_1 \times D_\infty$.*

Now we have the following result:

Proposition 12.3. *Let (W,S) be an irreducible, locally spherical Coxeter system that is not of (-1)-type nor of type A_5. Then all reflections of (W,S) are intrinsic reflections of W.*

Proof. When (W,S) has finite rank, the assertion follows immediately from Corollary 6.6 of [13]. We consider the remaining case, that is, (W,S) is of type A_∞, $A_{\pm\infty}$, C_∞, or D_∞. Then for each $s \in S$, Proposition 12.2 implies that $(W^{\perp s})_{\mathrm{fin}}$ is either trivial or generated by a single reflection conjugate to s in W (note that, for the case of type C_∞, the reflection $s_0s_1s_0$ is conjugate to s_1 in W; and for the case of type D_∞, the reflection $s_{0'}$ is conjugate to s_0 in W). Now Proposition 12.1 shows that s is an intrinsic reflection of W. This completes the proof. □

We also give the following two results. The first result is a generalization of (a part of) Proposition 10.7 in [13]; we note that the basic idea of the proof is also the same.

Proposition 12.4. *Let (W,S) be a locally spherical Coxeter system, M be an odd component of (W,S), and let $s \in M$. If $|Z(W)| = 1$ and the M-principal component of S is not of type A_5, then s is an intrinsic reflection of W.*

Proof. Let $J = C_0(M)$ denote the M-principal component of $E(M) = S$ (note that now S is locally spherical). Let $R \subseteq W$ be any Coxeter generating set. Since $|Z(W)| = 1$ by our hypothesis, Corollary 10.3 implies that $\langle J\rangle = \langle K\rangle$ for some irreducible component K of R. Moreover, by our hypothesis, J is irreducible and locally spherical, and J is not of (-1)-type nor of type A_5. Then Proposition 12.3 implies that s is an intrinsic reflection of $\langle J\rangle = \langle K\rangle$, therefore s is conjugate to an element of $K \subseteq R$. This completes the proof. □

The second result is a generalization of (a part of) Proposition 10.10 in [13]; again, the basic idea of the proof is also the same.

Proposition 12.5. *Let (W,S) be a locally spherical Coxeter system with $|Z(W)| = 2$, M be an odd component of (W,S), and let $s \in M$. Let $\widehat{J}$ be the M-principal component of S.*

1. *If $\widehat{J}$ is of type C_2, H_3, E_7, or $I_2(4k)$ with $k \geq 2$, then s is an intrinsic reflection of W.*
2. *If $\widehat{J}$ is of type A_1 and S has no component of type D_k or $I_2(k)$ with odd integer $k \geq 3$, then s is an intrinsic reflection of W.*

Proof. Let $R \subseteq W$ be any Coxeter generating set. By the hypothesis, $\widehat{J}$ is the only component of S of (-1)-type, therefore $\widehat{J} \subseteq S$ satisfies the hypothesis of Corollary 10.4 (where $I = \widehat{J}$). Then Corollary 10.4 implies that $\langle \widehat{J} \rangle = \langle K \rangle$ for some irreducible component K of R. Moreover, by the hypothesis and Proposition 6.5 of [13], all reflections of $(\langle \widehat{J} \rangle, \widehat{J})$ are intrinsic reflections of $\langle \widehat{J} \rangle = \langle K \rangle$. This implies that s is conjugate to some element of $K \subseteq R$. Hence s is an intrinsic reflection of W, concluding the proof. □

13 Proof of Theorem 2.2

As already explained, in order to prove Theorem 2.2, the remaining task is to prove Proposition 2.5. As in Proposition 2.5, let (W,S) be a Coxeter system, let $s \in S$, and let M be the odd component of (W,S) containing s. Let $J = C_0(M)$. Since the statement does not depend on the choice of s among the generators in M, we may assume without loss of generality (owing to Corollary 9.2) that the conditions in Corollary 9.2 are all satisfied. In particular, we have $\mathrm{LFC}_W(s) = \mathrm{LFC}_{\langle J \rangle}(s) \times \langle \Sigma(M) \rangle$ and $\mathrm{LFC}_{\langle J \rangle}(s) = \langle K \rangle$ for some $K \subseteq J = C_0(M) \subseteq S$, where $\Sigma(M)$ is the union of all locally spherical M-subsidiary components of $E(M)$. Moreover, owing to Lemma 5.5, the problem is also reduced to proving that s is an intrinsic reflection of $\mathrm{LFC}_W(s)$ in the current situation. By the hypothesis of Proposition 2.5, M is not mutable, therefore $\Sigma(M)$ has no component of (-1)-type.

We consider the first case of Proposition 2.5 where J is of type C_2, H_3, E_7, or $I_2(4k)$ for some $k \geq 2$. Then we have $\mathrm{LFC}_{\langle J \rangle}(s) = \langle J \rangle$ and $\mathrm{LFC}_W(s) = \langle J \rangle \times \langle \Sigma(M) \rangle$, therefore the Coxeter system $(\mathrm{LFC}_W(s), J \cup \Sigma(M))$ satisfies the hypothesis of Proposition 12.5. Hence by Proposition 12.5, s is an intrinsic reflection of $\mathrm{LFC}_W(s)$, as desired.

We consider the second case of Proposition 2.5 where J is locally spherical and is not of (-1)-type nor of type A_5. Then we have $\mathrm{LFC}_{\langle J \rangle}(s) = \langle J \rangle$ and $\mathrm{LFC}_W(s) = \langle J \rangle \times \langle \Sigma(M) \rangle$, therefore the Coxeter system $(\mathrm{LFC}_W(s), J \cup \Sigma(M))$ satisfies the hypothesis of Proposition 12.4. Hence by Proposition 12.4, s is an intrinsic reflection of $\mathrm{LFC}_W(s)$, as desired.

From now, we consider the remaining (i.e., the third) case of Proposition 2.5 where J is not locally spherical and M has no C_3-neighbors. By Corollary 9.2 and the assumption that M has no C_3-neighbors, there are the following three possibilities for $\mathrm{LFC}_{\langle J \rangle}(s) = \langle K \rangle$:

1. K is of type C_2;
2. K is of type $A_1 \times A_1$ and the two generators in K are conjugate to each other in W;
3. $K = \{s\}$.

In the first case, the Coxeter system $(\mathrm{LFC}_W(s), K \cup \Sigma(M))$ satisfies the hypothesis of Proposition 12.5. Hence by Proposition 12.5, s is an intrinsic reflection of $\mathrm{LFC}_W(s)$, as desired. In the second case, we have $Z(\mathrm{LFC}_W(s)) = \langle K \rangle$ and now the hypothesis of Proposition 11.3 is satisfied. Hence s is an intrinsic reflection of W by Proposition 11.3. Moreover, in the third case, we have $Z(\mathrm{LFC}_W(s)) = \langle s \rangle$ and now the hypothesis of Proposition 11.2 is satisfied. Hence by Proposition 11.2, s is either an intrinsic reflection of W or a right-angled generator in S. Now if s were a right-angled generator in S then we would have $C_0(M) = \{s\}$, which contradicts the current assumption that $J = C_0(M)$ is not locally spherical. Hence s is an intrinsic reflection of W.

Summarizing, s is an intrinsic reflection of W for all cases owing to Lemma 5.5. This completes the proof of Proposition 2.5, hence completes the proof of Theorem 2.2 as well.

14 On Reflection Independent Coxeter Groups

Following the terminology introduced by Bahls in [1], a Coxeter group W is called *reflection independent* if we have $S^W = R^W$ for any two Coxeter generating sets S, R of W. In this section, we give some results on reflection independent Coxeter groups of arbitrary ranks as an application of our results given in the previous part of this paper.

We start with the following fundamental property that relates intrinsic reflections to reflection independent Coxeter groups. Here we give a proof for the sake of completeness; see also Lemma 3.7 of the paper [3] by Brady, McCammond, Neumann, and the first author of this paper for another argument for this result.

Proposition 14.1. *Let (W, S) be a Coxeter system of arbitrary rank.*

1. *If $R \subseteq W$ is a Coxeter generating set for W and $S \subseteq R^W$, then we have $S^W = R^W$.*
2. *The Coxeter group W is reflection independent if and only if all generators in S are intrinsic reflections of W.*

Proof. For the first assertion, a general result of Dyer in Corollary 3.11 (ii) of [9] implies, when specialized to the Coxeter system (W, R) in our notation, the following: For any $T \subseteq R^W$ and $G = \langle T \rangle$, we have $G \cap R^W = T^G$. Now applying this result to $S = T$ gives the desired relation $R^W = S^W$ since $\langle S \rangle = W$. For the second assertion, the "only if" part is obvious from the definition of reflection independent Coxeter groups, while the "if" part follows from the first assertion. This completes the proof. $\square$

From now, we consider two natural classes of Coxeter systems to which Proposition 14.1 is applicable. For the first of the two classes, we recall that a Coxeter system (W, S) is called 2-*spherical* if we have $o(st) < \infty$ for any $s, t \in S$. In this case we also say that S is 2-spherical. Then we have the following result.

Proposition 14.2. *Let (W, S) be a 2-spherical and irreducible Coxeter system of arbitrary rank.*

1. *If (W, S) is not locally spherical, then we have $\mathrm{LFC}(s) = \langle s \rangle$ for any $s \in S$.*
2. *If $|W| = \infty$, then all generators in S are intrinsic reflections of W, hence W is reflection independent.*

Proof. For the first assertion, let $s \in S$, and let M denote the odd component of (W, S) containing s. We have $E(M) = S$ since S is 2-spherical and $C_0(M) = S$ since S is irreducible. Now owing to Theorem 9.1, it suffices to show that there is no focus, no half-focus and no C_3-neighbor of M in S.

First, assume for the contrary that there is either a focus (a, b) or a half-focus $\{a, b\}$ of M in S. Since $S = C_0(M)$ and (W, S) is 2-spherical, we have $S \setminus \{b\} \subseteq M$ by Condition 4 in Definition 7.1 or Conditions 1 and 5 in Definition 7.2, respectively. Then the odd graph $\Omega(S \setminus \{b\})$ is a tree, by Condition 1 in Definition 7.1 or Condition 2 in Definition 7.2, respectively. Moreover, this tree $\Omega(S \setminus \{b\})$ is in fact a path, by Condition 3 in Definition 7.1 or Condition 4 in Definition 7.2, respectively. Then Condition 2 in Definition 7.1 or Condition 3 in Definition 7.2, respectively, implies that (W, S) is of type C_n with $2 \le n \le \infty$ or D_n with $3 \le n \le \infty$, respectively. This contradicts the hypothesis that (W, S) is not locally spherical. Hence there is no focus and no half-focus of M in S.

Secondly, assume for the contrary that there is a C_3-neighbor b of M in S. Note that we have $o(bc) = 4$ for some $c \in M$ by Definition 7.3; let $a := a(b; c)$. Then we have $M = \{a, c\}$ by Condition 1 in Definition 7.3. On the other hand, Condition 2 in Definition 7.3 implies that $\{a, b, c\} \subseteq (S \setminus \{a, b, c\})^{\perp}$ (recall that $S = C_0(M)$), therefore $S = \{a, b, c\}$ since S is irreducible. Moreover, Condition 1 in Definition 7.3 implies that $S = \{a, b, c\}$ is of type C_3. This contradicts the hypothesis that (W, S) is not locally spherical. Hence there is no C_3-neighbor of M in S. This proves the first assertion.

For the second assertion, we note that s is not a right-angled generator since (W, S) is 2-spherical and non-spherical. Then, owing to Proposition 14.1, the assertion follows from the first assertion and Proposition

11.2 if (W,S) is not locally spherical, and the assertion follows from Proposition 12.3 if (W,S) is locally spherical. This completes the proof. □

Our second class of Coxeter systems considered here consists of Coxeter systems (W,S) for which the odd graph $\Omega(S)$ is connected, or equivalently (W,S) has a unique odd component S. In this case, we have the following result.

Proposition 14.3. *Let (W,S) be a Coxeter system of arbitrary rank that is not locally spherical. Suppose that the odd graph for (W,S) is connected. Then we have either $\mathrm{LFC}(s) = \langle s \rangle$ for any $s \in S$, or there is a half-focus $\{a,b\}$ of S and we have $\mathrm{LFC}(a) = \langle a,b \rangle$. Moreover, all generators in S are intrinsic reflections of W, hence W is reflection independent.*

Proof. By the hypothesis that S is the unique odd component of (W,S), it follows immediately from definition that there is no focus or no C_3-neighbor of S. Since $S = C_0(S)$ is not locally spherical by the hypothesis, the former assertion follows from Theorem 9.1. On the other hand, we note that s is not a right-angled generator since $\Omega(S)$ is connected and (W,S) is non-spherical. Now owing to Proposition 14.1, the latter assertion follows from the former assertion, Proposition 11.2, and Proposition 11.3. This completes the proof. □

15 Local Strong Rigidity of 2-spherical Coxeter Groups

Recall that a Coxeter group W is called *strongly rigid* if for any two Coxeter generating sets S, R of W, we have $R = S^w$ for some $w \in W$. One of the most remarkable results on the isomorphism problem for Coxeter groups known so far states that, if a Coxeter system (W,S) *of finite rank* is 2-spherical (see Section 14 for the terminology), irreducible, and non-spherical, then W is strongly rigid (see Proposition 15.7 below). In this section, we consider the question to which extent this result can be generalized to 2-spherical Coxeter systems of arbitrary rank. In particular, we prove Rigidity-Theorem stated in the introduction which shows (as a part of the result) that, for any such (W,S) of arbitrary rank except ones of types A_∞ and $A_{\pm\infty}$, any other Coxeter generating set R for W is locally conjugate to S in W. Part (a) of that theorem has already been settled in Proposition 14.2, so we are left with the proof of Part (b) and Part (c).

We start with a proof of Part (b) of Rigidity-Theorem for the case where (W,S) is locally spherical not of type A_∞ nor $A_{\pm\infty}$, i.e., (W,S) is of type C_∞ or D_∞. This is given by the following proposition.

Proposition 15.1. *Let (W,S) be a Coxeter system of type C_∞ or D_∞ and let $R \subseteq W$ be a Coxeter generating set for W. Then R is locally conjugate to S in W.*

Proof. By Proposition 14.2 we know that $R^W = S^W$. By Proposition 10.1, W does not admit a proper decomposition as a direct product and therefore (W,R) is irreducible. As (W,S) is assumed to be of type C_∞ or D_∞, the group W is locally finite. Since (W,R) is irreducible, it follows from Proposition 4.2 that it is of type A_∞, $A_{\pm\infty}$, C_∞, or D_∞. As $R^W = S^W$ it follows from Lemma 4.3 that (W,R) is of type C_∞ (respectively, D_∞) if (W,S) is of type C_∞ (respectively, D_∞). Now we write elements of S and of R as s_i and t_i, respectively, where the numberings are as in Figure 1. Then $\alpha\colon S \to R$ defined by $\alpha(s_i) = t_i$ is a bijection as (W,S) and (W,R) have the same type. Now Lemma 4.4 (respectively, Lemma 4.5) implies that S and R are locally conjugate via the map α if (W,S) is of type C_∞ (respectively, D_∞) by noting that any finite subset of S is included in some finite subset of type C_n (respectively, D_n) for a sufficiently large n. □

In view of Proposition 15.1, it remains to prove Parts (b) and (c) of Rigidity-Theorem for the case where (W,S) is not locally spherical.

15.1 Property (FA)

Following Serre [21] we say that a group G has *property (FA)*, if any action of G on a tree has a fixed point. We will apply the following result of Mihalik and Tschantz in [15].

Proposition 15.2. *Let (W, S) be a Coxeter system of finite rank. If G is a subgroup of W having property (FA) then there exist $w \in W$ and $J \subseteq S$ satisfying that $G^w \leq \langle J \rangle$ and J is 2-spherical. Moreover, $\langle K \rangle$ has property (FA) for any 2-spherical subset K of S.*

Proof. The former assertion is Lemma 25 of [15] and the latter assertion is Proposition 24 of [15]. $\square$

Corollary 15.3. *Let (W, S) be a 2-spherical Coxeter system of arbitrary rank and let R be a Coxeter generating set of W. Then (W, R) is 2-spherical.*

Proof. It suffices to show that xy has finite order for any $x, y \in R$. There exists a finite subset J of S with $x, y \in \langle J \rangle$, and there exists a finite subset K of R with $J \subseteq \langle K \rangle$. As J is finite and 2-spherical, the group $\langle J \rangle$ has property (FA) by Proposition 15.2. Again by Proposition 15.2 applied to the Coxeter system $(\langle K \rangle, K)$ of finite rank, there exist $L \subseteq K$ and $w \in \langle K \rangle$ satisfying that L is 2-spherical and $\langle J \rangle^w \leq \langle L \rangle$. Thus $\{x, y\}^w \subseteq \langle L \rangle$. Now, since $x, y \in R$ and $L \subseteq R$, Lemma 4.10 implies that $\{x, y\}^{wu} \subseteq L$ for some $u \in \langle L \rangle$. As L is 2-spherical, it follows that $o(xy) < \infty$. This completes the proof. $\square$

15.2 On Reflection Subgroups

It was mentioned in Section 4.5 that any reflection subgroup of a Coxeter group is again a Coxeter group. Here we need the following, more refined result on Coxeter generating sets for reflection subgroups, which has been obtained independently by V. Deodhar and M. Dyer.

Proposition 15.4. *Let (W, S) be a Coxeter system, and let G be a reflection subgroup of (W, S). Then there exists $R \subseteq G \cap S^W$ such that (G, R) is a Coxeter system and $S \cap G \subseteq R$.*

Proof. The first part of the assertion is the main result of [8] and Theorem 3.3 in [9]. The second part is an immediate consequence of the concrete description of the Coxeter generating set for G given in Theorem 3.3 of [9] (see also the first paragraph of the introduction of loc. cit.). $\square$

Corollary 15.5. *Let (W, S) be a Coxeter system, let G be a reflection subgroup of (W, S), and let R be a Coxeter generating set for G. If G is strongly rigid, then $(S \cap G)^u \subseteq R$ for some $u \in G$.*

Proof. By Proposition 15.4, there exists a Coxeter generating set R' for G with $S \cap G \subseteq R'$. As (G, R) is assumed to be strongly rigid, R' and R are conjugate in G, yielding the assertion. $\square$

15.3 The Rigidity Results

Lemma 15.6. *Let (W, S) be a 2-spherical irreducible Coxeter system that is not locally spherical, and let I be a finite subset of S. Then there exists a finite, irreducible and 2-spherical subset J of S satisfying that $I \subseteq J$ and J is non-spherical.*

Proof. As (W, S) is not locally spherical, there is a finite non-spherical subset K of S. We put $L := I \cup K$ and note that L is finite and non-spherical. As (W, S) is irreducible we can choose for any pair of elements in L a path joining them and define J to be the union of these paths. $\square$

Proposition 15.7. *Let (W, S) be a 2-spherical irreducible Coxeter system of finite rank that is non-spherical. Then W is strongly rigid.*

Proof. This is Theorem 1 in [11] and Theorem 1.2 in [5]; see also Corollary 1.3 in [6]. $\square$

Lemma 15.8. *Let (W, S) be a 2-spherical irreducible Coxeter system that is not locally spherical, and let R be a Coxeter generating set for W. Then (W, R) is also a 2-spherical irreducible Coxeter system that is not locally spherical, and $S^W = R^W$.*

Proof. By the hypothesis, Proposition 10.1 implies that the group W is directly indecomposable and hence (W, R) is also irreducible. Corollary 15.3 implies that (W, R) is also 2-spherical. As (W, S) is not locally spherical, W is not a locally finite group which implies that (W, R) is not locally spherical. Finally, Proposition 14.2 implies that W is reflection independent, therefore $S^W = R^W$. This completes the proof. $\square$

Proposition 15.9. *Let (W, S) be a 2-spherical irreducible Coxeter system that is not locally spherical. Let R be a Coxeter generating set for W, and let J be a finite subset of S. Then $J^w \subseteq R$ for some $w \in W$.*

Proof. By Lemma 15.8 the Coxeter system (W, R) is 2-spherical, irreducible and not locally spherical. As J is a finite subset of S, there exists a finite subset K' of R satisfying that $J \subseteq \langle K' \rangle$. Applying Lemma 15.6 to the Coxeter system (W, R) (with $I := K'$) we have a finite, irreducible and 2-spherical subset K of R satisfying that $J \subseteq \langle K \rangle$ and that K is non-spherical. As K is finite, it follows from Proposition 15.7 that the Coxeter system $(\langle K \rangle, K)$ is strongly rigid. Moreover, as $S^W = R^W$ (by Lemma 15.8), $\langle K \rangle$ is a reflection subgroup of (W, S). Now Corollary 15.5 implies that $(S \cap \langle K \rangle)^w \subseteq K$ for some $w \in \langle K \rangle$. As $J \subseteq S \cap \langle K \rangle$ and $K \subseteq R$ we have $J^w \subseteq R$. This completes the proof. $\square$

Lemma 15.10. *Let (W, S) be a Coxeter system, and let R be a Coxeter generating set for W. Let J be an irreducible, non-spherical subset of S, and let $w, v \in W$ satisfy $J^w \subseteq R$ and $J^v \subseteq R$. Then $s^w = s^v$ for each $s \in J$.*

Proof. We put $K := J^w$, $L := J^v$, and $u := w^{-1}v \in W$. Then $K, L \subseteq R$ and $L = K^u$. On the other hand, as $J \subseteq S$ is irreducible and non-spherical, both K and L are irreducible, non-spherical subsets of R. Now Lemma 4.9 applied to the Coxeter system (W, R) and two subsets L and $K = uLu^{-1}$ implies that $K = L$ and the element in $u\langle K \rangle$ of minimal length (with respect to the Coxeter generating set R), say uv with $v \in \langle K \rangle$, centralizes K. Now we have $t^u = t^{(uv)v^{-1}} = t^{v^{-1}}$ for each $t \in K$, therefore the map $t \mapsto t^{v^{-1}}$ with $v^{-1} \in \langle K \rangle$ gives a bijection $K \to K$. As K is irreducible and non-spherical, it follows that v^{-1} must be the identity element, therefore $uv = u$ centralizes K. Now $s^v = s^{wu} = s^w$ for each $s \in J$, as desired. $\square$

Proposition 15.11. *Let (W, S) be a 2-spherical irreducible Coxeter system that is not locally spherical. Let R be a Coxeter generating set for W. Then there exists an injective mapping $\alpha \colon S \to R$ satisfying the following condition (LC):*

(LC) For each finite subset J of S, there exists an element $c_J \in W$ satisfying $\alpha(s) = s^{c_J}$ for every $s \in J$.

Proof. Let $\mathcal{I}$ denote the set of finite subsets of S that are irreducible and non-spherical. By Lemma 15.6, each finite subset of S is included in some member of $\mathcal{I}$; in particular $S = \bigcup \mathcal{I}$. Therefore, it suffices to verify the condition (LC) only for each $J \in \mathcal{I}$.

Let $J \in \mathcal{I}$. By Proposition 15.9, there exists $c_J \in W$ satisfying $J^{c_J} \subseteq R$. Moreover, by Lemma 15.10 we have $s^u = s^{c_J}$ for any $u \in W$ with $J^u \subseteq R$ and any $s \in J$. This provides a canonical mapping $\alpha_J \colon J \to R$ with the property that $\alpha_J(s) = s^u$ for any $u \in W$ with $J^u \subseteq R$ and any $s \in J$.

Moreover, let $J, K \in \mathcal{I}$. Then Lemma 15.6 applied to $J \cup K$ implies that $J \cup K \subseteq L$ for some $L \in \mathcal{I}$. Now by the results in the previous paragraph, we have $L^{c_L} \subseteq R$, therefore $J^{c_L} \subseteq R$ and $K^{c_L} \subseteq R$; and we have $\alpha_J(s) = s^{c_L} = \alpha_K(s)$ for any $s \in J \cap K$. This enables us to define a mapping $\alpha \colon S \to R$ in a way that, for each $s \in S$, $\alpha(s) = \alpha_J(s)$ for some (or equivalently, any) $J \in \mathcal{I}$ with $s \in J$. Now it is obvious by the definition of α that the condition (LC) holds. Finally, for any distinct $s, t \in S$, by taking $J \in \mathcal{I}$ with $s, t \in J$ owing to Lemma 15.6, we have $\alpha(s) = \alpha_J(s) = s^{c_J} \neq t^{c_J} = \alpha_J(t) = \alpha(t)$. Hence α is injective. This completes the proof. $\square$

Now we are ready to give proofs of Part (b) and Part (c) of Rigidity-Theorem. As in the common hypothesis of these parts, let (W, S) be a 2-spherical irreducible Coxeter system that is non-spherical, and let R be a Coxeter generating set for W. Owing to Proposition 15.1, we may also assume that (W, S) is not locally spherical. By Lemma 15.8, (W, R) is also 2-spherical, irreducible and is not locally spherical. Proposition 15.11 applied to (W, S) and to (W, R) yields injective mappings $\alpha \colon S \to R$ and $\beta \colon R \to S$ satisfying the corresponding conditions (LC). Now let $s \in R$, and owing to Lemma 15.6 take a finite, irreducible, and non-spherical subset K of R containing s. Then $\beta(K)$ is also a finite, irreducible, and non-spherical subset of S since $\beta(K)$ is conjugate to K in W, therefore $\alpha(\beta(K)) \subseteq R$. Moreover, by the choice of α and β, there exist $u \in W$ and $v \in W$ satisfying that $\beta(t) = t^u$ and $\alpha(\beta(t)) = t^{uv}$ for any $t \in K$. Now Lemma 15.10 applied to R playing the role of S and to two elements $1, uv \in W$ implies that $t^{uv} = t$ for each $t \in K$, and in particular $\alpha(\beta(s)) = s^{uv} = s$. Hence the injection α is also surjective, therefore it is bijective. This completes the proof of Part (b) of the theorem.

For Part (c) of the theorem, let a subset $J \subseteq S$ be as in the statement. By replacing J with a subset of S yielded by Lemma 15.6 if necessary, we may assume without loss of generality that J is irreducible. Let $\alpha \colon S \to R$ be a bijection given by Part (b). By the condition (LC), there exists $u \in W$ satisfying $\alpha(s) = s^u$ for each $s \in J$. We show that $\alpha(t) = t^u$ for any $t \in S$. By the condition (LC) applied to $J \cup \{t\}$, there exists $v \in W$ satisfying $\alpha(s) = s^v$ for each $s \in J \cup \{t\}$. It follows that $s^u = \alpha(s) = s^v$ for any $s \in J$, therefore uv^{-1} centralizes J. On the other hand, J is irreducible and non-spherical and $J^{\perp} = \emptyset$ by the current assumption. Now the main result of [4] implies that the centralizer of J in W is trivial, therefore we have $u = v$ and hence $\alpha(t) = t^v = t^u$ as desired. This completes the proof of Part (c) of the theorem.

Acknowledgements. A large portion of the present work was done when the second author was with National Institute of Advanced Industrial Science and Technology. A part of this work was supported by JST PRESTO Grant Number JPMJPR14E8, JST CREST Grant Number JPMJCR14D6, and JSPS KAKENHI Grant Number JP19H01804 to the second author. The discussion for one of the main parts of this work was undertaken at Tambara Institute of Mathematical Sciences, The University of Tokyo. The authors would like to thank Itaru Terada for kindly organizing our discussion in Tambara and giving helpful comments.

References

[1] P. Bahls, A new class of rigid Coxeter groups, *Int. J. Algebra Comput.* **13** (2003) 87–94

[2] A. Björner, F. Brenti, *Combinatorics of Coxeter groups*, Springer (2005)

[3] N. Brady, J. P. McCammond, B. Mühlherr, W. D. Neumann, Rigidity of Coxeter groups and Artin groups, *Geom. Dedicata* **94** (2002) 91–109

[4] B. Brink, R. B. Howlett, Normalizers of parabolic subgroups in Coxeter groups, *Invent. Math.* **136** (1999) 323–351

[5] P.-E. Caprace, B. Mühlherr, Reflection rigidity of 2-spherical Coxeter groups, *Proc. London Math. Soc. (3)* **94** (2007) 520–542

[6] P.-E. Caprace, P. Przytycki, Bipolar Coxeter groups, *J. Algebra* **338** (2011) 35–55

[7] V. V. Deodhar, On the root system of a Coxeter group, *Commun. Algebra* **10** (1982) 611–630

[8] V. V. Deodhar, A note on subgroups generated by reflections in Coxeter groups, *Arch. Math. (Basel)* **53** (1989) 543–546

[9] M. Dyer, Reflection subgroups of Coxeter systems, *J. Algebra* **135** (1990) 57–73

[10] W. N. Franzsen, R. B. Howlett, Automorphisms of Coxeter groups of rank three, *Proc. Amer. Math. Soc.* **129** (2001) 2607–2616

[11] W. N. Franzsen, R. B. Howlett, B. Mühlherr, Reflections in abstract Coxeter groups, *Comment. Math. Helv.* **81** (2006) 665–697

[12] R. B. Howlett, Normalizers of Parabolic Subgroups of Reflection Groups, *J. London Math. Soc. (2)* **21** (1980) 62–80

[13] R. B. Howlett, B. Mühlherr, K. Nuida, Intrinsic Reflections and Strongly Rigid Coxeter Groups, *Proc. London Math. Soc.* **116**(3) (2018) 534–574

[14] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press (1990)

[15] M. Mihalik, S. Tschantz, Visual decompositions of Coxeter groups, *Groups Geom. Dyn.* **3** (2009) 173–198

[16] B. Mühlherr, K. Nuida, Intrinsic Reflections in Coxeter Systems, *J. Combin. Theory, Series A* **144** (2016) 326–360

[17] K. Nuida, On the direct indecomposability of infinite irreducible Coxeter groups and the Isomorphism Problem of Coxeter groups, *Commun. Algebra* **34**(7) (2006) 2559–2595

[18] K. Nuida, Almost central involutions in split extensions of Coxeter groups by graph automorphisms, *J. Group Theory* **10** (2007) 139–166

[19] K. Nuida, Locally parabolic subgroups in Coxeter groups of arbitrary ranks, *J. Algebra* **350** (2012) 207–217

[20] R. W. Richardson, Conjugacy classes of involutions in Coxeter groups, *Bull. Austral. Math. Soc.* **26** (1982) 1–15

[21] J.-P. Serre, *Trees*, Springer-Verlag (1980)