Intrinsic reflections in Coxeter systems

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Abstract

Let (W, S) be a Coxeter system and let $s \in S$. We call s a right-angled generator of (W, S) if $st = ts$ or st has infinite order for each $t \in S$. We call s an intrinsic reflection of W if $s \in R^W$ for all Coxeter generating sets R of W. We give necessary and sufficient conditions for a right-angled generator $s \in S$ of (W, S) to be an intrinsic reflection of W.

1 Introduction

Let (W, S) be a Coxeter system. We call an element $x \in W$ an *intrinsic reflection of* W if $x \in R^W := \{w^{-1}rw \mid r \in R, w \in W\}$ for all Coxeter generating sets R of W. This paper is a contribution to the following problem.

Problem: Let (W, S) be a Coxeter system and $s \in S$. Give necessary and sufficient condition for s to be an intrinsic reflection of W in terms of the diagram of (W, S).

This problem arises naturally in the context of the isomorphism problem for Coxeter groups which is still open at present. Substantial progress has been made by Caprace and Przytycki in [3] in this area. Combined with their result, the complete solution to the problem above would provide a characterization of all strongly rigid Coxeter systems in terms of their diagrams. This will be discussed in more detail in the paragraph on strong rigidity below.

In [9] the *finite continuation* of a finite order element in a finitely generated Coxeter group was introduced. The notion of an intrinsic reflection was not defined in that paper. However, its main purpose was to give a criterion which ensures that a generator $s \in S$ of a Coxeter system (W, S) is an intrinsic reflection of W. Indeed, Part b) of the Main result of [9] asserts that $s \in S$ is an intrinsic reflection of W if its finite continuation coincides with the subgroup of order 2 generated by s. In a forthcoming paper we intend to give a complete solution of the problem above and it is indeed the case that the criterion given in [9] plays a central role in our arguments. However, it turns out that the information provided by this criterion is rather limited if the generator $s \in S$ of the Coxeter system (W, S) is *right-angled* by which we mean that the order of st is in $\{1, 2, \infty\}$ for all $t \in S$. The purpose of this paper is to solve the above problem for right-angled generators.

There are several reasons for treating this special case in a separate paper: As already pointed out above, the general results about the finite continuation do not provide any particularly deep insights for right-angled generators. In fact, the corresponding information can be deduced more efficiently by direct ad-hoc arguments. Thus, we do not make use of the finite continuation here. This has the advantage that we do not have to assume that our Coxeter systems have finite ranks. Another reason for treating right-angled generators separately is the fact that there are specific tools which are only needed in this special case. Indeed, the case of right-angled generators is the only one where the *blowing down* procedure for Coxeter generating sets comes into play. The latter has been introduced in [12] and some of our intermediate results should be compared with those in [12]. However, our treatment is completely independent because our assumptions and goals are quite different from those in [12].

Intrinsic reflections and blowing down in spherical Coxeter systems

Before stating the main result of this paper it is convenient to provide first some basic information about intrinsic reflections in finite Coxeter groups.

Let (W, S) be a Coxeter system. We call (W, S) *spherical* if W is a finite group. A subset J of S is called a direct factor if $J \neq \emptyset$ and $[J, S \setminus J] = 1$; moreover (W, S) is called irreducible if S is the only direct factor of (W, S).

The irreducible spherical Coxeter systems are known. We denote their types (i.e. their diagrams) as in [10] with the only exception that we use here C_n instead of B_n. For each irreducible spherical Coxeter system (W, S) a description of W as abstract group is found in Appendix 5 of [20] and the following is a straightforward (but somewhat lengthy) exercise in finite group theory.

Fact 1: Let $(W, S), (W', S')$ be two irreducible spherical Coxeter systems such that W is isomorphic to W'. Then there exists an isomorphism from W onto W' mapping S onto S'.

For each irreducible spherical Coxeter system (W, S) the description of the automorphism group of the abstract group W is given in Theorem 31 of [8]. Combining this with the previous fact one deduces the following.

Fact 2: Let (W, S) be an irreducible spherical Coxeter system.

(i) If the center of W is trivial and (W, S) is not of type A_5, then each $s \in S$ is an intrinsic reflection of W.

(ii) If (W, S) is of type A_1, H_3, E_7 or $I_2(4k)$ for some $k \in \mathbf{N}$, then each $s \in S$ is an intrinsic reflection of W.

(iii) If (W, S) is not covered by (i) or (ii), then no $s \in S$ is an intrinsic reflection of W.

Remark: Let $3 \leq n \in \mathbf{N}$ be odd and let (W, S) be a Coxeter system of type C_n or $I_2(2n)$. Then W can be written as a direct product $W = A \times W'$ where A is a group of order 2 and W' has a Coxeter generating set S' such that (W', S') is of type D_n (resp. $I_2(n)$). Thus there is a Coxeter generating set R of W such that $|R| = |S| + 1$.

Let (W, S) be a Coxeter system and let R be a Coxeter generating set of W. As the previous remark shows it may happen that $|S| \neq |R|$. The question, to which extent the abstract group W determines the cardinality of a Coxeter generating set R was adressed by Mihalik and Ratcliffe in [12]. They define two procedures of manipulating Coxeter generating sets which they call *blowing up* and *blowing down*. These procedures rely on the examples described in the previous remark.

The investigation of intrinsic right-angled generators leads naturally to the consideration of a situation which is almost equivalent to their blowing down procedure. In our context it is convenient to introduce the notion of a *blowing down generator* for a right-angled generator s of a Coxeter system (W, S). The existence of such a blowing down generator for s ensures that one can find a Coxeter generating set R of W such that s is not in R^W and hence that s is not an intrinsic reflection of W.

The main result

The precise statement of our main result needs some preparation. In particular, the definition of a *blowing down generator* for a right-angled generator in a Coxeter system is somewhat technical.

Let (W, S) be a Coxeter system. For $s, t \in S$ we denote the order of st by m_{st} and we call $s \in S$ a *right-angled* generator of (W, S) if $\{m_{st} \mid t \in S\} \subseteq \{1, 2, \infty\}$. Moreover, for any generator $s \in S$ we put $s^{\perp} := \{t \in S \mid m_{st} = 2\}$ and $s^{\infty} := \{t \in S \mid m_{st} = \infty\}$.

Let $s \in S$ be a right-angled generator of (W, S). An *s-component* is an irreducible spherical component of the Coxeter system $(\langle s^\perp \rangle, s^\perp)$. We call $a \in s^\perp$ a *blowing down generator for s* if the following conditions are satisfied:

(BDG1) If C denotes the irreducible component of $(\langle s^\perp \rangle, s^\perp)$ containing a, then $(\langle C \rangle, C)$ is of type $I_2(2k+1)$ or D_{2k+1} for some $1 \leq k \in \mathbf{N}$; moreover, if ρ denotes the longest element in the Coxeter system $(\langle C \rangle, C)$, then $b := \rho a \rho \neq a$. (Note that $b \in C$ because ρ normalizes C.)

(BDG2) If $u_0, u_1, \ldots, u_n \in s^\infty$ are such that $u_{i-1}u_i$ has finite order for all $1 \leq i \leq n$, then there exists an element $x \in \{a, b\}$ such that $\{u_i \mid 0 \leq i \leq n\} \subseteq x^\infty$.

Main result: Let (W, S) be a Coxeter system of arbitrary rank and let $s \in S$ be a right-angled generator. Then s is an intrinsic reflection of W if and only if each s-component has trivial center and there is no blowing down generator for s. Moreover, if s is an intrinsic reflection and if R is a Coxeter generating set of W containing s, then s is a right-angled generator of (W, R).

The proof of the main result will be completed in the final section of this paper.

Some consequences and remarks

2-spherical Coxeter systems: A Coxeter system (W, S) is called *2-spherical* if the order of st is finite for all $s, t \in S$. Let (W, S) be a 2-spherical, irreducible Coxeter system such that W is an infinite group. Then Theorem 1 in [9] asserts that any fundamental generator $s \in S$ is an intrinsic reflection of W. Thus, if (W, S) is irreducible and $|W| = \infty$ and if there exists a generator $s \in S$ which is not an intrinsic reflection of W, then there have to exist $t, u \in S$ such that tu has infinite order. In some sense, right-angled generators violate the 2-sphericity condition for a Coxeter system to large extent. Thus, heuristically speaking, they are good candidates for generators which are not intrinsic reflections W.

Right-angled Coxeter-systems: A Coxeter system (W, S) is called *right-angled* if each $s \in S$ is right-angled. It is a consequence of our main result that a generator $s \in S$ of a right-angled Coxeter system (W, S) is an intrinsic reflection of W if and only if the Coxeter system $(\langle s^\perp \rangle, s^\perp)$ does not have a direct factor of type A_1. We do not know a reference for this statement in the literature but it is implicitly in [23] (for finite $|S|$) and [4].

Blowing-down generators: We already mentioned that *blowing down* for Coxeter generating sets was introduced in [12]. The conditions for the existence of a blowing down are given in Theorem 3.7 in [12]. These conditions are slightly more general than ours.

Strongly rigid Coxeter systems

We already mentioned that the problem of finding a characterization of intrinsic fundamental generators in a Coxeter system in terms of its diagram arises naturally in the context of the isomorphism problem for Coxeter groups. For more information about this open question we refer to [14] and to [18] for a more recent account including the infinite rank case.

In the context of the isomorphism problem the notion of strong rigidity for Coxeter systems has been introduced in [1]. A Coxeter system (W, S) is called *strongly rigid* if each Coxeter generating set R of W is conjugate to S in W. If (W, S) is strongly rigid, then the abstract group W determines the set of its reflections and hence each $s \in S$ is an intrinsic reflection; moreover, the abstract group W also determines the diagram and all automorphisms of the abstract group W are inner-by-graph. Thus, strongly rigid Coxeter systems form an interesting class of Coxeter systems which one would like to characterize in terms of their diagrams. An important step towards such a characterization is provided by a substantial result of Caprace and Przytycki in [3]. Their result gives in particular a characterization of all *strongly reflection rigid Coxeter systems* (see Definition 3.2 in [1]) in terms of their diagrams. It is a basic fact that a Coxeter system (W, S) is strongly rigid if and only if it is strongly reflection rigid and if each $s \in S$ is an intrinsic reflection of W. Thus, combined with the result of [3] the solution of the problem above would provide a characterization of all strongly rigid Coxeter systems in terms of their diagrams.

Organisation of the paper

In order to obtain our main result, we have to prove both implications. One of them is considerably harder to establish and the following Proposition is an equivalent formulation of it:

Proposition: Let (W, S) be a Coxeter system of arbitrary rank and let $s \in S$ be a right-angled generator such that each s-component has trivial center. If there exists a Coxeter generating set R of W such that s is not a reflection of (W, R), then there exists a blowing down generator for s.

The proof of the Proposition above will be accomplished in the final section of the paper.

The paper consists of two parts. In the first part we reduce the proof of the main result to this Proposition. This will be accomplished at the end of Section 4. Then, in the considerably more technical second part, we first provide several additional tools that will be only needed in the proof of the Proposition.

The main tools that we shall use are the following:

A detailed analysis of spherical Coxeter systems: We shall need some specific information about Krull-Remak-Schmidt decompositions of finite Coxeter groups. Furthermore, for the Coxeter groups of type $C_n, I_2(n)$ where $3 \leq n \in \mathbf{N}$ is odd, we have to study all Coxeter generating sets. Later on, the outcome of this analysis will lead us to three different cases in the proof of the Proposition above. These will be treated in Sections 6, 8 and 9 separately.

The Coxeter complex: We shall consider the Cayley graph of a Coxeter system (W, S) in order to prove Proposition 7.1 which is the key step in the proof of the Proposition above. The Cayley graph is a building and we shall use techniques from the theory of buildings. It is for this reason, that we call the Cayley graph the *Coxeter complex* of (W, S).

Conjugacy theorems and Property FA: Let (W, S) be a Coxeter system and $J \subseteq S$. We call J spherical (resp. 2-spherical) if $\langle J \rangle$ is a finite group (resp. if st has finite order for all $s, t \in J$). In the proof of Proposition 7.1 we shall use the result that each finite subgroup of W is conjugate to a subgroup of $\langle J \rangle$ for some spherical $J \subseteq S$. There is also a characterization of the subgroups U of W which are conjugate to subgroups $\langle J \rangle$ for 2-spherical $J \subseteq S$. It turns out that this is the case if and only if U is an FA-group. This will be explained and applied in Section 9.

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2 Coxeter systems

In this paper we allow Coxeter systems of arbitrary rank. Whenever 'finite rank' is required it will be mentioned explicitly. Dropping the assumption of finite rank does not lead to any serious problems for most of the basic results on Coxeter groups that we shall use here. But there are also exceptions to this since the corresponding statements have to be modified in order to hold also in the infinite rank case (e.g. parabolic closure of a subgroup) and their proofs need some additional argument. Moreover, even if a result is also valid in the infinite rank case, it is not obvious to find it in this generality in the literature, since finite rank is often explicitly required or tacitly assumed. In this section we collect the basic results on Coxeter systems that will be used in later and sketch their proofs in the infinite rank case if finite rank is not assumed.

Let (W,S) be a Coxeter system. For $s,t \in S$ we denote the order of st by m_{st}. The *length* of an element $w \in W$ with respect to the generating set S is denoted by $\ell(w)$. A subset J of S is called a *direct factor of* (W,S) if $J \neq \emptyset$ and $[J, S \setminus J] = 1$. The Coxeter system (W,S) is called *irreducible* if S is the only direct factor of (W,S). We put $S^W := \{w^{-1}sw \mid s \in S, w \in W\}$. The elements of S^W are called the *reflections of* (W,S).
In the following lemma we recall several basic facts about Coxeter systems. We shall often use them in the sequel without explicitly referring to this lemma.

Lemma 2.1 *Let (W,S) be a Coxeter system and $J \subseteq S$. Then $(\langle J \rangle, J)$ is a Coxeter system and $S^W \cap \langle J \rangle = J^{\langle J \rangle}$. Moreover, if $\ell_J : \langle J \rangle \to \mathbf{N}$ denotes its length function, then $\ell_J = \ell \mid_{\langle J \rangle}$. Finally, if $K \subseteq S$, then $\langle J \rangle \cap \langle K \rangle = \langle J \cap K \rangle$.*

PROOF: See, for instance, Theorem 5.5 in [10] for the case where (W,S) has finite rank. It is straightforward to reduce the infinite rank case to the finite rank case. □

Let (W,S) be a Coxeter system and let $J \subseteq S$. An *irreducible component of* J is a direct factor C of $(\langle J \rangle, J)$ such that $(\langle C \rangle, C)$ is irreducible.

Lemma 2.2 *Let (W,S) be a Coxeter system and let $J \subseteq S$ be a direct factor of (W,S).*

(i) If $K := S \setminus J$ then $W = \langle J \rangle \times \langle K \rangle$ and $\ell(uv) = \ell(u) + \ell(v)$ for all $u \in \langle J \rangle$ and $v \in \langle K \rangle$.

(ii) If $K \subseteq J$ is a direct factor of $(\langle J \rangle, J)$, then K is a direct factor of (W,S) as well. In particular, the irreducible components of (W,S) form a partition of S.

(iii) If there are finitely many irreducible components $C_1, \ldots, C_m$ of (W,S), then $W = \oplus_{1 \leq i \leq m} \langle C_i \rangle$; moreover, if $v_i \in \langle C_i \rangle$ for $1 \leq i \leq m$, then $\ell(v_1 v_2 \ldots v_m) = \sum_{1 \leq i \leq m} \ell(v_i)$.

PROOF: Assertion (i) follows from the previous Lemma and Assertion (ii) is straightforward from the definitions and Assertion (iii) follows from Assertions (i) and (ii) by induction on m. □

Spherical Coxeter systems

A Coxeter system (W,S) is called *spherical* if W is a finite group. A subset J of S is called *spherical* if $(\langle J \rangle, J)$ is a spherical Coxeter system. Note that any spherical Coxeter system is always of finite rank.

Lemma 2.3 *Let (W, S) be a spherical Coxeter system with $S \neq \emptyset$. Then there exists a unique element $\rho \in W$ such that $\ell(\rho) \geq \ell(w)$ for all $w \in W$. The element ρ is an involution and $S^\rho = S$.*

Proof: See, for instance, Exercise 2 of Paragraph 5.6 in [10]. □

Let (W, S) be a spherical Coxeter system. Then we call the unique element $\rho \in W$ of Lemma 2.3 *the longest element* of (W, S).

Let (W, S) be a Coxeter system. Then (W, S) is called of (-1)-*type* if it is spherical and if its longest element is contained in the center of W. A subset J of S is said to be of (-1)-*type* if the Coxeter system $(\langle J \rangle, J)$ is of (-1)-type.

Lemma 2.4 *Let (W, S) be an irreducible spherical Coxeter system and let $\rho \in W$ be the longest element in (W, S). If $|Z(W)| > 1$, then $Z(W) = \langle \rho \rangle$. In particular, (W, S) is of (-1)-type if and only if $Z(W) = \langle \rho \rangle$.*

Proof: This is deduced from the geometric representation of (W, S). See for instance Exercise 1 of Paragraph 6.3 in [10]. □

Lemma 2.5 *Let (W, S) be a Coxeter system of finite rank. Then (W, S) is spherical (resp. of (-1)-type) if and only if all irreducible components of (W, S) are spherical (resp. of (-1)-type).*

Suppose that (W, S) is spherical and let ρ be the longest element of (W, S). Let $C_1, \ldots, C_m$ be the irreducible components of (W, S) and let ρ_i be the longest element of $(\langle C_i \rangle, C_i)$ for $1 \leq i \leq m$. Then $\rho = \rho_1 \rho_2 \ldots \rho_m$ and $\ell(\rho) = \sum_{i=1}^{m} \ell(\rho_i)$. If (W, S) is of (-1)-type, then the center of W is an elementary abelian subgroup of W of order 2^m which is generated by the set $\{\rho_i \mid 1 \leq i \leq m\}$.

Proof: This is straightforward from Lemma 2.2 and Lemma 2.4. □

Lemma 2.6 *Let (W, S) be a spherical Coxeter system and let $a \neq b \in S^W$ be two reflections of (W, S). Then there exist an element $w \in W$ and a subset J of S such that $|J| = 2$ and such that $\langle a, b \rangle^w \leq \langle J \rangle$. Moreover, if $ab = ba$, then J is of (-1)-type and $(ab)^w$ is the longest element in $(\langle J \rangle, J)$.*

Proof: This is seen from the geometric representation of (W, S). We omit the details. □

Parabolic subgroups, finite subgroups and involutions

Lemma 2.7 *Let (W, S) be a Coxeter system and let I, J be finite subsets of S. Suppose that there exists an element $w \in W$ such that $w^{-1}\langle I\rangle w = \langle J\rangle$. Then there exists an element $v \in \langle I\rangle w$ such that $v^{-1}Iv = J$. In particular, the Coxeter systems $(\langle I\rangle, I)$ and $(\langle J\rangle, J)$ are isomorphic and we have $|I| = |J|$.*

Proof: This follows from Proposition 12 in [9] for Coxeter systems of finite rank. If (W, S) has infinite rank, we can find a finite subset K of S such that $I \cup J \subseteq K$ and such that $w \in \langle K\rangle$. Thus we can reduce the infinite rank case to the finite rank case by arguing in the Coxeter system $(\langle K\rangle, K)$. □

Let (W, S) be a Coxeter system and let J be a spherical subset of S. We denote the longest element of $(\langle J\rangle, J)$ by ρ_J.

Lemma 2.8 *Let (W, S) be a Coxeter system, let $J \subseteq S$ be finite and suppose that $I \subseteq S$ is of (-1)-type. Then the following hold.*

(i) If $w \in W$ is such that $w^{-1}\rho_I w \in \langle J\rangle$, then $w^{-1}\langle I\rangle w \leq \langle J\rangle$.

(ii) If J is of (-1)-type, then $\{w \in W \mid w^{-1}\rho_I w = \rho_J\} = \{w \in W \mid w^{-1}\langle I\rangle w = \langle J\rangle\}$.

Proof: In the finite rank case, Assertion (i) is Lemma 20 and Assertion (ii) is Lemma 21 in [9]. As in the proof of the previous lemma, the infinite rank case is reduced to the finite rank case by considering a finite subset K of S such that $I \cup J \cup \{w\} \subseteq \langle K\rangle$. □

Definition: Let (W, S) be a Coxeter system. A subgroup $P \leq W$ is called a *parabolic subgroup* of (W, S) if there exist $J \subseteq S$ and $w \in W$ such that $P = \langle J\rangle^w$. The *parabolic closure* of a subset $X \subseteq W$ in (W, S) is the intersection of all parabolic subgroups of (W, S) containing X. It is denoted by $Pc_S(X)$.

Remark: The notion of the parabolic closure was introduced by Krammer in [11].

Proposition 2.9 *Let (W, S) be a Coxeter system of finite rank, $K \subseteq S$ and $X \subseteq W$. Then the following hold:*

(i) The parabolic closure $Pc_S(X)$ is a parabolic subgroup of (W, S);

(ii) if P is a parabolic subgroup of (W, S), then $P \cap \langle K\rangle$ is a parabolic subgroup of $(\langle K\rangle, K)$;

(iii) if $X \subseteq \langle K\rangle$ and $Pc_K(X)$ denotes the parabolic closure in $(\langle K\rangle, K)$, then $Pc_K(X) = Pc_S(X)$.

Proof: Assertion (i) is an immediate consequence of the discussion following Proposition 2.1.4 in [11]; for a formal statement and proof we refer to Theorem 1.2 in [21]. Assertion (ii) follows from Corollary 7 in [7] and Assertion (iii) is an immediate consequence of Assertion (ii). □

Corollary 2.10 *Let (W, S) be a Coxeter system and let $X \subseteq W$ be finite. Then $Pc_S(X)$ is a parabolic subgroup of (W, S). In particular, if $U \leq W$ is a finitely generated subgroup, then $Pc_S(U)$ is a parabolic subgroup of (W, S).*

Proof: As X is a finite subset of W, there exists a finite subset K of S such that $X \subseteq \langle K \rangle$. We denote its parabolic closure in $(\langle K \rangle, K)$ by $Pc_K(X)$. It follows from Assertion (ii) of Proposition 2.9 that $Pc_K(X)$ is a parabolic subgroup of $(\langle K \rangle, K)$. Note that, by definition, each parabolic subgroup of $(\langle K \rangle, K)$ is a parabolic subgroup of (W, S). Therefore, $Pc_K(X)$ is also a parabolic subgroup of (W, S). Thus, the corollary is proved if we show $Pc_K(X) = Pc_S(X)$. As each parabolic subgroup of $(\langle K \rangle, K)$ is a parabolic subgroup of (W, S) we have $Pc_S(X) \subseteq Pc_K(X)$. Thus, the corollary is proved if we show $Pc_K(X) \subseteq Pc_S(X)$ which in turn follows from the following.

Claim: If P is a parabolic subgroup of (W, S) containing X, then it contains $Pc_K(X)$ as well.

Proof of the Claim: Let P be a parabolic subgroup of (W, S) which contains X. By definition there exist $J \subseteq S$ and $w \in W$ such that $P = w\langle J \rangle w^{-1}$. As $X \subseteq P$ we have $Y := w^{-1}Xw \subseteq \langle J \rangle$. As X is finite, the set Y is finite as well and therefore there exists a finite subset J_1 of J such that $Y \subseteq \langle J_1 \rangle$. We have also a finite subset M of S such that $w \in \langle M \rangle$. Now $L := K \cup J_1 \cup M$ is a finite subset of S. Let $Pc_L(X)$ denote the parabolic closure of X in $(\langle L \rangle, L)$. Since $K \subseteq L$ and $X \subseteq \langle K \rangle$, it follows by Assertion (iii) of Proposition 2.10 (with $(W, S) = (\langle L \rangle, L)$) that $Pc_K(X) = Pc_L(X)$. As $J_1 \subseteq L$ and $w \in \langle L \rangle$, the group $P_1 := w\langle J_1 \rangle w^{-1}$ is a parabolic subgroup of $(\langle L \rangle, L)$ which contains X. Thus it follows that $Pc_K(X) = Pc_L(X) \leq P_1 \leq P$ which yields the claim. □

Remark: In a Coxeter system (W, S) of arbitrary rank it is not true, that the parabolic closure of a subset X is a parabolic subgroup. A counter example is described in the introduction of [17], where the problem of sensibly generalizing the concept of the parabolic closure to Coxeter systems of arbitrary rank is addressed.

Lemma 2.11 *Let (W, S) be a Coxeter system and let $U \leq W$ be a finite subgroup of W. Then there exist a spherical subset J of S and an element $w \in W$ such that $U^w := w^{-1}Uw \leq \langle J \rangle$. In particular, the parabolic closure $Pc_S(U)$ is a finite group.*

PROOF: This follows from Assertion (a) of Proposition 3.2.1 in [11] in the finite rank case. The reduction of the general case to the finite rank case is straightforward. □

Corollary 2.12 *Let (W,S) be a Coxeter system, let $U \leq W$ be a subgroup and let $x \in W$ be such that $\langle U, x\rangle$ is a finite subgroup of W. Then $\langle Pc_S(U), x\rangle$ is also a finite subgroup of W.*

PROOF: Let $H := \langle U, x\rangle$. By Lemma 2.11 there exist a spherical subset $J \subseteq S$ and $w \in W$ such that $H^w \leq \langle J\rangle$. We have $U^w \leq \langle J\rangle$ and therefore $Pc_S(U^w) \leq \langle J\rangle$. Hence $(\langle Pc_S(U), x\rangle)^w = \langle Pc_S(U^w), x^w\rangle \leq \langle J\rangle$ is a finite group which yields the claim. □

Lemma 2.13 *Let (W,S) be a Coxeter system and let $a \neq b \in S^W$ be such that ab has finite order. Then there exist an element $w \in W$ and a subset J of S such that $|J| = 2$ and $\langle a, b\rangle^w \leq \langle J\rangle$. Moreover, if $ab = ba$, then J is of (-1)-type and $(ab)^w$ is the longest element in $(\langle J\rangle, J)$.*

PROOF: As ab has finite order, the group $\langle a, b\rangle$ is finite. By Lemma 2.11 there exist a spherical subset K of S and an element $w \in W$ such that $\langle a, b\rangle^w \leq \langle K\rangle$. By Lemma 2.1 a^w, b^w are reflections of $(\langle K\rangle, K)$ and we can apply Lemma 2.6 to get the result. □

Corollary 2.14 *Let (W,S) be a Coxeter system, let $I \subseteq S$ be of (-1)-type and let r be the longest element in $(\langle I\rangle, I)$. If $v \in W$ is such that $\langle v, r\rangle$ is a finite subgroup of W, then $\langle \{v\} \cup I\rangle$ is a finite subgroup of W as well.*

PROOF: Since $\langle v, r\rangle$ is a finite subgroup of W, there exist a spherical subset J of S and $w \in W$ such that $\langle v, r\rangle^w \leq \langle J\rangle$. It follows from Lemma 2.8 that $I^w \subseteq \langle J\rangle$. As $\langle J\rangle$ is a finite group containing v^w and I^w, the group $\langle \{v\} \cup I\rangle$ is finite as well. □

Lemma 2.15 *Let (W,S) be a Coxeter system and let $r \in W$ be an involution. Then there exist a subset J of S and an element $w \in W$ such that J is of (-1)-type and such that $w^{-1}rw = \rho_J$. Moreover, if $K \subseteq S$ is of (-1)-type and $v \in W$ is such that $v^{-1}rv = \rho_K$, then $|K| = |J|$.*

PROOF: The first assertion is a well known result of Richardson. For Coxeter systems of finite rank it can be found, for instance, in Paragraph 8.3 in [10] and the reduction of the general case to finite rank causes no problems. The second assertion follows from Assertion (ii) of Lemma 2.8 and Lemma 2.7.

□

Let (W, S) be a Coxeter system and let $r \in W$ be an involution. By the previous lemma there exists $J \subseteq S$ of (-1)-type such that r is conjugate to ρ_J. The *S-rank* of r is defined to be the cardinality of J which makes sense in view of the second assertion of the previous lemma. Note that an involution has S-rank one if and only if it is a reflection of (W, S).

Proposition 2.16 *Let (W, S) be a Coxeter system and let $J \subseteq S$ be a spherical subset of S and let r be the longest element of $(\langle J \rangle, J)$. Then the following hold:*

(i) If J is of (-1)-type, then $C_W(r) = N_W(\langle J \rangle)$.

(ii) Suppose that $[s, J] = 1$ for all $s \in S \setminus J$ having the property that $\langle \{s\} \cup J \rangle$ is a finite group. Then $N_W(\langle J \rangle) = \langle J \rangle \times \langle J^{\perp} \rangle$.

PROOF: Assertion (i) is a consequence of Assertion (ii) of Lemma 2.8. Assertion (ii) is a consequence of the main result in [2] in the finite rank case. As in the proof of Lemma 2.7 the infinite rank case is easily settled using the fact that the assertion holds in the finite rank case. □

3 Coxeter generating sets

Let (W, S) be a Coxeter system. In this section we shall investigate conditions on the diagram of (W, S) which ensure that the Coxeter generating set S can be replaced by another one. In the first paragraph we provide some very basic observations in the situation where the Coxeter system admits a visible decomposition as a direct product or as free product with amalgamation. In the second paragraph we provide specific information in the spherical case which will play an important role in the sequel.

Direct products and free products with amalgamation

We recall some definitions from basic group theory. Let G be a group and let A, B be subgroups. We call G the *direct product of A and B* if A and B are normal subgroups, $G = AB$ and $|A \cap B| = 1$. We call G the *free product with amalgamation of A and B* if $G = \langle A, B \rangle$ and if the following universal property is satisfied:

(FPWA) If H is a group and $\varphi_A : A \to H$, $\varphi_B : B \to H$ are homomorphisms such that $\varphi_A \mid_{A \cap B} = \varphi_B \mid_{A \cap B}$, then there exists a homomorphism $\varphi : G \to H$ such that $\varphi \mid_A = \varphi_A$ and $\varphi \mid_B = \varphi_B$.

We shall need the following basic observation about free products with amalgamation.

Lemma 3.1 *Let G be a group and let A, B be subgroups of G such that G is the free product with amalgamation of A and B. Then ab has infinite order for all $(a, b) \in (A \setminus B) \times (B \setminus A)$.*

PROOF: This can be seen by using the normal form for elements in free products with amalgamation. See for instance Theorem 1 in Paragraph 1.2 in [22]. □

Lemma 3.2 *Let (W, S) be a Coxeter system and let $K, L \subseteq S$ be such that $S = K \cup L$.*

(i) If $K \cap L = \emptyset$ and $m_{kl} = 2$ for all $(k, l) \in K \times L$, then W is the direct product of $\langle K \rangle$ and $\langle L \rangle$.

(ii) If $m_{kl} = \infty$ for all $(k, l) \in (K \setminus L) \times (L \setminus K)$, then W is the free product with amalgamation of $\langle K \rangle$ and $\langle L \rangle$.

PROOF: Note first that W is generated by $\langle K \rangle$ and $\langle L \rangle$ by the assumption that $S = K \cup L$. Let $J = K \cap L$. Then $\langle J \rangle = \langle K \rangle \cap \langle L \rangle$ by Lemma 2.1.

Under the assumptions of Assertion (i) it follows $|\langle K \rangle \cap \langle L \rangle| = 1$ and that $\langle K \rangle$ and $\langle L \rangle$ centralize each other. As we already observed that W is generated by $\langle K \rangle$ and $\langle L \rangle$ the first assertion follows.

Let H be a group and let $\varphi_K : \langle K \rangle \to H$ and $\varphi_L : \langle L \rangle \to H$ be homomorphisms which coincide on $\langle K \rangle \cap \langle L \rangle$. Then φ_K and φ_L coincide on J and therefore we have a mapping $\varphi : S \to H$ such that $\varphi \mid_K = \varphi_K$ and $\varphi \mid_L = \varphi_L$. As φ_K (resp. φ_L) is a homomorphism, it follows that $(\varphi(a)\varphi(b))^{m_{ab}} = 1$ for all $a, b \in K$ (resp. for all $a, b \in L$). As $m_{kl} = \infty$ for all $(k, l) \in (K \setminus L) \times (L \setminus K)$, it follows from the universal property of Coxeter systems that φ is the restriction of a homomorphism φ_S from W to H. Thus we have shown that (FPWA) holds which finishes the proof of Assertion (ii). □

Proposition 3.3 *Let G be a group, let A, B be subgroups and let K (resp. L) be a Coxeter generating set of A (resp. B).*

(i) If G is the direct product of A and B, then $S := K \cup L$ is a Coxeter generating set of G.

(ii) If G is the free product with amalgamation of A and B and J is a Coxeter generating set of $A \cap B$ contained in K and L, then $S := K \cup L$ is a Coxeter generating set of G.

PROOF: Assertion (i) is obvious. In order to prove Assertion (ii) we first remark that ab has infinite order for all $(a,b) \in (A \setminus B) \times (B \setminus A)$ by Lemma 3.1.

In view of Lemma 2.1 we have $k \notin \langle J \rangle$ for each $k \in K \setminus L$ and $l \notin \langle J \rangle$ for each $l \in L \setminus K$. Thus kl has infinite order for all $(k,l) \in (K \setminus L) \times (L \setminus K)$.

For $s, t \in S$ we let m_{st} denote the order of st. By what we have said so far we already know that $m_{kl} = \infty$ for all $(k,l) \in (K \setminus L) \times (L \setminus K)$.

Let H be a group and let $\alpha : S \to H$ be a map such that $(\alpha(x)\alpha(y))^{m_{xy}} = 1$ for all $x, y \in S$ with $m_{xy} \neq \infty$. As K (resp. L) is a Coxeter generating set of A (resp. B), there exists a unique homomorphism φ_A (resp. φ_B) from A (resp. B) to H such that $\varphi_A(k) = \alpha(k)$ for all $k \in K$ (resp. $\varphi_B(l) = \alpha(l)$ for all $l \in L$). As $\varphi_A(j) = \alpha(j) = \varphi_B(j)$ for each $j \in J$ and as $\langle J \rangle = A \cap B$, it follows that $\varphi_A |_{A \cap B} = \varphi_B |_{A \cap B}$. As G is the free product with amalgamation of A and B we have a homomorphism $\varphi : G \to H$ such that $\varphi |_A = \varphi_A$ and $\varphi |_B = \varphi_B$ and hence $\varphi |_S = \alpha$. This shows that (G,S) is indeed a Coxeter system and finishes the proof of Assertion (ii). □

Corollary 3.4 *Let (W,S) be a Coxeter system, let $K, L \subseteq S$ be such that $S = K \cup L$ and let K' (resp. L') be a Coxeter generating set of $\langle K \rangle$ (resp. $\langle L \rangle$). Then $S' := K' \cup L'$ is a Coxeter generating set of W if one of the following conditions is satisfied.*

(i) $K \cap L = \emptyset$ and $m_{kl} = 2$ for all $(k,l) \in K \times L$;

(ii) $K \cap L \subseteq K' \cap L'$ and $m_{kl} = \infty$ for all $(k,l) \in (K \setminus L) \times (L \setminus K)$.

PROOF: Let $A := \langle K \rangle$ and $B := \langle L \rangle$.

Under the assumptions of Assertion (i) it follows from Lemma 3.2 that W is the direct product of A and B and the claim follows then from the first assertion of Proposition 3.3.

Under the assumptions of Assertion (ii) it follows from Lemma 3.2 that W is the free product with amalgamation of A and B. By Lemma 2.1 we also know that $\langle K \cap L \rangle = A \cap B$. Hence the second assertion of Proposition 3.3 finishes the proof of the second statement of the corollary. □

Lemma 3.5 *Let (W,S) be a Coxeter system and let R be a Coxeter generating set of W. If $R \subseteq S^W$, then $R^W = S^W$.*

Proof: This is a consequence of a result on subgroups of Coxeter groups generated by reflections which has been found independently by Deodhar [5] and Dyer [6]. For an explicit statement and proof we refer to Corollary A.2 in [3]. □

Spherical Coxeter systems

Lemma 3.6 *Let (W, S) be a spherical Coxeter system, let $s \in S$ be such that $s \in Z(W)$, let $C := S \setminus \{s\}$ and let ρ be the longest element of $(\langle C \rangle, C)$. Then the following hold for all $1 \leq k \in \mathbf{N}$:*

(i) Suppose that $(\langle C \rangle, C)$ is of type $I_2(2k+1)$ and that $a \in C$. Then $b := \rho a \rho \in C$ and $(S \setminus \{s, b\}) \cup \{s\rho\}$ is a Coxeter generating set of type $I_2(4k+2)$ of W. Moreover, if R is any Coxeter generating set of type $I_2(4k+2)$ of W, then $\{a, b, s\rho\} \subseteq R^W \subseteq a^W \cup (s\rho)^W$.

(ii) Suppose that $(\langle C \rangle, C)$ is of type D_{2k+1} and that $a \in C$ is such that $b := \rho a \rho \neq a$. Then $b \in C$ and $(S \setminus \{s, b\}) \cup \{s\rho\}$ is a Coxeter generating set of type C_{2k+1} of W. Moreover, if R is any Coxeter generating set of type C_{2k+1} of W, then $s\rho \in R^W$ and $R^W \cap \{c, sc\} \neq \emptyset$ for each $c \in C$; if $a \in R^W$ we have $R \subseteq (s\rho)^W \cup a^W$ and if $sa \in R^W$ we have $R \subseteq (s\rho)^W \cup (sa)^W$.

Proof: As s is in the center of W, we have $(as\rho)^2 = a\rho a\rho = ab$ and hence the order of $as\rho$ is $2m_{ab}$. It readily follows that R satisfies the Coxeter relations in (i) and (ii). By checking the orders of the corresponding finite Coxeter groups, it follows that R is indeed a Coxeter generating set. The second statement in Assertion (i) follows from the fact that all non-central involutions are reflections in a dihedral group. The second statement in Assertion (ii) follows by considering the automorphism group of the Coxeter groups of type C_{2k+1} (see for instance Theorem 31 in [8]). □

Proposition 3.7 *Let (W, S) be an irreducible spherical Coxeter system. Suppose that there are subgroups A and B such that $1 < |A| \leq |B|$ and such that W is the direct product of A and B. Then $A = Z(W)$ and one of the following holds:*

(i) (W, S) is of type E_7 and H_3 and B is not a Coxeter group.

(ii) (W, S) is of type $I_2(4k+2)$ for some $1 \leq k \in \mathbf{N}$ and B is a Coxeter group. Moreover, all Coxeter generating sets of B are of type $I_2(2k+1)$.

(iii) (W, S) is of type C_{2k+1} for some $1 \leq k \in \mathbf{N}$ and B is a Coxeter group. Moreover, all Coxeter generating sets of B are of type D_{2k+1}.

Proof: Let (W,S) be an irreducible spherical Coxeter system. Then the decompositions of W as a non-trivial direct product can be determined by going through the list. This is done in the final section of [19]. It follows from Fact 1 of the introduction that all irreducible Coxeter generating sets of a finite Coxeter group are conjugate in its automorphism group. This observation yields the second statements of Assertions (ii) and (iii). $\square$

Corollary 3.8 *Let (W,R) be an irreducible spherical Coxeter system and let $W = A \times B$ be a non-trivial decomposition of W as a direct product such that $|A| \leq |B|$. Then (W,R) is of (-1)-type and $A = \langle s \rangle$ where s is the longest element in (W,R) and B admits no proper decomposition as a direct product.*

If B is a Coxeter group with Coxeter generating set C and if ρ denotes the longest element in (B,C), then $s\rho \in R^W$. Moreover, one of the following holds:

I) There exists $1 \leq k \in \mathbf{N}$ such that (W,R) is of type $I_2(4k+2)$ and (B,C) is of type $I_2(2k+1)$. Moreover $C \subseteq R^W$ and $R \subseteq (s\rho)^W \cup a^W$ for any $a \in C$.

D) There exists $1 \leq k \in \mathbf{N}$ such that (W,R) is of type C_{2k+1} and (B,C) is of type D_{2k+1}. Moreover $C \subseteq R^W$ and $R \subseteq (s\rho)^W \cup a^W$ for any $a \in C$.

$\bar{D}$) There exists $1 \leq k \in \mathbf{N}$ such that (W,R) is of type C_{2k+1} and (B,C) is of type D_{2k+1}. Moreover $\bar{C} := \{sc \mid c \in C\} \subseteq R^W$ and $R \subseteq (s\rho)^W \cup (sa)^W$ for any $a \in C$.

Proof: This follows from Lemma 3.6 and Proposition 3.7. $\square$

4 Right-angled generators

In this section we prove basic facts about right-angled generators of Coxeter systems. In the finite rank case, several of these facts follow from the more general results about the finite continuation in [9]. Although the results there do not apply directly to the infinite rank case, our arguments are nevertheless inspired by that paper. Combining our results on right-angled generators with the information of the previous sections we will accomplish the 'easy' direction of the main result which is Proposition 4.11. Furthermore, we shall give a brief outline on the proof of the opposite direction of the main result.

Definition: Let (W,S) be a Coxeter system. For $s \in S$ we put $s^\perp := \{t \in S \mid m_{st} = 2\}$ and $s^\infty := \{t \in S \mid m_{st} = \infty\}$ and we call s a *right-angled generator of* (W,S) if $S = \{s\} \cup s^\perp \cup s^\infty$.

Convention for this section: Throughout this section (W, S) is a Coxeter system (possibly of infinite rank) and $s \in S$ is a right-angled generator of (W, S). Moreover, $\pi : W \longrightarrow \{+1, -1\}$ is the unique homomorphism mapping s onto -1 and s' onto $+1$ for all $s \neq s' \in S$ and V denotes the kernel of π.

Lemma 4.1 *The centralizer of s in W is the group $\langle \{s\} \cup s^{\perp} \rangle$.*

PROOF: This follows from Assertion (ii) of Proposition 2.16 with $J := \{s\}$. $\square$

Lemma 4.2 *We have $s^W \cap \langle \{s\} \cup s^{\perp} \rangle = \{s\}$.*

PROOF: Let $w \in W$ be such that $t = s^w \in \langle \{s\} \cup s^{\perp} \rangle$. As s is not in V which is a normal subgroup of W, it follows that t is not in V. As $t \in \langle \{s\} \cup s^{\perp} \rangle$, there exists an element $u \in \langle s^{\perp} \rangle \leq C_W(s)$ such that $t = su$. As $t^2 = s^2 = [s, u] = 1$, we have $u^2 = 1$.

Suppose $u \neq 1_W$. By Lemma 2.15 applied to $(\langle s^{\perp} \rangle, s^{\perp})$ there exists a non-empty (-1)-set $J \subseteq s^{\perp}$ and an element $x \in \langle s^{\perp} \rangle \leq C_W(s)$ such that u^x is the longest element in $(\langle J \rangle, J)$. As $[s, x] = 1$ it follows that $(s^w)^x = (su)^x = su^x$ is the longest element of the (-1)-set $K := \{s\} \cup J$ which is a contradiction, because $|K| \geq 2$ and s is a reflection of (W, S). $\square$

Lemma 4.3 *Let $x \in W$ be such that $\langle s, x \rangle$ is a finite subgroup of W. Then $[s, x] = 1$ and in particular $x \in \langle \{s\} \cup s^{\perp} \rangle$.*

PROOF: We first consider the case where $x \in V$. By Lemma 2.11 there exist a spherical subset J of S and an element $w \in W$ such that $\langle s, x \rangle^w \leq \langle J \rangle$. As $\langle s, x \rangle$ is not contained in V which is a normal subgroup of W, $\langle s, x \rangle^w$ is not contained in V. As $S \setminus \{s\} \subseteq V$ it follows that $s \in J$. Since J is spherical and contains s, it follows that $J \subseteq \{s\} \cup s^{\perp}$. It follows that $s^w \in \langle J \rangle \leq \langle \{s\} \cup s^{\perp} \rangle$ and applying Lemma 4.2 we see that $s^w = s$ which means $[s, w] = 1$. Note that $x^w \in V \cap \langle \{s\} \cup s^{\perp} \rangle = \langle s^{\perp} \rangle$ and hence $[s, x]^w = [s^w, x^w] = [s, x^w] = 1$ which implies $[s, x] = 1$.

Suppose now that x is not in V. Then $v := sx \in V$ and $\langle s, v \rangle = \langle s, x \rangle$ is a finite subgroup of W. By what we know already it follows that $1 = [s, v] = [s, sx]$ which implies $[s, x] = 1$.

The last assertion of the lemma follows from Lemma 4.1. $\square$

Proposition 4.4 *Let $r \in \langle s^{\perp} \rangle$ be an involution and $x \in W$ be such that $\langle sr, x \rangle$ is a finite subgroup of W. Then $[s, x] = 1$ and in particular $x \in \langle \{s\} \cup s^{\perp} \rangle$.*

Proof: As in the proof of Lemma 4.3 we consider first the case where $x \in V$. As $r \in \langle s^\perp \rangle$ there exists a (-1)-set $J \subseteq s^\perp$ and $y \in \langle s^\perp \rangle$ such that $u := r^y$ is the longest element in $(\langle J \rangle, J)$. Setting $K := J \cup \{s\}$, it follows that K is of (-1)-type and that su is the longest element in $(\langle K \rangle, K)$. As $y \in \langle s^\perp \rangle$ it follows that $s^y = s$ and therefore $\langle su, x^y \rangle$ is a finite group. It follows from Corollary 2.14 that $\langle K \cup \{x^y\} \rangle$ is a finite group which implies that $\langle s, x^y \rangle$ is a finite group. Applying now Lemma 4.3 we have $1 = [s, x^y] = [s^y, x^y] = [s, x]^y$ and hence $[s, x] = 1$.

Suppose now that x is not in V and put $v := srx$. Then $v \in V$ and $\langle sr, v \rangle = \langle sr, x \rangle$ is a finite group. By what we know from the first case, it follows that $1 = [s, v] = [s, srx]$ and as $[s, r] = 1$ is follows that $[s, x] = 1$.

The last assertion of the lemma follows from Lemma 4.1. □

Corollary 4.5 *Let $r \in \langle s^\perp \rangle$ be an involution and let $u \in s^\infty$. Then $(sr)u$ has infinite order.*

Proof: Assume by contradiction that $(sr)u$ has finite order. As sr and u are both involutions, it follows that $\langle sr, u \rangle$ is a finite subgroup of W. Thus we are in the position to apply Proposition 4.4 which yields $[s, u] = 1$ and hence a contradiction. □

s-components

We first recall the definition of an s-component: An s-component is a spherical irreducible component of $(\langle s^\perp \rangle, s^\perp)$.

Proposition 4.6 *Let C be an s-component of (-1)-type and let ρ be the longest element in $(\langle C \rangle, C)$. Then $R := \{s\rho\} \cup (S \setminus \{s\})$ is a Coxeter generating set of W and s is not a reflection of (W, R).*

Proof: We define the map $\alpha : S \to W$ by $\alpha(s) := s\rho$ and $\alpha(t) := t$ for all $t \in S \setminus \{s\}$.

We claim that the order of $\alpha(u)\alpha(v)$ is equal to the order of uv for all $u, v \in S$. This is obvious if $u \neq s \neq v$ and if $u = s = v$ since ρ is an involution contained in $\langle s^\perp \rangle \leq C_W(s)$ and $\alpha(s) = s\rho$. It remains to consider the case where $u = s \neq v$. As C is an irreducible component of $(\langle s^\perp \rangle, s^\perp)$ of (-1)-type, it follows that ρ is in the center of $\langle s^\perp \rangle$. Thus, if $v \in s^\perp$, we have $[\alpha(s), \alpha(v)] = [s\rho, v] = 1$ and hence the order of $\alpha(s)\alpha(v) = s\rho v$ is equal 2 which is also the order of sv. If $v \in s^\infty$, then $\alpha(s)\alpha(v) = s\rho v$ has infinite order by Corollary 4.5. Thus our claim is proved.

In view of the above and by the universal property of (W, S), there is a unique endomorphism τ of W such that $\tau \mid_S = \alpha$ and its square is easily seen to be the identity on W. Thus τ is an automorphism and $R := \tau(S)$ is a Coxeter generating set of W.

Assume, by contradiction, that $s \in R^W$. Then it follows that $S \subseteq R^W$ because $S \setminus \{s\} \subseteq R$. By Lemma 3.5 this implies $S^W = R^W$. As $s\rho \in R$ it follows that $s\rho \in S^W$. This is a contradiction because $s\rho$ is the longest element of $(\langle \{s\} \cup C \rangle, \{s\} \cup C)$ and $|\{s\} \cup C| > 1$. □

Corollary 4.7 *If there exists an s-component of (-1)-type, then s is not an intrinsic reflection of W.*

Blowing down generators

We first recall the definition of a blowing down generator for s: We call $a \in s^\perp$ a *blowing down generator for s* if the following conditions are satisfied:

(BDG1) If C denotes the irreducible component of $(\langle s^\perp \rangle, s^\perp)$ containing a, then $(\langle C \rangle, C)$ is of type $I_2(2k+1)$ or D_{2k+1} for some $k \in \mathbf{N}$; moreover, if ρ denotes the longest element in the Coxeter system $(\langle C \rangle, C)$, then $b := \rho a \rho \neq a$.

(BDG2) If $u_0, u_1, \ldots, u_n \in s^\infty$ are such that $u_{i-1}u_i$ has finite order for all $1 \leq i \leq n$, then there exists an element $x \in \{a, b\}$ such that $\{u_i \mid 0 \leq i \leq n\} \subseteq x^\infty$.

Let a be a blowing down generator for s and let C, ρ and $b = \rho a \rho$ be as in the previous definition. Then we call a a *proper* blowing down generator for s if $s^\infty \subseteq b^\infty$.

Proposition 4.8 *Suppose that $a \in s^\perp$ is a blowing down generator for s. Then there exists a Coxeter generating set S_1 of W with the following properties.*

(i) $\{s\} \cup s^\perp \subseteq S_1$;

(ii) $S_1^W = S^W$;

(iii) s is a right-angled generator of (W, S_1);

(iv) a is a proper blowing down generator for s with respect to the generating set S_1.

Proof: Let $K_0 := \{t \in s^\infty \mid m_{bt} \neq \infty\}$ and for $1 \leq n \in \mathbf{N}$ let $K_n := \{u \in s^\infty \mid m_{tu} \neq \infty$ for some $t \in K_{n-1}\}$ and put $K := \cup_{n \geq 0} K_n$. It follows from the construction of the set K that $m_{kl} = \infty$ for all $(k, l) \in K \times (s^\infty \setminus K)$. As a is a blowing down generator for s it follows also that $K \subseteq a^\infty$. As C is an irreducible component of $(\langle s^\perp \rangle, s^\perp)$ we have also that $(\{s\} \cup s^\perp) \setminus C \subseteq C^\perp$. We are now in the position to apply a diagram twist as described in [1] Definition 4.4. with $V := C$ and $U := K$. Setting $S_1 = (S \setminus K) \cup K'$ with $K' := \{\rho x \rho \mid x \in K\}$ we obtain a new Coxeter generating set of W by Theorem 4.5 of [1]. By the definition of S_1 we have $\{s\} \cup s^\perp \subseteq S_1 \subseteq S^W$, which implies $S^W = S_1^W$ by Lemma 3.5. A straight forward checking reveals that s is also a right-angled generator of (W, S_1) and that a is indeed a proper blowing down generator for s with respect to S_1. □

Proposition 4.9 *Let $a \in s^\perp$ be a proper blowing down generator for s. Let C be the irreducible component of $(\langle s^\perp \rangle, s^\perp)$ containing a, let ρ be the longest element in $(\langle C \rangle, C)$ and let $b := \rho a \rho$. Then $R := (S \setminus \{s, b\}) \cup \{s\rho\}$ is a Coxeter generating set of W and $s \notin R^W$.*

PROOF: Let $I := \{s\} \cup C$ and $I_1 := (I \setminus \{s, b\}) \cup \{s\rho\}$. It follows from Lemma 3.6 that I_1 is a Coxeter generating set of $\langle I \rangle$. Let $X := \{s\} \cup s^\perp \setminus I$. Then $m_{ix} = 2$ for all $(i, x) \in I \times X$. By Assertion (i) of Corollary 3.4 it follows that $I_1 \cup X$ is a Coxeter generating set of $\langle \{s\} \cup s^\perp \rangle$.

We now set $K := \{s\} \cup s^\perp$. By what we have just proved it follows that $K_1 := (K \setminus \{b, s\}) \cup \{s\rho\}$ is a Coxeter generating set of K. Setting $L := (K \setminus \{b, s\}) \cup s^\infty$ we have $J := K \cap L = K \setminus \{s, b\}$. Thus $K \setminus L = \{s, b\}$ and $L \setminus K = s^\infty \subseteq b^\infty$. We conclude that $m_{kl} = \infty$ for all $(k, l) \in (K \setminus L) \times (L \setminus K)$. Thus we are in the position to apply Assertion (ii) of Corollary 3.4 in order to see that $(S \setminus \{s, b\}) \cup \{s\rho\}$ is a Coxeter generating set of W.

As $a \in R$, we have also $b = \rho a \rho \in R^W$. Assume, by contradiction, that $s \in R^W$. Then $S \subseteq R^W$ which implies $S^W = R^W$ by Lemma 3.5 and hence $s\rho \in R$ is a reflection of (W, S). As $\rho \neq 1$ Lemma 4.2 yields a contradiction. Hence s is not a reflection of (W, R) which concludes the proof of the proposition. □

Corollary 4.10 *If there exists a blowing down generator $a \in s^\perp$ for s, then s is not an intrinsic reflection of (W, S).*

PROOF: Let $a \in s^\perp$ be a blowing down generator for s and let S_1 be as in Proposition 4.8. Now, by Proposition 4.9 applied to (W, S_1) there exists a Coxeter generating set R of W such that s is not a reflection of (W, R). □

Remark on the proof of the main result

Corollaries 4.7 and 4.10 yield the following Proposition.

Proposition 4.11 *If s is an intrinsic reflection of (W, S), then each s-component has trivial center and there are no blowing down generators for s.*

Proposition 4.11 provides one direction of the main result. In the remainder of the paper we shall prove the Proposition of the introduction which is just a reformulation of the other direction. In order to do this we have to establish the existence of a blowing down generator for s. Here, checking Axiom (BDG2) requires most of the work. In Proposition 5.1 we shall divide up the problem into three cases which will be treated separately in the subsequent sections. One of these cases is rather easy to handle. In order to settle the remaining two cases we need an additional tool which is Proposition 7.1. Its proof uses the geometry of the Cayley graph of (W, S).

5 Reduction to three cases

As already pointed out, the proof of the opposite direction of the main result splits into three cases. In this section we shall establish this case distinction. More specifically, the goal of this subsection is to prove the following.

Proposition 5.1 *Let (W, S) be a Coxeter system, let $s \in S$ be a right-angled generator of (W, S) such that there is no s-component of (-1)-type and suppose that there is a Coxeter generating set T of W such that s is not a reflection of (W, T). Then there exists a Coxeter generating set R of W, an irreducible subset J of R of (-1)-type and an s-component C such that s is the longest element of $(\langle J \rangle, J)$, $\langle J \rangle = \langle s \rangle \times \langle C \rangle$ and $s\rho \in J^{\langle J \rangle}$ where ρ denotes the longest element in $(\langle C \rangle, C)$. Moreover, one of the following holds.*

- *I) There exists $1 \leq k \in \mathbf{N}$ such that $(\langle J \rangle, J)$ is of type $I_2(4k+2)$ and $(\langle C \rangle, C)$ is of type $I_2(2k+1)$. Moreover $C \subseteq J^{\langle J \rangle}$ and $J \subseteq (s\rho)^{\langle J \rangle} \cup a^{\langle J \rangle}$ for any $a \in C$.*
- *D) There exists $1 \leq k \in \mathbf{N}$ such that $(\langle J \rangle, J)$ is of type C_{2k+1} and $(\langle C \rangle, C)$ is of type D_{2k+1}. Moreover $C \subseteq J^{\langle J \rangle}$ and $R \subseteq (s\rho)^{\langle J \rangle} \cup a^{\langle J \rangle}$ for any $a \in C$.*
- *$\bar{D}$) There exists $1 \leq k \in \mathbf{N}$ such that $(\langle J \rangle, J)$ is of type C_{2k+1} and $(\langle C \rangle, C)$ is of type D_{2k+1}. Moreover $\bar{C} := \{sc \mid c \in C\} \subseteq J^{\langle J \rangle}$ and $R \subseteq (s\rho)^{\langle J \rangle} \cup (sa)^{\langle J \rangle}$ for any $a \in C$.*

On decompositions of finite Coxeter groups as direct products

Proposition 5.2 *Let (W, S) be a spherical Coxeter system and let $s \in S$ be such that $Z(W) = \langle s \rangle$. Suppose that A and B are subgroups of W such that $W = A \times B$. Suppose further that $\langle s \rangle$ is properly contained in A and that there exists a subset J of A such that (A, J) is an irreducible Coxeter system. Then there exists an irreducible component $C \neq \{s\}$ of (W, S) such that $A = \langle s \rangle \times \langle C \rangle$.*

In order to establish the proof of this proposition it is convenient to recall some basic results on direct product decompositions of groups and in particular, the Krull-Remak-Schmidt Theorem.

Definition: A group G is called *indecomposable* if $|G| > 1$ and if G is not the inner direct product of two non-trivial subgroups. A family $(H_i)_{1 \leq i \leq n}$ of subgroups of G is called a *Remak-decomposition* of G if G is the inner direct product of the $(H_i)_{1 \leq i \leq n}$ and if H_i is indecomposable for $1 \leq i \leq n$.

The following observation is an immediate consequence of Corollary 3.8.

Lemma 5.3 *Let (W, S) be an irreducible spherical Coxeter system and let $H_1, \ldots, H_n$ be non-trivial subgroups of W such that $W = H_1 \times H_2 \ldots \times H_n$. Then $n \leq 2$.*

The following is obvious.

Lemma 5.4 *Any finite group admits a Remak decomposition.*

Definition: Let G be a group and $\alpha \in Aut(G)$. Then α is called a *central automorphism of G* if α induces the identity on $G/Z(G)$, i.e. if $\alpha(g) \in gZ(G)$ for all $g \in G$.

We shall need the following version of the Krull-Remak-Schmidt Theorem.

Proposition 5.5 *Let G be a finite group and let $(H_i)_{1 \leq i \leq n}$ and $(K_j)_{1 \leq j \leq m}$ be Remak decompositions of G. Then $n = m$ and there exists a permutation $\pi \in Sym(n)$ and a central autormorphism α of G such that $\alpha(H_i) = K_{\pi(i)}$ for all $1 \leq i \leq n$.*

PROOF: This is a special case of Theorem 3.3.8 in [16]. □

Corollary 5.6 *Let G be a finite group and let $(H_i)_{0 \leq i \leq n}$ be a Remak decomposition of G such that $H_0 = Z(G)$. Let $A, B \leq G$ be such that $G = A \times B$ and $Z(G) \leq A$. Then there exist a subset I of $\{1, \ldots, n\}$ such that $A = Z(G) \times \langle H_i \mid i \in I \rangle$.*

Proof: Let $(X_i)_{1\le i\le k}$ and $(Y_j)_{1\le j\le l}$ be Remak decompositions of A and B. Then $(X_1,\ldots,X_k,Y_1,\ldots,Y_l)$ is a Remak decomposition of G. As any automorphism of G stabilizes $Z(G)=H_0$ it follows from Proposition 5.5 that $X_i=Z(G)$ for some $1\le i\le k$ because of the assumption $Z(G)\le A$. Without loss of generality we may assume $X_k=Z(G)$. We now put $K_0=Z(G)=H_0$, $K_i:=X_i$ for $1\le i\le k-1$ and $K_i:=Y_{i-k+1}$ for $k\le i\le m:=l+k-1$. It follows from Proposition 5.5 that $m=n$ and that there are a permutation $\pi\in Sym(n)$ and a central automorphism α of G such that $\alpha(H_i)=K_i$ for $1\le i\le n$. Setting $I:=\pi^{-1}(\{1,\ldots,k-1\})$ the assertion follows. □

Lemma 5.7 *Let (W,S) be a spherical Coxeter system and let $s\in S$ be such that $Z(W)=\langle s\rangle$. Let $C_1,\ldots,C_n$ be the irreducible components of (W,S) distinct from $C_0:=\{s\}$. Then $(\langle C_i\rangle)_{0\le i\le n}$ is a Remak decomposition of W.*

Proof: Let $1\le i\le n$. The Coxeter system $(\langle C_i\rangle,C_i)$ is an irreducible Coxeter system. Moreover, if $z\in Z(\langle C_i\rangle)$, then $z\in Z(W)\cap\langle C_i\rangle$ which is the trivial group. It follows from Corollary 3.8 that $\langle C_i\rangle$ is indecomposable which finishes the proof. □

Proof of Proposition 5.2: Let $C_1,\ldots,C_n$ be the irreducible components of (W,S) distinct from $C_0:=\{s\}$. By Lemma 5.7 $(\langle C_i\rangle)_{0\le i\le n}$ is a Remak decomposition of W.

Let A and B be subgroups of W satisfying the hypothesis of the Proposition. Thus we have that $\langle s\rangle=Z(W)$ is properly contained in A. As $(\langle C_i\rangle)_{0\le i\le n}$ is a Remak decomposition of W Corollary 5.6 yields a subset I of $\{1,\ldots,n\}$ such that $A=Z(W)\times\langle C_i\mid i\in I\rangle$. As $Z(W)$ is properly contained in A we have $I\neq\emptyset$. By our assumption that there is an irreducible Coxeter generating set J of A it follows from Lemma 5.3 that $|I|\le 1$. Thus, $A=Z(W)\times\langle C_i\rangle$ for some $1\le i\le n$ which is precisely the claim of the proposition.

More on right-angled generators

Convention: Throughout this subsection (W,S) is a Coxeter system and $s\in S$ is a right-angled generator of (W,S). We let $\pi:W\to\{+1,-1\}$ be the unique homomorphism mapping s onto -1 and s' onto $+1$ for all $s'\in S$ which are distinct from s and we denote its kernel by V. Moreover, we assume that there is no s-component of (W,S) which is of (-1)-type.

Lemma 5.8 *The center of $\langle s^\perp\rangle$ is trivial.*

Proof: This follows from the fact, that there is no irreducible component of $(\langle s^{\perp}\rangle, s^{\perp})$ of (-1)-type. $\square$

Lemma 5.9 *Let T be a Coxeter generating set of W such that s is not a reflection of (W,T). Then there exists a Coxeter generating set R of W and a (-1)-subset J of R such that $|J| \geq 2$ and s is the longest element of $(\langle J\rangle, J)$.*

Proof: By Lemma 2.15 there exists a (-1)-subset K of T and an element $w \in W$ such that s^w is the longest element in $(\langle K\rangle, K)$. As s is not in T^W, we have $|K| \geq 2$. Setting $v := w^{-1}$, $R := T^v$ and $J := K^v$ the claim follows. $\square$

Proposition 5.10 *Let R be a Coxeter generating set of W and let $J \subseteq R$ be of (-1)-type such that s is the longest element of $(\langle J\rangle, J)$. Then the following hold:*

(i) If $r \in R \setminus J$ is such that $\langle\{r\} \cup J\rangle$ is a finite group, then $[J, r] = 1$.

(ii) Let $K := \{r \in R \mid r \notin J \text{ and } [r, J] = 1\}$. Then $\langle\{s\} \cup s^{\perp}\rangle = C_W(s) = N_W(\langle J\rangle) = \langle J\rangle \times \langle K\rangle$.

(iii) J is an irreducible subset of R.

(iv) if $|J| \geq 2$ then there exists a spherical irreducible component of C of $(\langle s^{\perp}\rangle, s^{\perp})$ such that $\langle J\rangle = \langle s\rangle \times \langle C\rangle$.

Proof: As $\langle J\rangle$ is a finite subgroup containing s we have $\langle J\rangle \leq \langle\{s\} \cup s^{\perp}\rangle$ and also that $\langle J\rangle$ is not contained in V. Thus there exists a $\sigma \in J$ such that $\sigma \notin V$ which we can write as $\sigma = s\rho$ for some involution $\rho \in \langle s^{\perp}\rangle$. Suppose that $r \in R \setminus J$ is such that $r\sigma$ has finite order, then $\langle \sigma, r\rangle$ is a finite subgroup of W and by Proposition 4.4 we obtain that $[s, r] = 1$. As s is the longest element in $(\langle J\rangle, J)$, it follows that $r \in C_W(s) = N_W(\langle J\rangle)$. We conclude that $[r, J] = 1$. This yields Assertion (i).

As s is the longest element of $(\langle J\rangle, J)$ we have $C_W(s) = N_W(\langle J\rangle)$ by Assertion (i) of Proposition 2.16; Assertion (ii) of Proposition 2.16 and Assertion (i) yield $N_W(\langle J\rangle) = \langle J\rangle \times \langle K\rangle$; finally we have $C_W(s) = \langle\{s\} \cup s^{\perp}\rangle$ by Lemma 4.1. This finishes the proof of Assertion (ii).

By Assertion (ii) we have decomposition of $\langle\{s\} \cup s^{\perp}\rangle$ as a direct product $\langle J\rangle \times \langle K\rangle$. Assume by contradiction, that J is not irreducible. Then, by Lemma 2.5, the center of $\langle J\rangle$ is of order at least 4, because $(\langle J\rangle, J)$ is of (-1)-type. It follows that the center of $\langle\{s\} \cup s^{\perp}\rangle$ has order at least 4. Moding out $\langle s\rangle$ this yields that $\langle s^{\perp}\rangle$ has a non-trivial center and contradicts Lemma 5.8. This finishes the proof of Assertion (iii).

Let $P := Pc_S(\langle J\rangle)$ be the parabolic closure of $\langle J\rangle$ in (W, S). As $\langle J\rangle$ is a finite normal subgroup of $\langle \{s\} \cup s^\perp\rangle$, the group P is also a finite, normal subgroup of $\langle\{s\}\cup s^\perp\rangle$. As P is a parabolic subgroup of the Coxeter system $(\langle\{s\}\cup s^\perp\rangle, \{s\}\cup s^\perp)$, it follows that there are irreducible spherical components $C_1, \ldots, C_n$ of $(\langle s^\perp\rangle, s^\perp)$ such that $P = \langle s\rangle \times \langle C_1\rangle \times \ldots \times \langle C_n\rangle$. For $1 \leq i \leq n$ the set C_i is an s-component and since there are no s-components of (-1)-type by assumption, it follows that the center of $\langle C_i\rangle$ is trivial. Thus $\langle s\rangle$ is the center of P.

Setting $Q := \langle K\rangle \cap P$ we have a direct decomposition $P = \langle J\rangle \times Q$. We are now in the position to apply Proposition 5.2 with $W := P, A := \langle J\rangle$ and $B := Q$. It follows that there exists an $1 \leq i \leq n$ such that $\langle J\rangle = \langle s\rangle \times \langle C_i\rangle$. As C_i is an irreducible spherical component of $(\langle s^\perp\rangle, s^\perp)$ this finishes the proof of Assertion (iv). □

Proof of Proposition 5.1: By Lemma 5.9 there exists a Coxeter generating set R of W and a (-1)-subset J of R such that $|J| \geq 2$ and such that s is the longest element of $(\langle J\rangle, J)$. By Assertion (iv) of Proposition 5.10 there exists an irreducible spherical component C of $(\langle s^\perp\rangle, s^\perp)$ such that $\langle J\rangle = \langle s\rangle \times \langle C\rangle$. Applying now Corollary 3.8 with $(W, R) := (\langle J\rangle, J), A := \langle s\rangle$ and $B := \langle C\rangle$, yields the proposition.

6 The case $\bar{D}$

Convention: Throughout this section (W, S) is a Coxeter system, $s \in S$ is a right-angled reflection of (W, S), $\pi : W \to \{+1, -1\}$ is the homomorphism which sends s onto -1 and t onto $+1$ for all $t \in S \setminus \{s\}$ and $V \leq W$ is its kernel. Moreover, $C \subseteq s^\perp$ is a s-component of type D_{2k+1}, ρ is the longest element of $(\langle C\rangle, C)$ and $a \in C$ is such that $\rho a\rho \neq a$. Finally, $R \subseteq W$ is a Coxeter generating set of W and $J \subseteq R$ is of type C_{2k+1} and such that $\langle J\rangle = \langle s\rangle \times \langle C\rangle$ and $J \subseteq (s\rho)^{\langle J\rangle} \cup (sa)^{\langle J\rangle}$.

The goal of this section is to prove the following.

Proposition 6.1 *The order of cu is infinite for all $c \in C$ and all $u \in s^\infty$. In particular, a is a blowing down generator for s.*

Lemma 6.2 *Suppose that $r \in J$ and $x \in W$ are such that $\langle r, x\rangle$ is a finite subgroup of W. Then x normalizes $\langle J\rangle$.*

Proof: As ρ and a are in V, it follows that $s\rho$ and sa are not in V and therefore $r \in (s\rho)^{\langle J\rangle} \cup (sa)^{\langle J\rangle}$ is not in V because V is a normal subgroup of W. On the other hand, $r \in \langle J\rangle = \langle s\rangle \times \langle C\rangle$ which implies that there is an involution $\omega \in \langle C\rangle \leq \langle s^\perp\rangle$

such that $r = s\omega$. By our assumption, $\langle s\omega, x\rangle$ is a finite subgroup of W. Thus it follows by Proposition 4.4 that $[s, x] = 1$ and in particular $x \in \langle \{s\} \cup s^\perp\rangle$.

As C is an irreducible component of $(\langle \{s\} \cup s^\perp\rangle, \{s\} \cup s^\perp)$ it follows that x normalizes $\langle C\rangle$. Now, x centralizes $\langle s\rangle$ and normalizes $\langle C\rangle$. Thus it normalizes $\langle J\rangle = \langle s\rangle \times \langle C\rangle$ and we are done. □

Lemma 6.3 *Let $1 \neq y \in \langle J\rangle$ and $x \in W$ be such that $\langle y, x\rangle$ is a finite group. Then $x \in \langle \{s\} \cup s^\perp\rangle$.*

Proof: Let $Y := Pc_R(y)$. As $1 \neq y \in \langle J\rangle$ and $J \subseteq R$, there exist $w \in \langle J\rangle$ and $J_1 \subseteq J$ such that $J_1 \neq \emptyset$ and $Y^w = \langle J_1\rangle$. By Corollary 2.12 the group $\langle Y, x\rangle$ is a finite group and therefore $\langle Y^w, x^w\rangle$ is finite as well.

Let $r \in J_1 \subseteq \langle J_1\rangle = Y^w$. Then $\langle r, x^w\rangle \leq \langle Y^w, x^w\rangle$ is a finite group and therefore x^w normalizes $\langle J\rangle$ by Lemma 6.2. As $w \in \langle J\rangle$ it follows that x normalizes $\langle J\rangle$ and hence also $Z(\langle J\rangle) = \langle s\rangle$. Applying Lemma 4.1 we obtain $x \in \langle \{s\} \cup s^\perp\rangle$. □

Proof of Proposition 6.1: Let $c \in C$ and $u \in S$. Then $1 \neq c \in \langle J\rangle$. Thus, if cu has finite order, then $\langle c, u\rangle$ is a finite group. By Lemma 6.3 it follows that $u \in \langle \{s\} \cup s^\perp\rangle$ and hence $u \in \{s\} \cup s^\perp$ because $u \in S$. We conclude that cu has infinite order for all $u \in s^\infty$.

7 On the Cayley graph of (W, S)

The goal of this section is to prove the following proposition.

Proposition 7.1 *Let (W, S) be a Coxeter system, $\{\tau, a\} \subseteq S^W$ be such that $a\tau$ has finite order and such that $b := a^\tau \neq a$. Let $\sigma \in \langle a, \tau\rangle \cap S^W$ be such that $\sigma \neq \tau$ and $[\tau, \sigma] = 1$.*

Suppose that $(U_0, U_1, \ldots, U_k)$ is a sequence of subgroups of W such that the following hold:

(i) $\langle U_{i-1}, U_i\rangle$ is a finite group for all $1 \leq i \leq k$;

(ii) $\langle U_i, \tau\rangle$ and $\langle U_i, \sigma\rangle$ are both infinite groups for all $0 \leq i \leq k$.

Then there exists an $x \in \{a, b\}$ such that $\langle U_i, x\rangle$ is an infinite group for all $0 \leq i \leq k$.

Remark: Our proof of the proposition uses the Cayley graph associated with (W, S) which is in fact the unique thin building of type (W, S). We shall apply several basic facts about buildings in our reasoning. It is convenient to adapt the language of buildings for the Cayley graph in order to be able to give references in the literature. Thus, the vertices of the Cayley graph will be called *chambers*, the edges of the Cayley graph will be called *panels* and right-cosets of standard parabolic subgroups will be called *residues*.

Galleries and convexity in $\Sigma(W, S)$

Let (W, S) be a Coxeter system. The Coxeter complex associated with (W, S) is defined to be the pair $\Sigma(W, S) := (\mathcal{C}, \mathcal{P})$ where $\mathcal{C} := W$ and $\mathcal{P} := \{\{sw, w\} \mid s \in S, w \in W\}$. The elements of $\mathcal{C}$ are called the *chambers* of $\Sigma(W, S)$ and the elements of $\mathcal{P}$ are called the *panels* of $\Sigma(W, S)$.

As S generates W the graph $\Sigma(W, S)$ is connected. Let $c, d \in \mathcal{C}$. A *gallery from c to d of length $m \in \mathbf{N}$* is a sequence $\gamma = (c = c_0, c_1, \ldots, c_m = d)$ such that $\{c_{i-1}, c_i\} \in \mathcal{P}$ for all $1 \leq i \leq m$. The *distance between c and d* is the length of a gallery joining them of minimal length; it is denoted by $\ell(c, d)$ and we observe that $\ell(c, d) := \ell(cd^{-1})$ where $\ell : W \to \mathbf{N}$ denotes the length function of (W, S). Note that $(\mathcal{C}, \ell)$ is a metric space and that therefore we have a natural notion of a *convex subset of $\mathcal{C}$*.

Residues in $\Sigma(W, S)$

We continue to assume that (W, S) is a Coxeter system and we let $\Sigma(W, S) = (\mathcal{C}, \mathcal{P})$ be its Coxeter complex.

A subset R of $\mathcal{C}$ is called a *residue* of $\Sigma(W, S)$ if there are a subset J of S and $w \in W$ such that $R = \langle J \rangle w$. As $\langle J \cap K \rangle = \langle J \rangle \cap \langle K \rangle$ for all $J, K \subseteq S$, the set $J \subseteq W$ in the definition of a residue R is uniquely determined by R. This set is called the *type of R* and the *rank of R* is defined to be the cardinality of its type. We observe that the residues of rank 1 are precisely the panels of $\Sigma(W, S)$. A residue is called *spherical* if its type is spherical; hence a residue R is spherical if and only if R is a finite set.

Lemma 7.2 *Let $R \subseteq \mathcal{C}$ be a residue. Then R is a convex subset of $(\mathcal{C}, \ell)$.*

PROOF: This is Proposition 3.24 in [24]. □

For a chamber $c \in \mathcal{C}$ and $w \in W$ we put $c^w := cw$ where cw is the product in W. In this way we get an action $\mathcal{C} \times W \to \mathcal{C}, (c, w) \mapsto c^w$ which is regular on the set of chambers and type-preserving on the set of residues.

Remark: Note that in our setup, the group W acts from the right on $\Sigma(W, S)$.

Lemma 7.3 *Let $R \subseteq \mathcal{C}$ be a residue, let $J \subseteq S$ be its type and let $w \in R$. Let $\mathcal{P}_R := \{P \in \mathcal{P} \mid P \subseteq R\}$. Then the following hold:*

(i) the stabilizer of R in W is the group $w^{-1}\langle J\rangle w$;

(ii) the map $x \mapsto x^w$ is an isomorphism from $\Sigma(\langle J\rangle, J)$ onto $\Sigma_R := (R, \mathcal{P}_R)$;

(iii) if $|J| = 2$ and $\mathcal{E} \subseteq \mathcal{P}_R$ has cardinality at least 3, then the graph $(R, \mathcal{P}_R \setminus \mathcal{E})$ has at least 3 connected components.

PROOF: Assertions (i) and (ii) are straightforward. Suppose that $|J| = 2$ which means that $\langle J\rangle$ is a dihedral group. Then $\Sigma(\langle J\rangle, J)$ is isomorphic to $(\mathbf{Z}, \{\{z, z+1\} \mid z \in \mathbf{Z}\})$ (if $|\langle J\rangle| = \infty$) or to a circuit of length $2k$ for some $2 \leq k \in \mathbf{N}$ (if $\langle J\rangle$ is a finite group). In both cases one verifies that removing at least three edges produces at least three connected components and Assertion (iii) is thus a consequence of Assertion (ii). □

Walls and roots in $\Sigma(W, S)$

We continue to assume that (W, S) is a Coxeter system and we let $\Sigma(W, S) = (\mathcal{C}, \mathcal{P})$ be its Coxeter complex. We recall that $S^W := \{w^{-1}sw \mid w \in W, s \in S\}$ is the set of reflections of (W, S).

Lemma 7.4 *Let $P \in \mathcal{P}$. Then $Stab_W(P) = \langle t\rangle$ for some $t \in S^W$.*

PROOF: This follows from Assertion (i) of Lemma 7.3 and the fact, that panels are precisely the residues of rank 1. □

Let $t \in S^W$ be a reflection of (W, S). The *wall of t* is defined to be the set M_t of all panels stabilized by t; hence $M_t := \{P \in \mathcal{P} \mid P^t = P\}$. Furthermore, we define the graph $\Sigma_t := (\mathcal{C}, \mathcal{P} \setminus M_t)$.

Lemma 7.5 *Let $t \in S^W$ be a reflection. Then Σ_t has two connected components which are interchanged by t. If $u \in S^W$ is a reflection distinct from t, then $M_t \cap M_u = \emptyset$; in particular, each panel $Q \in M_u$ is contained in one of the two connected components of Σ_t.*

PROOF: This follows from Propositions 3.11 and 3.12 in [24]. □

Let $t \in S^W$ be a reflection of (W, S). The two connected components of $\Sigma_t(W, S)$ are called the *roots associated with t*. For a chamber $c \in \mathcal{C}$ we denote the root associated to t which contains c by $H(t, c)$; more generally, if $\emptyset \neq X \subseteq \mathcal{C}$ is such that $H(t, x) = H(t, y)$ for all $x, y \in X$, then we denote the unique root associated with t containing the set X by $H(t, X)$.

A *root* of (W, S) is a set of chambers $\emptyset \neq \alpha \subseteq \mathcal{C}$ such that there exists a reflection $t \in S^W$ with $\alpha = H(t, \alpha)$ and $\Phi(W, S)$ denotes the set of all roots of (W, S).

Proposition 7.6 *The following hold:*

(i) Roots are convex subsets of $(\mathcal{C}, \ell)$.

(ii) if $X \subseteq \mathcal{C}$ *is a convex subset of* $(\mathcal{C}, \ell)$ *then* X *is the intersection of all roots containing* X.

(iii) Suppose that $X \subseteq \mathcal{C}$ *is a connected subset of* $\Sigma(W, S)$ *and that* $t \in S^W$ *is such that* X *contains no panel on the wall of* t. *Then there exists a root associated with* t *containing* X. *In particular,* $H(t, X)$ *is well defined.*

(iv) Let $R \subseteq \mathcal{C}$ *be a residue and let* $t \in S^W$ *be a reflection. Then* t *stabilizes* R *if and only if there is a panel* $P \in M_t$ *such that* $P \subseteq R$. *If this is not the case, then* R *is contained in a unique root associated with* t, *i.e.* $H(t, R)$ *is well defined.*

Proof: Assertion (i) is Proposition 3.19 in [24] Assertion (ii) is (29.20) in [25].

Let (x_0, x_1, x_2) be path of length 2 in X. Then the panels $P := \{x_0, x_1\}$ and $Q := \{x_1, x_2\}$ are not in M_t by our assumption and hence $H(t, P)$ and $H(t, Q)$ is well defined. It follows that $H(t, x_0) = H(t, P) = H(t, x_1) = H(t, Q) = H(t, x_2)$. By induction of the length of a path in X joining two chambers x and y in X it follows that $H(t, x) = H(t, y)$ for any two chambers in X. Thus Assertion (iii) holds.

Let $R \subseteq \mathcal{C}$, let $J \subseteq R$ be its type and $t \in S^W$. If there exists a panel $P \in M_t$ which is contained in R, then P is stabilized by t and since R is the unique J-residue containing P, R is stabilized by t as well. Suppose now that R does not contain a panel of wall of t. Since R is convex, it is connected and Assertion (iii) yields that $H(t, R)$ is well defined. Now $H(t, R^t) = (H(t, R))^t = \mathcal{C} \setminus H(t, R)$ and hence $R^t \neq R$. □

Lemma 7.7 *Let* $t \neq u \in S^W$ *be such that* $tu = ut$. *Then* t *stabilizes both roots associated with* u.

Proof: Let P be a panel in M_t. As $t \neq u$ and $P \in M_t$ the root $H(u, P)$ is well defined and we have $(H(u, P))^t = H(u^t, P^t) = H(u, P)$. Hence t stabilizes $H(u, P)$ and hence also $-H(u, P) := \mathcal{C} \setminus H(u, P)$. □

Lemma 7.8 *Let $t \in S^W$ and let $R \subseteq \mathcal{C}$ be a residue of rank 2. Then $|\{P \in M_t \mid P \subseteq R\}| \leq 2$.*

PROOF: Let α and $-\alpha$ be the two roots associated with t. As R is convex (by Lemma 7.2) and as roots are convex (by Assertion (i) of Proposition 7.6), it follows that $R \cap \alpha$ and $R \cap -\alpha$ are convex and in particular connected. Setting, as in Lemma 7.3, $\mathcal{P}_R := \{P \in \mathcal{P} \mid P \subseteq R\}$ it follows that the graph $(R, \mathcal{P}_R \setminus M_t)$ has at most two connected components. Thus Assertion (iii) of Lemma 7.3 yields $|\{P \in M_t \mid P \subseteq R\}| \leq 2$ and we are done. □

Projections in $\Sigma(W, S)$

We continue to assume that (W, S) is a Coxeter system and we let $\Sigma(W, S) = (\mathcal{C}, \mathcal{P})$ be its Coxeter complex.

Lemma 7.9 *Let $R \subseteq \mathcal{C}$ be a residue and $c \in \mathcal{C}$. Then there exists a unique chamber $d \in R$ such that $\ell(c, x) = \ell(c, d) + \ell(d, x)$ for all $x \in R$.*

PROOF: This is Theorem 3.22 in [24]. □

Let $R \subseteq \mathcal{C}$ be a residue and $c \in \mathcal{C}$. The unique chamber d in Lemma 7.9 is called *the projection of c onto R* and it will be denoted by $proj_R c$. For any subset X of $\mathcal{C}$ we put $proj_R X := \{proj_R x \mid x \in X\}$.

Lemma 7.10 *Let R be a residue and $t \in S^W$ be such that $R^t = R$. Then the following hold:*

(i) $H(t, c) = H(t, proj_R c)$ for each chamber $c \in \mathcal{C}$;

(ii) $proj_R P \in M_t$ for all $P \in M_t$.

PROOF: $H(t, c)$ is a convex set of chambers by Assertion (i) of Proposition 7.6. As $R^t = R$, there is a panel $P \in M_t$ which is contained in R by Assertion (iv) of Proposition 7.6 and hence there is a chamber $d \in R \cap H(t, c)$. As there is a minimal gallery from c to d passing through $proj_R c$ we have $proj_R c \in H(t, c)$ which yields $H(t, c) = H(t, proj_R c)$ and hence Assertion (i).

Let x, y be the two chambers in P and let $x' := proj_R x$, $y' := proj_R y$. Without loss of generality we may assume that $\ell(y, y') \leq \ell(x, x')$. Assume, by contradiction, that $\ell(x', y') \geq 2$. Then $\ell(x, y') = \ell(x, x') + \ell(x', y') \geq \ell(x, x') + 2 \geq \ell(y, y') + 2 = \ell(y, y') + \ell(x, y) + 1 > \ell(x, y')$. Thus $\ell(x', y') \leq 1$. As $R^t = R$ and $x^t = y$, we have $x'^t = y' \neq x'$ and hence also $y'^t = x'$. Thus $Q := \{x', y'\}$ is a panel contained in R and stabilized by t. As $Q = proj_R P$ this finishes the proof of Assertion (ii). □

Proposition 7.11 *Let $R \subseteq \mathcal{C}$ be a residue of rank 2 and let $t, u, v \in S^W$ be pairwise distinct reflections such that $R^t = R^u = R^v = R$ and $uv = vu$. Then there exist roots $\alpha, \beta \in \Phi(W, S)$ such that the following holds:*

(i) α is associated with u and β is associated with v;

(ii) any panel $P \in M_t$ is contained in $(\alpha \cap \beta) \cup (-\alpha \cap -\beta)$.

If α and β are as above, then any panel in M_{utu} is contained in $(\alpha \cap -\beta) \cup (-\alpha \cap \beta)$.

Proof: As t stabilizes R there exists a panel $P \in M_t$ which is contained in R. Since $u \neq t \neq v$ the roots $\alpha := H(u, P)$ and $\beta := H(v, P)$ are well defined. Since $u \neq v$ and $r := uv = vu$ it follows that r is in the center of the stabilizer of R in W by Lemma 2.13. We have in particular $rt = tr$ and hence r stabilizes the wall M_t of t. As r stabilizes also the residue R, we have that $Q := P^r$ is a panel in the wall of t which is also contained in R.

Note also that $\alpha^r = (\alpha^v)^u = \alpha^u = -\alpha$ by Lemma 7.7, and similarly $\beta^r = -\beta$. As $Q = P^r \subseteq (\alpha \cap \beta)^r = (-\alpha) \cap (-\beta)$ it follows in particular $Q \neq P$. It follows by Lemma 7.8 that $\{X \in M_t \mid X \subseteq R\} = \{P, Q\}$. Let $Y \in M_t$ be a panel on the wall of t. Then, by Assertion (ii) of Lemma 7.10 we have $proj_R Y \in \{P, Q\}$. If $proj_R Y = P$, then $H(u, Y) = H(u, P) = \alpha$ and $H(v, Y) = H(v, P) = \beta$ (by Assertion (i) of Lemma 7.10) and hence $Y \subseteq \alpha \cap \beta$. Similarly, we obtain $Y \subseteq (-\alpha) \cap (-\beta)$ if $proj_R Y = Q$. This finishes the proof of the first assertion.

Let $P' := P^u$. As P is a panel in the wall of t we have $P' \in M_{utu}$ and as u stabilizes R and $P \subseteq R$, the panel P' is also contained in R. Now $H(u, P') = H(u, P^u) = (H(u, P))^u = \alpha^u = -\alpha$ and $H(v, P') = H(v, P^u) = (H(v^u, P))^u = (H(v, P))^u = \beta$. It follows now from the first assertion that each panel in M_{utu} is contained $((-\alpha) \cap \beta) \cup (\alpha \cap (-\beta))$ and we are done. □

Finite subgroups of W

We continue to assume that (W, S) is a Coxeter system and we let $\Sigma(W, S) = (\mathcal{C}, \mathcal{P})$ be its Coxeter complex.

Lemma 7.12 *Let $U \leq W$ be a finite subgroup of W. Then U stabilizes a spherical residue of $\Sigma(W, S)$.*

Proof: This follows from Lemma 2.11 and Assertion (i) of Lemma 7.3. □

For a finite subgroup $U \leq W$ we let $Sph(U)$ denote the set of all spherical residues stabilized by U.

Lemma 7.13 *Let $U \leq W$ be a finite subgroup and let $t \in S^W$ be such that $\langle U, t \rangle$ is an infinite group. Then there exists a unique root associated with t which contains each residue in $Sph(U)$.*

PROOF: This is Lemma 2.6. in [15]. □

Let $U \leq W$ be a finite subgroup of W and $t \in S^W$ be such that $\langle U, t \rangle$ is an infinite group. Then the unique root associated with t which contains each spherical residue stabilized by U is denoted by $H(t, U)$.

Proposition 7.14 *Let $t \in S^W$ and $(U_0, U_1, \ldots, U_k)$ be a sequence of subgroups such that*

- *$\langle U_{i-1}, U_i \rangle$ is a finite group for all $1 \leq i \leq k$;*
- *$\langle U_i, t \rangle$ is an infinite group for all $0 \leq i \leq k$.*

Then $H(t, U_i) = H(t, U_j)$ for all $0 \leq i, j \leq k$.

PROOF: We proceed by induction on k. For $k = 0$ the assertion is trivial and we may assume $k > 0$. By induction it suffices to show that $H(t, U_{k-1}) = H(t, U_k)$. As $V := \langle U_{k-1}, U_k \rangle$ is a finite subgroup of W by assumption, there exists a spherical residue R stabilized by V (by Lemma 7.12). As $U_k \leq V$ and $\langle U_k, t \rangle$ is infinite, it follows that $\langle V, t \rangle$ is infinite and $R \subseteq H(t, V)$; we have in particular, $H(t, V) = H(t, R)$. As U_{k-1} and U_k are both subgroups of V, it follows that R is also stabilized by these groups. Hence we have $H(t, U_{k-1}) = H(t, R) = H(t, U_k)$. □

Corollary 7.15 *Let $R \subseteq \mathcal{C}$ be a residue of rank 2 and let $t, u, v \in S^W$ be pairwise distinct reflections such that $R^t = R^u = R^v = R$ and $uv = vu$. Let $(U_0, U_1, \ldots, U_k)$ be a sequence of subgroups such that*

- *$\langle U_{i-1}, U_i \rangle$ is a finite group for all $1 \leq i \leq k$;*
- *$\langle U_i, u \rangle$ and $\langle U_i, v \rangle$ are both infinite groups for all $0 \leq i \leq k$.*

Then there exists $x \in \{t, utu\}$ such that $\langle U_i, x \rangle$ is an infinite group for all $0 \leq i \leq k$.

PROOF: By Proposition 7.14 there exists a root γ associated to u such that $\gamma = H(u, U_i)$ for all $0 \leq i \leq k$ and a root δ associated to v such that $\delta = H(v, U_i)$ for all $0 \leq i \leq k$. Using Proposition 7.11 we see that there exists a reflection $x \in \{t, utu\}$ such that $\gamma \cap \delta$ contains no panel of the wall M_x associated with x. We claim that $\langle U_i, x \rangle$ is an infinite group for all $0 \leq i \leq k$.

Assume, by contradiction, that $V := \langle U_i, x\rangle$ is a finite subgroup of W. Then V stabilizes a spherical residue T by Lemma 7.13 and since $x \in V$, there exists a panel $P \in M_x$ which is contained in T. On the other hand, T is a spherical residue stabilized by U_i because $U_i \leq V$. Hence we have $T \subseteq H(u, U_i) = \gamma$ and $T \subseteq H(v, U_i) = \delta$ which yields $P \subseteq \gamma \cap \delta$. Hence there exists $P \in M_x$ such that $P \subseteq \gamma \cap \delta$ which contradicts our choice of x. $\square$

Proof of Proposition 7.1: Since $a, \tau \in S^W$ and $a\tau$ has finite order, there exists a rank 2 residue $R \subseteq \mathcal{C}$ in $\Sigma(W, S)$ which is stabilized by $\langle a, \tau\rangle$. By the hypothesis of the proposition we have $b := a^\tau \neq a$, $\sigma \in \langle a, \tau\rangle \cap S^W$, $\sigma \neq \tau$ and $[\sigma, \tau] = 1$. As $\sigma \in \langle a, \tau\rangle \leq Stab_W(R)$ we have $R^a = R^\tau = R^\sigma = R$. Also, since $a^\tau \neq a$ and $\sigma^\tau = \sigma$ we have that a, τ and σ are pairwise distinct. Thus, applying Corollary 7.15 with $a := t, u := \tau$ and $v := \sigma$ provides the proposition.

8 The case D

Convention: Throughout this section (W, S) is a Coxeter system and $s \in S$ is a right-angled reflection of (W, S). Moreover, $C \subseteq s^\perp$ is a s-component of type D_{2k+1}, ρ is the longest element of $(\langle C\rangle, C)$ and $a \in C$ is such that $b := \rho a \rho \neq a$. We put $\tau := s\rho$ and $\sigma := abs\rho = ab\tau$. Finally, $R \subseteq W$ is a Coxeter generating set of W and $J \subseteq R$ is of type C_{2k+1} and such that $\langle J\rangle = \langle s\rangle \times \langle C\rangle$ and $J \subseteq (s\rho)^{\langle J\rangle} \cup a^{\langle J\rangle}$ and $a, \tau \in R^W$.

The goal of this section is to prove the following.

Proposition 8.1 *The generator a is a blowing down generator for s.*

Lemma 8.2 *The following hold.*

(i) $ab = ba \neq 1$ *and* ab *is an involution in* $\langle s^\perp\rangle$*;*

(ii) $\rho ab = ab\rho \neq 1$ *and* $ab\rho$ *is an involution in* $\langle s^\perp\rangle$*;*

(iii) $\rho s = s\rho$*,* τ *is an involution,* $a^\tau = b$*;*

(iv) $\sigma = \tau^a$*,* $\sigma \neq \tau$ *and* $\sigma\tau = \tau\sigma$*;*

(v) if $u \in s^\infty$*, then* σu *and* τu *are both of infinite order;*

(vi) $\sigma \in R^W$*.*

Proof: As ρ is the longest element of $(\langle C\rangle, C)$ which is of type D_{2k+1} and $a \in C$ is such that $b = a^\rho \neq a$, it follows that $ab = ba$ and as $a \neq b$ are both involutions, it follows that ab is an involution. As C is a spherical irreducible component of $(\langle s^\perp\rangle, s^\perp)$ and $a, b \in C$, it follows that $ab \in \langle s^\perp\rangle$. This concludes the proof of Assertion (i).

As ρ is an involution and $a^\rho = b$, we have $b^\rho = a$ and hence $(ab)^\rho = ba = ab$ where the last equality follows from Assertion (i). Thus ab and ρ are commuting involutions which shows that $(\rho ab)^2 = 1$; As $(\langle C\rangle, C)$ is of type D_{2k+1} with $1 \leq k$, it follows that the length of ρ with respect to the generator set C is $2k(2k+1) \geq 6$. On the other hand we have the length of ab with respect to C is 2 because $a \neq b \in C$ and therefore $\rho \neq ab$ which shows that $ab\rho$ is an involution. As $\rho, a, b \in \langle C\rangle$ and C is an irreducible spherical component of $(\langle s^\perp\rangle, s^\perp)$, it follows that $ab\rho \in \langle s^\perp\rangle$. This concludes the proof of Assertion (ii).

As $\rho \in \langle s^\perp\rangle$ and $s \notin \langle s^\perp\rangle$ it follows that $s \neq \rho$ and $s\rho = \rho s$. As s and ρ are both involutions, $\tau = s\rho$ is an involution as well. As $a \in s^\perp$ we have $[a, s] = 1$. It follows $a^\tau = \tau a \tau = (s\rho)a(s\rho) = a^\rho = b$ which finishes (iii).

We have $\tau^a = (s\rho)^a = as\rho a = sa\rho a = s\rho(\rho a\rho)a = s\rho a^\rho a = s\rho ba = \sigma$, since we have already established $[a, s] = [\rho, s] = [a, b] = 1$ in the previous parts of this proof. We have also $\tau^s = s(s\rho)s = \rho s = s\rho = \tau$. Assume by contradiction that $\tau = \sigma$. Then $s\rho ab = s\rho$ and hence $ab = 1$ implying $a = b$ which yields a contradiction. Finally, since $[a, s] = [b, s] = [ab, \rho] = 1$ we have $\sigma\tau = s\rho absρ = s\rho s(ab)\rho = s\rho s\rho ab = \tau\sigma$ and we are done with (iv).

We first remark that ρ and ρab are both involutions in $\langle s^\perp\rangle$ by Assertion (ii). As $\tau = s\rho$ and $\sigma = s\rho ab$, Assertion (v) follows from Corollary 4.5.

Finally, Assertion (vi) follows from Assertion (iv) of the Lemma. □

Proposition 8.3 *Let $(u_0, u_1, \ldots, u_n)$ be a sequence in s^∞ such that $u_{i-1}u_i$ has finite order for all $1 \leq i \leq n$. There exists $x \in \{a, b\}$ such that xu_i has infinite order for all $0 \leq i \leq n$.*

Proof: We have $\{\tau, a\} \subseteq R^W$. Moreover $a\tau$ has finite order since a and τ are both contained in the finite subgroup $\langle J\rangle$. We have also $b = a^\tau \neq a$ and Assertion (iv) of Lemma 8.2 yields $\sigma = \tau^a$ and hence $\sigma \in \langle \tau, a\rangle \cap R^W$ by Assertion (vi) of Lemma 8.2. By Assertion (iv) of Lemma 8.2 we have $\sigma \neq \tau$ and $[\tau, \sigma] = 1$.

Let $U_i := \langle u_i\rangle$ for $0 \leq i \leq n$. Then $\langle U_{i-1}, U_i\rangle = \langle u_{i-1}, u_i\rangle$ is a finite group for $1 \leq i \leq n$ because the u_i are involutions and $u_{i-1}u_i$ has finite order by hypothesis. Finally, Assertion (v) of Lemma 8.2 yields that $\langle \tau, U_i\rangle$ and $\langle \sigma, U_i\rangle$ are infinite groups for $0 \leq i \leq n$ because $u_i \in s^\infty$.

We are now in the position to apply Proposition 7.1 with $S := R$. It asserts that there is an element $x \in \{a, b\}$ such that $\langle x, U_i\rangle$ is an infinite group for all

$0 \leq i \leq n$. As $U_i = \langle u_i \rangle$ with an involution u_i for $0 \leq i \leq n$ it follows that there exists an $x \in \{a, b\}$ such that xu_i has infinite order for $0 \leq i \leq n$. □

Proof of Proposition 8.1: The generator a satisfies Axiom (BDG1) by the general assumptions of this section and Proposition 8.3 yields that a satisfies Axiom (BDG2) as well.

9 The case I

Convention: Throughout this section (W, S) is a Coxeter system, $s \in S$ is a right-angled reflection of (W, S). Moreover, $C = \{a, b\} \subseteq s^\perp$ is a s-component of type $I_2(2k+1)$, ρ is the longest element of $(\langle C \rangle, C)$ and $\tau := s\rho$. Finally, $R \subseteq W$ is a Coxeter generating set of W and $J \subseteq R$ is of type $I_2(4k+2)$ and such that $\langle J \rangle = \langle s \rangle \times \langle C \rangle$ and $J \subseteq (s\rho)^{\langle J \rangle} \cup a^{\langle J \rangle}$ and $a, \tau \in R^W$.

The goal of this section is to prove the following.

Proposition 9.1 *The generator a is a blowing down generator for s.*

Similarly as in the case D, the proof of this proposition consists essentially of checking the conditions of Proposition 7.1. However, in the I_2-case there is an additional difficulty which requires some extra work. It is the proof of Lemma 9.8 where we need an additional argument with respect to the D-case. This lemma could be established at much lesser cost if we would exclude the case $k = 1$.

Coxeter systems and FA-groups

The following definition is due to Serre (see Paragraph 6.1 in [22]):

Definition: A group G is called an *FA-group* if it satisfies the following condition:

(FA) If G acts without inversion on a non-empty tree $T = (V, E)$, then G fixes a vertex $v \in V$.

Lemma 9.2 *Finite groups are FA-groups.*

PROOF: This is a special case of Example 6.3.1 in [22]. □

Definition: Let (W, S) be a Coxeter system of finite rank. A subset J of S is called *2-spherical* if st has finite order for all $s, t \in J$. A parabolic subgroup P of (W, S) is called 2-spherical, if $P = \langle J \rangle^w$ for some 2-spherical subset of S and some $w \in W$.

The following result is due to Mihalik and Tschantz.

Proposition 9.3 *Let (W,S) be a Coxeter system of finite rank. Then the following hold:*

(i) If $J \subseteq S$ is 2-spherical, then $\langle J\rangle$ is a FA-group.

(ii) If $U \leq W$ is a finitely generated FA-group, then $Pc_S(U)$ is 2-spherical.

PROOF: Assertion (i) is Proposition 24 and Assertion (ii) is Lemma 25 in [13]. $\square$

Corollary 9.4 *Let W be a Coxeter group and let $S \subseteq W$ and $R \subseteq W$ be Coxeter generating sets. Let $J \subset S$ be a finite 2-spherical subset of S and let $s \in S$ be such that $s \in Pc_R(\langle J\rangle)$. Then $J \cup \{s\}$ is also a 2-spherical subset of S.*

PROOF: As J is finite and 2-spherical, $\langle J\rangle$ is a finitely generated FA-group. By Assertion (ii) of Proposition 9.3 $P := Pc_R(\langle J\rangle)$ is a 2-spherical parabolic subgroup of (W,R). (Note that $P = Pc_{R'}(\langle J\rangle)$ for a finite subset R' of R, since J is finite.) It is of finite rank and in particular a finitely generated FA-group. Moreover, $s \in P$ by assumption. Again by Assertion (ii) of Proposition 9.3 $Q := Pc_S(P)$ is a 2-spherical parabolic subgroup of (W,S) containing $\langle\{s\} \cup J\rangle$. Thus there exists a 2-spherical subset K of S and an element $w \in W$ such that $(\{s\} \cup J)^w \leq \langle K\rangle$. Hence, by Assertion (ii) of Proposition 2.7 $\{s\} \cup J$ is a 2-spherical subset of S. $\square$

Finite paths in s^∞

Lemma 9.5 *We have $b = a^\tau$, $\langle J\rangle = \langle a,\tau\rangle$, $\rho \neq \tau \neq a \neq \rho \in R^W \cap \langle J\rangle$ and $[\tau,\rho] = 1$.*

PROOF: As $a,b \in s^\perp$ and $\tau = s\rho$ we have $a^\tau = a^{s\rho} = a^\rho = b$ where the last equality follows from the fact that ρ is the longest element of the system $(\langle a,b\rangle, \{a,b\})$ which is of type $I_2(2k+1)$.

As, a and ρ are in $\langle C\rangle$ and $\tau = s\rho$ it follows that $\langle a,\tau\rangle \leq \langle s\rangle \times \langle C\rangle = \langle J\rangle$. On the other hand, $b = a^\tau \in \langle a,\tau\rangle$ which implies that $\langle C\rangle \leq \langle a,\tau\rangle$. We have in particular, $\rho \in \langle a,\tau\rangle$ and therefore also $s = \tau\rho \in \langle a,\tau\rangle$ which implies that $\langle J\rangle = \langle s\rangle \times \langle C\rangle \leq \langle a,\tau\rangle$ which finishes the proof of the second equation. As $b = a^\tau$ it follows that $b \in \langle J\rangle \cap R^W$ and in particular that $\rho \in \langle a,b\rangle \leq \langle J\rangle$. As $(\langle a,b\rangle, \{a,b\})$ is of type $I_2(2k+1)$ and ρ is its longest element, it follows that $\rho \in a^W$ and hence $\rho \in \langle J\rangle \cap R^W$. As $a^\tau = b \neq a$ it follows that $a \neq \tau$ and as $\tau\rho = s$ and $\rho^2 = 1$ it follows that $\tau \neq \rho$. Finally, as ρ is the longest element of the system $(\langle a,b\rangle, \{a,b\})$, it follows that $a \neq \rho$. As $\tau \in R$ we have $\tau^2 = 1$ and as $a^\tau = b$ we have $b^\tau = a$. It follows that τ normalizes $\{a,b\}$ and hence centralizes the longest element of the system $(\langle a, b\rangle, \{a, b\})$ which is ρ. □

Lemma 9.6 *We have $Pc_R(\langle a, b\rangle) = \langle J\rangle$ and in particular $s \in Pc_R(\langle a, b\rangle)$.*

PROOF: As $\langle a, b\rangle = \langle C\rangle \leq \langle J\rangle$, the group $Pc_R(\langle a, b\rangle)$ is a parabolic subgroup of (W, R) contained in $\langle J\rangle$ and $\langle J\rangle$ is a parabolic subgroup of (W, R) of rank 2. As the order of parabolic subgroup of rank one is 2 and the order of $\langle a, b\rangle$ is 4k+2, it follows that $Pc_R(\langle a, b\rangle) = \langle J\rangle$. □

Lemma 9.7 *Let $u \in S \setminus \{a, b, s\}$ be such that au and bu have finite order. Then $u \in s^{\perp}$.*

PROOF: By the previous lemma we have $s \in Pc_R(\langle a, b\rangle)$ As au and bu are of finite order the set $K := \{a, b, u\}$ is a 2-spherical subset of S. Thus we can apply Corollary 9.4 to see that $\{s\} \cup K$ is 2-spherical. As s is a right-angled generator it follows $[s, u] = 1$. □

Lemma 9.8 *Let $u \in s^{\infty}$. Then at least one of the elements ua and ub has infinite order.*

PROOF: This follows from Lemma 9.7. □

Lemma 9.9 *For each $u \in s^{\infty}$ the orders of $u\tau$ and $u\rho$ are infinite.*

PROOF: We have $a, b \in s^{\perp}$ and therefore $\rho \in \langle a, b\rangle$ is an involution contained in $\langle s^{\perp}\rangle$. By Corollary 4.5 it follows that $\tau u = (s\rho)u$ has infinite order for each $u \in s^{\infty}$.

We have $a, b, u \in S$ and by Lemma 9.8 we know at least one of the elements au and bu has infinite order. As ρ is the longest element in the system $(\langle a, b\rangle, \{a, b\})$, one verifies using the geometric representation (or the solution of the word problem in Coxeter groups) that the order of ρu is also infinite. □

Proposition 9.10 *Let $(u_0, u_1, \ldots, u_n)$ be a sequence in s^{∞} such that $u_{i-1}u_i$ has finite order for all $1 \leq i \leq n$. There exists $x \in \{a, b\}$ such that xu_i has infinite order for all $0 \leq i \leq n$.*

PROOF: We have $\{\tau, a\} \subseteq R^W$. Moreover the order of $a\tau$ is finite since a and τ are both contained in the finite subgroup $\langle J\rangle$. We have also $b = a^{\tau} \neq a$. By Lemma 9.5 we have also $\rho \in \langle J\rangle \cap R^W$ and $[\tau, \rho] = 1$.

Let $U_i := \langle u_i \rangle$ for $0 \leq i \leq n$. Then $\langle U_{i-1}, U_i \rangle = \langle u_{i-1}, u_i \rangle$ is a finite group for $1 \leq i \leq n$ because the u_i are involutions and $u_{i-1}u_i$ has finite order by hypothesis. Moreover, Lemma 9.9 yields that $\langle \tau, U_i \rangle$ and $\langle \rho, U_i \rangle$ are infinite groups for $0 \leq i \leq n$ because $u_i \in s^\infty$.

We are now in the position to apply Proposition 7.1 with $S := R$ and $\sigma := \rho$. It asserts that there is an element $x \in \{a, b\}$ such that $\langle x, U_i \rangle$ is an infinite group for all $0 \leq i \leq n$. As $U_i = \langle u_i \rangle$ with an involution u_i for $0 \leq i \leq n$ it follows that there exists an $x \in \{a, b\}$ such that xu_i has infinite order for $0 \leq i \leq n$. □

Proof of Proposition 9.1: The generator a satisfies Axiom (BDG1) by the general assumptions of this section and Proposition 9.10 yields that a satisfies Axiom (BDG2) as well.

10 Proof of the main result

We first recall the Proposition of the introduction.

Proposition 10.1 *Let (W, S) be a Coxeter system of arbitrary rank and let $s \in S$ be a right-angled generator such that each s-component has trivial center. If there exists a Coxeter generating set T of W such that s is not a reflection of (W, T), then there exists a blowing down generator for s.*

PROOF: In view of the hypothesis of the Proposition we are in the position to apply Proposition 5.1. Thus, there is a Coxeter generating set R of W, an irreducible subset J of R of (-1)-type and a s-component C such that we are in one of the cases I, D or $\bar{D}$ described in Proposition 5.1. If we are in case I, (resp. D,$\bar{D}$) Proposition 9.1 (resp. 8.1, 6.1) asserts that there exists a blowing down generator for s. □

The first assertion of the main result follows from Propositions 4.11 and 10.1. The second assertion follows from Lemma 4.3 applied to the Coxeter system (W, R).

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